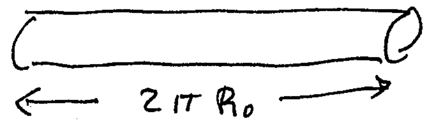
Lecture 15

Kink instability

Treatment of δW for kink in cylindrical approximation for tokamak: $R_0 \, d\zeta = dz$

Start with:



$$\delta W_F = \frac{1}{2}\int dr \Big[Q_\perp^2 + B^2 [\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\kappa|^2 + \gamma p |\nabla\cdot\xi|^2 - 2(\xi_\perp\cdot\nabla p)(\kappa\cdot\xi_\perp^*) - J_\parallel (\xi_\perp^*\times b)\cdot Q_\perp \Big]$$

Equilibrium Condition

$$\frac{\partial}{\partial r}\left(p + \frac{B_\theta^2 + B_\zeta^2}{2}\right) + \frac{B_\theta^2}{r} = 0;$$

Perturbation $\xi = \xi(r)\, e^{im\theta - in\zeta}$

Choose as displacement coordinates:

$$\xi = \xi\, e_r + \eta\, e_\eta + \xi_\parallel\, b$$

$$e_\eta = b \times \nabla\psi / |\nabla\psi|$$

$$= (B_\zeta\, e_\theta - B_\theta\, e_\zeta)/B \qquad \hat{e}_\zeta$$

Choose $\xi_\parallel$ so that $\nabla\cdot\xi = 0$

Find: $\xi_\parallel = i\dfrac{B}{F}\,\nabla\cdot\xi_\perp$, $k = \dfrac{m}{r}\hat{e}_\theta - \dfrac{n}{R_0}\hat{e}_\zeta$

$$F = k\cdot B = \frac{m B_\theta}{r} - \frac{n}{R_0} B_z = (m - nq(r))\, B_\theta / r$$

After this substitution we have $\delta W(\xi, \eta)$, with the expression algebraic in η. Then takin variation with respect to η, gives

$$\eta = \frac{1}{r k_0^2 B}\left[G(r\xi)' + \frac{2n}{R_0} B_\theta \xi\right]$$

$$k_0^2 = \frac{n^2}{R_0^2} + \frac{m^2}{r^2} \; ; \quad \underset{\sim}{G} = \underset{\sim}{k} \times \underset{\sim}{B} = \frac{m B_\xi}{r} + \frac{n}{R_0} B_\theta$$

$$\underset{\sim}{k} = \frac{m}{r}\hat{\theta} - \frac{n}{R_0}\hat{\xi}$$

Leads to (after appropriate integration by parts)

$$\frac{\delta W_F}{2\pi^2 R_0} = \left[\int_0^a \left(f \xi'^2 + g \xi^2\right) dr + \left[\frac{k^2 r^2 B_z^2 - m^2 B_\theta^2}{k_0^2 r^2}\right] \xi_a^2\right.$$

$$\text{if } f = \frac{r F^2}{k_0^2} \; ; \quad g = 2\frac{k^2}{k_0^2} p' + \frac{(k_0^2 r^2 - 1)}{k_0^2 r^2} r F^2 - \frac{2k^2}{r k_0^4}\left(\frac{n}{R_0} B_\xi + \frac{m B_\theta}{r}\right) F$$

$$k^2 = \frac{m^2}{r^2} + n^2/R_0^2$$

To this we need to add the vacuum contribution to δW the magnetic field, satisfying

$$\hat{\underset{\sim}{B}}_1 = \underset{\sim}{\nabla} \hat{V}_1 \; , \quad \nabla^2 V_1 = 0$$

With $\hat{\underset{\sim}{B}}_1 \cdot \underset{\sim}{n} = 0$ on surrounding conductor $(r = b)$
← check
and $\hat{B}_{1r}(a) = \underset{\sim}{B} \cdot \underset{\sim}{\nabla} \xi - (\underset{\sim}{\eta} \cdot \underset{\sim}{\nabla}) \underset{\sim}{B}_1 \Big|_a$, leading to

/2

$$\hat{V}_1 = A\left[K_m\left(\frac{nr}{R_0}\right) - \frac{K_m'\left(\frac{nb}{R_0}\right)}{I_m'\left(\frac{nb}{R_0}\right)} I_m\left(\frac{nr}{R_0}\right)\right]$$

$$\cdot \exp\left[im\theta - n\varphi\right]$$

$$A = \frac{iF(a)\,\xi_a}{K_m\left(\frac{na}{R_0}\right)}\left[1 - \frac{K_m'\left(\frac{nb}{R_0}\right)}{K I_m'\left(\frac{nb}{R_0}\right)}\,\frac{I_m\left(\frac{na}{R_0}\right)}{K_m\left(\frac{na}{R_0}\right)}\right]^{-1}$$

The volume integral over the vacuum region, can be expressed as a surface term at the plasma - vacuum interface through the relation;

$$\delta W_V = \frac{1}{2}\int d^3r\,|\hat{B}_1|^2 = \frac{1}{2}\int d^3r\,\nabla\cdot(\hat{V}_1^*\nabla\hat{V}_1)$$

$$= -\frac{1}{2}\int dS\,\hat{V}_1^*\,\underset{\sim}{n}\cdot\underset{\sim}{\nabla} V_1$$

This result in relation

$$\frac{\delta W_V}{2\pi R_0} = \pi\,\frac{r^2 F^2 \Lambda}{|m|}\,\xi_a^2$$

$$\Lambda = \frac{+|m|\,K_m\left(\frac{na}{R_0}\right)}{\frac{na}{R_0}K_m'\left(\frac{na}{R_0}\right)}\left[\frac{1 - K_m'\left(\frac{nb}{R_0}\right)I_m\left(\frac{na}{R_0}\right)/I_m'\left(\frac{nb}{R_0}\right)K_m\left(\frac{na}{R_0}\right)}{1 - K_m'\left(\frac{nb}{R_0}\right)I_m'\left(\frac{na}{R_0}\right)/I_m'\left(\frac{nb}{a}\right)K_m'\left(\frac{na}{R_0}\right)}\right]$$

$$\simeq \frac{1+(a/b)^{2|m|}}{1-(a/b)^{2|m|}}, \quad \text{if } |n|\frac{b}{R_0} \ll 1$$

Therefore:

$$\frac{\delta W}{2\pi^2 R_0} = \delta W_F + \delta W_V$$

$$= \int_0^a dr\,(f\xi'^2 + g\xi^2)\,dr + \left[\frac{n^2 r^2}{R_0^2} B_\theta^2 \frac{\left(1 - \frac{m^2}{n^2 q(a)^2}\right)}{k_0^2 r^2} + \frac{r^2 \Lambda F^2}{|m|}\right]\Bigg|_{r=a} \xi^2(a)$$

Note drive for instability if

$$\frac{m}{n\,q(a)} > 1$$

$$\Lambda \geq 1 \,, \quad f = \frac{r^2}{k_0^2 R_0^2}\left(\frac{m}{q(r)} - \frac{n}{R}\right)^2 B_\theta^2 > 0$$

$g(r) \equiv$ can have drive:

$\xi(r)$ satisfies Euler equation

$$\frac{d}{dr} f \frac{d\xi(r)}{dr} - g\,\xi = 0$$

<u>Oscillation Theorem (Newcomb)</u>

If starting at the origin, with regularity boundary condition, $\xi(r)$ has a zero for $r \leq a$, then $\delta W < 0$; and we have instability. Applies only to an internal mode!

Tokamak model for kink instabilities:

$\epsilon = a/R$

Take $n \frac{r_0}{R_0} \ll 1$, $\frac{P}{B^2} \approx \left(\frac{a}{R_0}\right)^2 \simeq \epsilon^2 \equiv \left[\frac{a}{R}\right]^2$

The result:

$$\delta W = \frac{\pi B_\xi^2}{R_0} \int^a dr \left[\left(r\frac{d\xi}{dr}\right)^2 + (m^2-1)\xi^2\right]\left(\frac{n}{m} - \frac{1}{q(r)}\right)^2 r\,dr$$

$$+ \left[\frac{2}{q_0}\left(\frac{n}{m} - \frac{1}{q(a)}\right) + (1+m\Lambda)\left(\frac{n}{m} - \frac{1}{q}\right)^2\right] a^2 \xi_a^2$$

$$\Lambda = \frac{1+\left(\frac{a}{b}\right)^{2m}}{1-\left(\frac{a}{b}\right)^{2m}}$$

kink drive for instability if

$\frac{m}{n} / q(a) > 1$

'pitch' of perturbation greater than pitch of surface field line's pitch

Whether this drive is enough for instability depends on shape of current profile:

One can solve for growth rate, keeping kinetic energy term and find, (5)

$$\frac{d}{dr}\left[(\rho\omega^2 - F^2)\, r \frac{d}{dr}(r\xi)\right] - \left[m^2(\rho\omega^2 - F^2) - r\frac{dF^2}{dr}\right]\xi = 0$$

with boundary condition at $r = a$

$$\frac{d}{dr}(r\xi) = \frac{m(m-nq_a)^2}{\rho\omega^2 a^2/B_\theta^2 - (m-nq_a)^2}\left(\Lambda - \frac{2}{m-nq_a}\right)\xi$$

and one finds the following growth rates; and stability boundaries:

ideal mhd instability is the kink mode.
It leads to a kinking of the magnetic
a boundary. The driving force comes
gradient of the toroidal current,
pect-ratio tokamak with low $\beta(\sim \varepsilon^2)$ the
rturbation having a radial displacement

$$\Big)^2 + (m^2-1)\xi^2\Big]\left(\frac{n}{m}-\frac{1}{q}\right)^2 r\,dr$$

$$+ (1+m\lambda)\left(\frac{n}{m}-\frac{1}{q_a}\right)^2\Big]a^2\xi_a^2\Big\} \qquad 6.3.1$$

$(a/b)^{2m})$,

ie plasma, b the radius of a perfectly
fety factor, and the subscript a indicates
iode $m = 1$ is a special case and is dealt

g wall at the surface of the plasma, the
$= 0$. It is seen from eqn 6.3.1 that in this
asma is stable. For any position of the
ig $b \to \infty$, δW is positive for all modes
in increasing function of r, modes having
the plasma will have $m/n < q_a$ and will

na $q \propto r^2$, modes with resonant surfaces
$m/n > q_a$. Such modes can therefore be
vard way to determine stability for a
the eigenmode equation

$$\Big] - \left[m^2(\rho\omega^2 - F^2) - r\frac{dF^2}{dr}\right]\xi = 0$$

density, $F = (m - nq)B_\theta/r\mu_0^{1/2}$, and the
ence exp $-i\omega t$. The boundary condition
gin is $\xi \propto r^{m-1}$ and that at the plasma

$$\frac{nq_a)^2}{-(m-nq_a)^2}\left(\lambda - \frac{2}{m-nq_a}\right)\xi, \quad (r = a).$$

its the requirement of pressure balance
the plasma boundary remains a flux

Figure 6.3.1 gives growth rates, $\gamma(=\omega_i)$, for a parabolic current distribution and Fig. 6.3.2 shows the regime of stability for the class of current profiles $j = j_0(1-(r/a)^2)^\nu$. The broad band of instability in the upper part of this diagram is due to the destabilizing effect of the current gradient close to the surface of the plasma arising from the flatness of the current profiles. The protrusions of instability into the lower part of the diagram, where the current profiles are more peaked, are due to the proximity of the resonant surfaces of the various modes to the surface of the plasma and hence to the current gradients within the plasma.

The onset of stability as the resonant surface reaches the plasma surface from outside, that is $q_a \to m/n$, is a misleading feature resulting from the ideal mhd model. It can be seen from Section 6.7 on the stability of tearing modes that the inclusion of resistive effects removes this stability boundary and leads to instabilities with resonant surfaces inside the plasma.

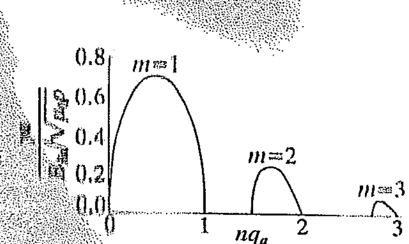
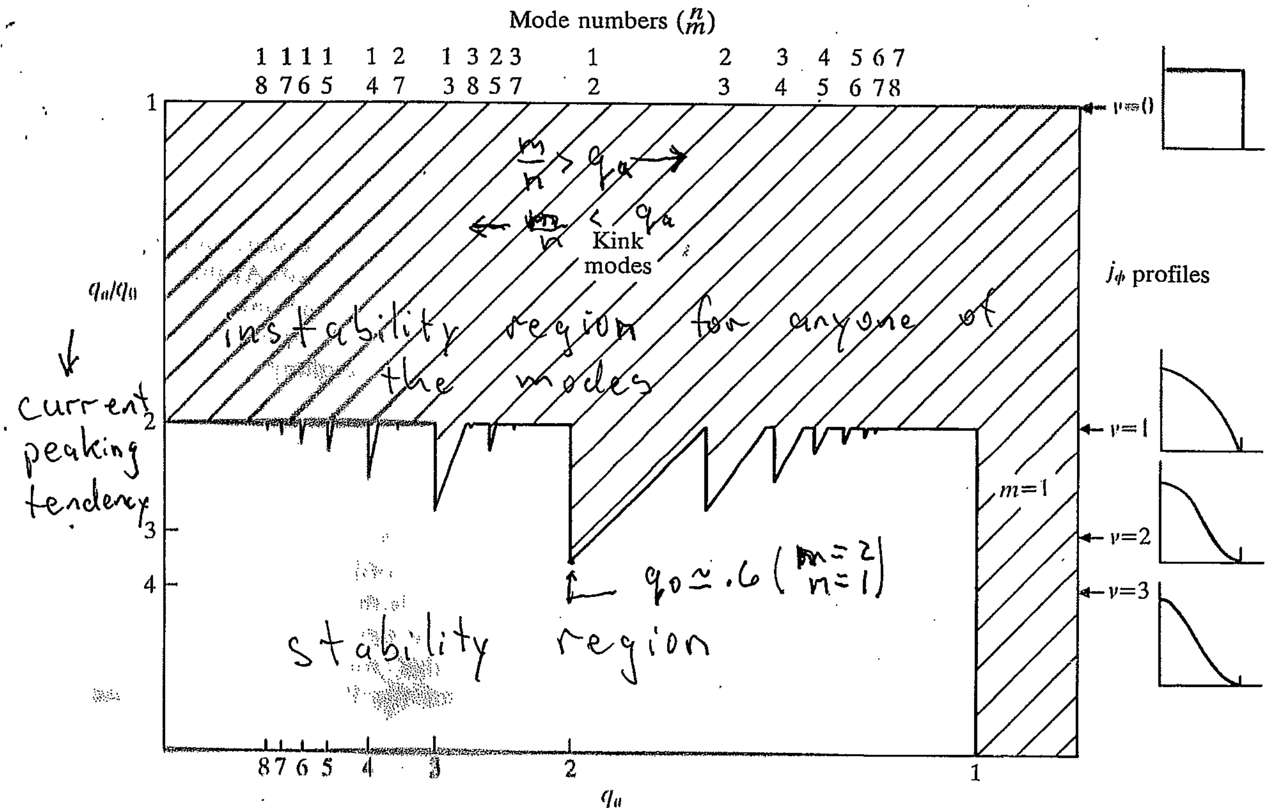


Fig. 6.3.1 Growth rate of modes $m = 1$, 2, and 3 for the parabolic current profile $j = j_0(1-(r/a)^2)$, as a function of nq_a.

Fig. 6.3.2 Stability diagram for kink modes for the current distributions $j = j_0(1-(r/a)^2)^\nu$. The vertical axis measures the peaking of the current as given by q_a/q_0 $(=\nu+1)$ and the horizontal coordinate is proportional to $1/q_a$ and therefore to the total current.



(7)

Internal kink mode ($m=n=1$)

$$\delta W = \frac{\pi^2 B_\phi^2}{R_0} \int_0^a dr\, r \left[\left(r \frac{d\xi}{dr} \right)^2 + (m^2-1)\,\xi^2 \right] \left(\frac{n}{m} - \frac{1}{q} \right)^2$$

It is quite clear that a minimizing perturbation is $\frac{d\xi}{dr} = 0$; $\xi =$ constant

Note $\frac{d\xi}{dr} \equiv$ finite, is line bending.

If we wish an internal mode ($\xi(a)=0$), there is a chance not to excite line bending energy if $\xi(r) \to 0$, around resonant surface where $q(r) \to \frac{m}{n} = 1$

Take $\xi(r) = \xi_0 \quad r < r_1 - \epsilon$; $\xi(r) = 0, \; r > r_1$

$$\xi(r) = \lim_{\epsilon \to 0} \frac{\xi_0 (r - r_1)}{\epsilon}, \quad r - r_1 < \epsilon \qquad q(r_1) = 1$$

Energy of bending $\left(\frac{n}{m} - \frac{1}{q(r)} \right)^2 \simeq \frac{(r_1 - r)^2\, q'^2(r_1)}{q^4}$

$$\delta W \propto \lim_{\epsilon \to 0} r_1^3 \int_{r_1-\epsilon}^{r_1} dr_1\, \frac{\xi_0^2}{\epsilon^2}\, q'^2(r_1)\,(r-r_1)^2 = r_1^3\, \epsilon\, \xi_0^2\, q'^2(r_0) \int_0^1 dx\, x^2$$

$$\to 0$$