Lecture # 14

Interchange,

Extended Energy Principle

Kink Modes

$$\delta W = \frac{1}{2}\int d^3r \left[|Q_\perp|^2 + B^2 |\nabla\cdot\xi_\perp + 2\xi\cdot\kappa|^2 + \gamma p |\nabla\cdot\xi|^2 - 2(\xi_\perp\cdot\nabla p)(\kappa\cdot\xi_\perp^*) - J_\parallel (\xi_\perp^* \times b)\cdot Q_\perp \right]$$

Pressure Driven Mode Term

$$-2(\xi_\perp\cdot\nabla p)(\kappa\cdot\xi^*)$$

Always stable at low β in pressure confined in a minimum B region of space.

Possible in a 3-D mirror configuration. Impossible in a toroidal system

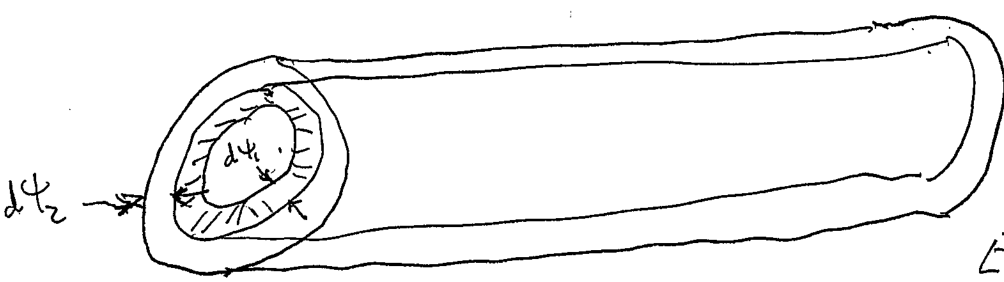
Parallel current other possible source of instability

Important for a tokamak

Not an issue for a stellarator whose flux surfaces arise without plasma current

In a tokamak we cannot have an ideal magnetic well. At most we can have some average magnetic well, and what can it be.

At low beta, one can arrange for $\nabla\cdot\xi_\perp + \kappa\cdot\xi = 0$ (no magnetic compression) and we also want to eliminate line bending stabilization. This leads to a most destabilizing perturbation, at low β, when the fluids, between two adjacent flux tubes, interchange



Ideal MHD constraint $E + v\times B = 0$ demands that the enclosing flux in a fluid remain constant

$$\therefore \quad d\psi_1 = d\psi_2$$

Now, before interchange

$$\Delta E_1 = \frac{\Delta V_1 P_1}{\gamma - 1} \quad \left(\begin{array}{l} \Delta E_1 = \cancel{m} \, T N/2 \quad \left(\eta \equiv \#\text{ of degrees of freedom} \right) \\ \gamma = \eta + 2/\eta \\ \to \frac{\eta}{2} = \frac{1}{\gamma - 1} \end{array} \right)$$

$$\Delta E_2 = \frac{\Delta V_2 P_2}{\gamma - 1}$$

$$\Delta V = \oint \frac{d\Psi}{B} dl = \Delta\Psi \oint \frac{dl}{B} \quad (\text{Here } d\Psi = \underset{\sim}{B} \cdot d\underset{\sim}{A})$$

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$$\Delta E_1 = \frac{\Delta\Psi}{\gamma - 1} \int \frac{P_1}{B} dl$$

$$\Delta E_2 = \frac{\Delta\Psi}{(\gamma - 1)} \int \frac{\left(P_1 + d\Psi \frac{\partial P_1}{\partial \Psi}\right) dl}{B} \; ; \quad \boxed{\begin{array}{l} \Delta V_2 = \Delta V_1 + \frac{\partial \Delta V_1}{\partial \Psi} \Delta\Psi \\ = \Delta\Psi \oint \frac{dl}{B} + \Delta\Psi^2 \frac{\partial}{\partial \Psi} \oint \frac{dl}{B} \end{array}}$$

When interchange takes place

$$P_1 \Delta V_1^{\gamma} = P_1' \Delta V_2^{\gamma} \; ; \qquad P_1' = P_1 \, \Delta V_1^{\gamma} / \Delta V_2^{\gamma}$$

$$P_2 \Delta V_2^{\gamma} = P_2' \Delta V_1^{\gamma} \qquad P_2' = P_2 \, \Delta V_2^{\gamma} / \Delta V_1^{\gamma}$$

New energy

$$\frac{P_1' \Delta V_2}{\gamma - 1} + \frac{P_2' \Delta V_1}{\gamma - 1}$$

$$V'(\psi) \equiv \frac{\partial V}{\partial \psi}(\psi) = \frac{\partial}{\partial \psi}\left(\int^{\psi} d\psi' \oint \frac{dl}{B}\right) \quad ; \quad P'(\psi) \equiv \frac{\partial P}{\partial \psi} ;$$

$$\boxed{V''(\psi) \equiv \frac{\partial^2}{\partial \psi^2}\int^{\psi} d\psi' \oint \frac{dl}{B}}$$

$$\Delta V_1 = \int_{\psi-\Delta\psi}^{\psi} d\psi' V'(\psi') = \Delta\psi\, V'(\psi) - \frac{\Delta\psi^2}{2} V''(\psi)$$

$$\Delta V_2 = \Delta\psi\, V'(\psi) + \frac{\Delta\psi^2}{2} V''(\psi) = \int_{\psi}^{\psi+\Delta\psi} d\psi' V'(\psi')$$

$$\bar{P}_1 \equiv \int_{\psi-\Delta\psi}^{\psi} P(\psi')d\psi'/\Delta\psi \quad ; \quad \bar{P}_2 \equiv \int_{\psi}^{\psi+\Delta\psi} d\psi' P(\psi')/\Delta\psi$$

$$\bar{P}_1' = \bar{P}_1 \left(\frac{\Delta V_1}{\Delta V_2}\right)^{\gamma} \quad ; \quad \bar{P}_2' = P_2 \left(\frac{\Delta V_2}{\Delta V_1}\right)^{\gamma}$$

$$\delta W = \left(\bar{P}_1' \Delta V_2 + \bar{P}_2' \Delta V_1 - \bar{P}_1 \Delta V_1 - \bar{P}_2 \Delta V_2\right)/(\gamma - 1)$$

$$= \gamma P(\psi)\left(\frac{V''(\psi)}{V'(\psi)}\right)^2 + \frac{\partial P(\psi)}{\partial \psi}\left(\frac{V''(\psi)}{V'(\psi)}\right)$$

Sufficient condition for stability

$$\boxed{\frac{\partial P}{\partial \psi} \frac{V''(\psi)}{V'(\psi)} > 0}$$

or for stability:

$$\frac{\partial P}{\partial \psi} \frac{\frac{\partial}{\partial \psi}\oint \frac{dl}{B}}{\oint \frac{dl}{B}} = \frac{-\frac{\partial P}{\partial \psi}\oint \frac{dl}{B^2}\frac{\partial B}{\partial \psi}}{\oint \frac{dl}{B}} > 0$$

This is appropriate average of $\frac{\partial B}{\partial \psi}$ over flux tube

A miracle of tokamaks:

The Shafronov shift enables the property that

$$\frac{\partial}{\partial\psi}\oint\frac{dl}{B} = -\int\frac{dl}{B^2}\frac{\partial B}{\partial\psi} < 0$$

if $q(r) > 1$ (stability to plasma confinement)

while

$$\frac{\partial}{\partial\psi}\oint\frac{dl}{B} = -\int\frac{dl}{B^2}\frac{\partial B}{\partial\psi} > 0$$

(no stability guarantee)

if $q(r) < 1$

Extended Energy Principle

When we derived the energy principle we assumed that the plasma was in contact with a rigid wall and that $\underset{\sim}{\xi} \cdot \underset{\sim}{n} = 0$, which allowed us to neglect boundary terms.

This limitation can be overcome with what is known as the extended energy principle, where the energy is the sum of three terms

$$\delta W = \underset{\substack{| \\ \text{Fluid Part} \\ \text{of } \delta W}}{\delta W_F} + \underset{\substack{| \\ \text{Surface} \\ \text{Part}}}{\delta W_S} + \underset{\substack{| \\ \text{Vacuum Part}}}{\delta W_V}$$

$$\delta W_F = \frac{1}{2} \int d^3r \left[|Q|^2 - \xi_\perp^* \cdot \underset{\sim}{J} \times \underset{\sim}{Q} + \gamma p |\nabla \cdot \xi|^2 + (\underset{\sim}{\xi_\perp} \cdot \nabla p) \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi_\perp^*} \right]$$

$$\delta W_S = \frac{1}{2} \int dS \, |\underset{\sim}{\hat{n}} \cdot \underset{\sim}{\xi_\perp}|^2 \left[\left[\underset{\sim}{\nabla} \left(p + \frac{B^2}{2} \right) \right] \right] \cdot \underset{\sim}{\hat{n}}$$

(jump of equilibrium surface quantities)

$$\delta W_V = \frac{1}{2} \int d^3r \, |\underset{\sim}{B}_{1V}|^2$$

δW_S usually not important as not present in <u>diffusive</u> system

$\underset{\sim}{B}_{1V}$ is the perturbed vacuum field. If the vacuum region were filled with a cold, non-current carrying plasma, δW_V would be replaced with

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$\underset{\sim}{B}_{1r} \rightarrow \underset{\sim}{Q} = \underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B}) = \underset{\sim}{B}_{1p}$

whereas $\quad \underset{\sim}{B}_{1r} = \underset{\sim}{\nabla} \Phi$ with $\nabla^2 \Phi = 0$

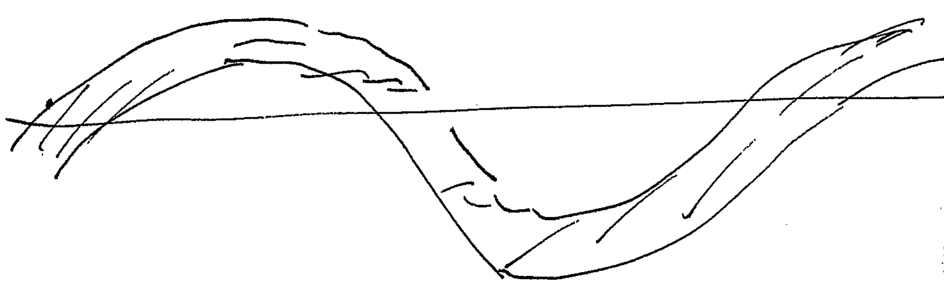
Both are magnetic fields, and magnetic field energy, and you might think there is no difference between the two, but indeed there is a profound difference

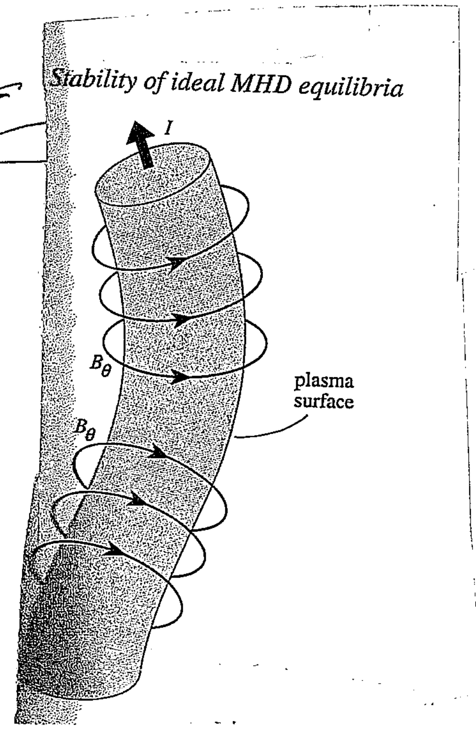
$\underset{\sim}{B}_{1p} = \underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B})$, a consequence of the ideal ohm's law, implies the constraint the the plasma moves with the fluid. "Flux conserved" Further, in order to have exponential instability bounded, as $\underset{\sim}{\xi}$ needs to be bounded, otherwise $\omega^2 = - \delta W / \int \rho \underset{\sim}{\xi} \underset{\sim}{\xi} \rightarrow 0$

If we demand that $\underset{\sim}{B}_p = \underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B})$ we find that the

MHD equations may demand that $\xi \to \infty$. If we rule out such a solution we could conclude stability, whereas if we allowed a vacuum solution, we would conclude instability.

This case occurs frequently with current driven modes, where we can get a kink instability.





Imagine a quasi-toroidal plasma

$$L = 2\pi R_0 \quad , \quad R_0 \phi = Z$$

$$\xi_r = \xi(r) \exp\left[-i \frac{n Z}{R_0} + im\theta\right]$$

$$= \xi(r) \exp\left[-i \left(n \phi - \frac{m}{q(r)}\theta\right)\right]$$

It turns out that the kink is stable if $m/n < q(r_a)$
($r_a \equiv$ radius of plasma edge)

Kink in perturbation moves around in poloidal direction more slowly than surface field line moves around in poloidal direction

Now, consider a case where $\frac{m}{n} > q(r_a)$, and consider a discharge where $q(r)$ increases radially within discharge. In vacuum

$q(r) = r^2$ $\left(B_\theta \propto \frac{I}{r} \therefore q(r) = \frac{B_\varphi r}{B_\theta R_0} \propto r^2 \right)$.

Hence, at some radius (if there is no conducting wall in the way) we reach a point where $\frac{m}{n} \to q(r_c)$ and then $\frac{m}{n} < q(r > r_c)$.

As this region is a vacuum, there is no constraint on B_{1r} (indeed ξ makes no sense here), and ~~there~~ is nothing special about a region where the pitch ($\frac{m}{n}$) of the perturbation is the same as the pitch of the field line ($q(r_c)$).

One then often finds that a kink instability is calculated when this happens.

Now, what would be different if instead of a vacuum, we had a "cold" currentless plasma in the void region.

Then $\underset{\sim}{B}_p = \underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B})$ (if ideal MHD applies). When one attempts to solve for $\underset{\sim}{\xi}$ ~~at~~ "resonant" ($\frac{m}{n} = q(r)$) surface, one finds, due to the resonance, that $\xi(r_{res}) \to \infty$, and growth rate, (that may have been calculated if this region were in a vacuum) goes to zero!

Specifically, how does this arise in the energy principle?

In the δW_F expression, we can minimize by setting $\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi}=0$ in the expression

$$\delta W_F = \frac{1}{2}\int d^3r \left[|Q_\perp|^2 + B^2\left(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi} + 2\underset{\sim}{\xi}_\perp\cdot\underset{\sim}{\kappa}\right) + \gamma P|\nabla\cdot\xi|^2 - 2(\xi_\perp\cdot\nabla p)\,\kappa\cdot\xi_\perp^* - J_\parallel(\underset{\sim}{\xi}_\perp^*\times\underset{\sim}{b})\cdot Q_\perp \right]$$

Take variation of δW with respect to $\xi_\parallel$, yields

$$\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\left(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi}\right)=0$$

so that $\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi}=0$ unless

$$B\cdot\nabla = 0 \qquad \underset{\sim}{B}\cdot\nabla \equiv \frac{B_0}{R}\left(n - \frac{m}{q(r)}\right)$$

for $\xi_\parallel \propto \exp[-in\zeta + im\theta]$

and at resonant surface

$\frac{m}{n} = q(r)$, we can no longer set $\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi}=0$. This constraint in fact prevents ideal MHD kink modes to occur, when the resonant surface is in the plasma, (it in fact yields $\delta_r \to \infty$ if we force perturbation through resonant surface).

This is a strange limitation. It dissapears when we introduce resistivity into Ohm's law

$$\underset{\sim}{E} + \underset{\sim}{V} \times \underset{\sim}{B} = \eta \underset{\sim}{j}$$

Then MHD predictions of kink instabilities, through energy arguments, apply, but growth rates go to zero as $\eta \to 0$. The class of such instabilities are called resistive instabilities