Lecture #13

Interchange Instability

MHD Energy Principle

Demonstrated self - Adjointness

$$-\omega^2 \rho \underset{\sim}{\xi} = \underset{\approx}{F} \cdot \underset{\sim}{\xi}$$

$$\underset{\approx}{F} \cdot \underset{\sim}{\xi} = -\underset{\sim}{\nabla} p_1 + \underset{\sim}{Q} \times \underset{\sim}{B}_0 + (\underset{\sim}{\nabla} \times \underset{\sim}{Q}) \times \underset{\sim}{B}_0$$

$$p_1 = -\underset{\sim}{\xi} \cdot \nabla p_0 - \gamma \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}$$

$$\underset{\sim}{Q} = \underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B}_0) \equiv B_1$$

With boundary condition $\underset{\sim}{\xi} \cdot \underset{\sim}{\hat{n}} = 0$

$$-\omega^2 \int d^3r \, \rho \underset{\sim}{\eta} \cdot \underset{\sim}{\xi}$$

$$= \int d^3r \, \underset{\sim}{\eta} \cdot \underset{\approx}{F} \cdot \underset{\sim}{\xi} = \int d^3r \, \underset{\sim}{\xi} \cdot \underset{\approx}{F} \cdot \underset{\sim}{\eta}$$

If $\underset{\sim}{\eta} = \underset{\sim}{\xi}$

$-\frac{\omega^2}{2} \int d^3r \, \rho \underset{\sim}{\xi} \cdot \underset{\sim}{\xi} \equiv$ perturbed ~~potential~~ kinetic energy $\equiv KE$

$-\frac{1}{2} \int d^3r \, \underset{\sim}{\xi} \cdot \underset{\approx}{F} \cdot \underset{\sim}{\xi} =$ perturbed potential energy $\equiv PE = \delta W$

In excited state

$$\boxed{KE + PE = \text{constant}}$$

(1)

The most ~~direct~~ commonly ~~way~~ used form (without much manipulation, just taking $\underset{\sim}{\xi}^* \cdot \underset{\approx}{F}$ and integrating, with some obvious parts integrations)

$$\delta W = \frac{1}{2} \int d^3r \left[|\underset{\sim}{Q}|^2 + \gamma p |\nabla \cdot \underset{\sim}{\xi}|^2 - \xi^* \cdot \left[\underset{\sim}{J} \times \underset{\sim}{Q} + \nabla (\xi \cdot \nabla p) \right] \right]$$

($\xi^* \cdot \nabla (\gamma p) \nabla \cdot \xi$ & $\xi^* \cdot (\nabla \times \underset{\sim}{Q}) \times \underset{\sim}{B}$ integrated by parts)

If one then writes

$\underset{\sim}{\xi} = \underset{\sim}{\xi}_\perp + \xi_\parallel \underset{\sim}{b}$, it follows that

$$\xi_\parallel^* \, \underset{\sim}{b} \cdot \left[\underset{\sim}{J} \times \underset{\sim}{Q} + \underset{\sim}{\nabla} (\xi \cdot \nabla p) \right] = 0$$

(see Eq. A-4 of Freidberg handout)

$$\boxed{\delta W = \frac{1}{2} \int d^3r \left[|Q|^2 - \underset{\sim}{\xi}_\perp^* \cdot \underset{\sim}{J} \times \underset{\sim}{Q} + \gamma p |\nabla \cdot \underset{\sim}{\xi}|^2 + (\xi_\perp \cdot \underset{\sim}{\nabla} p) \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}_\perp^* \right]}$$

Standard Form of energy principle

A more "intuitive" form of δW obtained with further manipulation

$$|Q|^2 = |Q_\perp|^2 + |Q_\parallel|^2$$

$$\xi_\perp^* \cdot J \times Q = J_\parallel (\xi_\perp^* \times b) \cdot Q_\perp + Q_\parallel \, \xi_\perp^* \cdot J_\perp \times b$$

$$J_\perp = b \times \nabla p / B$$

$$Q_\parallel = b \cdot \nabla \times (\xi_\perp \times B) = b \cdot (B \cdot \nabla \xi_\perp - \xi_\perp \cdot \nabla B - B \nabla \cdot \xi_\perp)$$

$$= -B(\nabla \cdot \xi_\perp + 2 \xi_\perp \cdot \kappa) + \xi_\perp \cdot \nabla p / B$$

(home work step)

Substitution into δW, leads to (HW)

"δW intuitive"

$$\boxed{\delta W = \frac{1}{2} \int d^3r \left[|Q_\perp|^2 + B^2 |\nabla \cdot \xi_\perp + 2 \xi_\perp \cdot \kappa|^2 + \gamma p |\nabla \cdot \xi|^2 - 2 (\xi_\perp \cdot \nabla p)(\kappa \cdot \xi_\perp^*) - J_\parallel (\xi_\perp^* \times b) \cdot Q_\perp \right]}$$

Notice 3 positive definite terms

If $x = 0$, $J_{\parallel} = 0$

(slab homogeneos MHD system)

the 3- positive definite terms drive the 3 basic MHD waves (together with kinetic energy)

(a) $\underset{\sim}{Q}_{\perp} = 0$, $\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} = 0$

$$\omega^2 = (k V_A)^2; \quad V_A^2 = B^2/\rho$$

(magnetic compressional perturbation ($\delta \underset{\sim}{B} \parallel \underset{\sim}{B}_0$)

(b) $\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}_{\perp} = 0$, $\underset{\sim}{\nabla} \cdot \underset{\sim}{\rho} = 0$

$$\omega^2 = k_{\parallel}^2 V_A^2;$$

(magnetic shear perturbation)

$\delta B_{\perp} \perp \underset{\sim}{B}_0$

(magnetic shear $\cong$ line bending)

(c) $\underset{\sim}{\nabla} \cdot \xi_{\perp} = 0$, $\underset{\sim}{Q}_{\perp} = 0$

$$\omega^2 = k_{\parallel}^2 C_s^2 \qquad C_s^2 = \gamma p/\rho$$

sound speed perturbation

H.W. Demonstrate these modes, starting from $\delta W + KE = 0$, for spatially homogeneous plasma

By confining plasma in a spatially dependent field we introduce $\tilde{J}_{\parallel}$ and $\underset{\sim}{\kappa}$ which leads to possible non-positive definite terms, and thus the possibility $\delta W < 0$

These are in fact the fundamental sources of MHD instability in a confined plasma.

First let us consider the possible curvature drive

Recall

$$\underset{\sim}{\kappa} \equiv \frac{\nabla_{\perp} B}{B} \quad (\text{low-}\beta \text{ plasma})$$

(more precisely $\underset{\sim}{\kappa} = \nabla_{\perp}\left(\frac{B^2}{2} + p\right) / B^2$)

The destabilizing term is

$$\delta W = -2 \int d^3r \, (\xi_\perp \cdot \nabla p)(\kappa \cdot \xi)$$

$$= -2 \int d^3r \, (\xi_\perp \cdot \nabla \psi)(\kappa \cdot \xi)\, p'$$

$p' = \frac{\partial p}{\partial \psi} < 0$ (pressure falling off)

If $(\xi_\perp \cdot \nabla \psi)\,[\kappa_\psi \nabla \psi \cdot \xi + \kappa_g \xi \cdot B \times \nabla \psi] > 0$, stability

$$\kappa = \kappa_\psi \nabla \psi + \kappa_g \, B \times \nabla \psi$$

Look at

$(\xi_\perp \cdot \nabla \psi)(\xi_\perp \cdot \nabla \psi)\, \kappa_\psi$ term (other term can be shown zero at low beta)

If $\kappa_\psi > 0$, this is stabilizing: At low beta

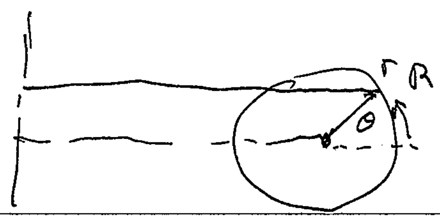
$$\kappa_\psi = \frac{1}{2B^2}\frac{\partial B^2}{\partial \psi} > 0$$

Fundamental relationship for very strong MHD stability

Mod $|B^2|$ increases everywhere from confinement region.

Can this be done?

Not in a tokamak



$$B \approx B_0 \left(1 - \frac{r}{R}\cos\theta\right)$$

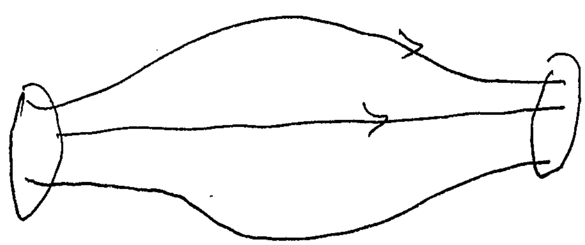
B increases with increasing r (4)

for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

but ~~in~~creases with increasing r

for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

What about a simple mirror machine



$B = B_0 \left(1 + \frac{Z^2}{L^2}\right)$ near mid-plane

$$B^2 = B_0^2 \left(1 - \frac{r^2}{L^2} + \frac{2Z^2}{L^2}\left(1 + \frac{r^2}{2L^2}\right)\right)$$

decreasing mod-B

$$\underset{\sim}{B} = \underset{\sim}{\nabla} \Phi$$

$$\Phi = B_0\left(z + \frac{z^3}{3L^2}\right) + \Phi_1(r, z)$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = \underset{\sim}{\nabla}^2 \Phi = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi_1}{\partial r} = -\frac{2 B_0 z}{L^2}$$

$$\frac{\partial \Phi_1}{\partial r} = -\frac{B_0 z r}{L^2}$$

$$\Phi_1 = -\frac{B_0 z r^2}{2L^2}$$

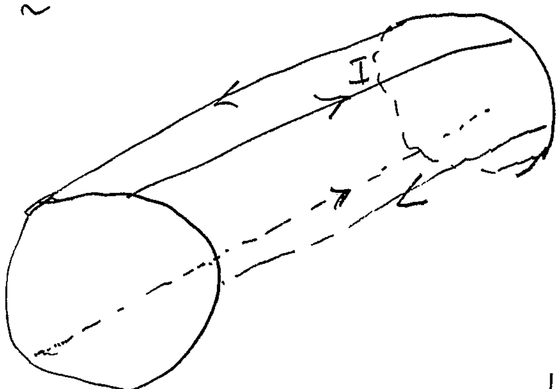
$$\underset{\sim}{\nabla} \Phi = B_0\left(1 + \frac{z^2}{L^2} - \frac{r^2}{2L^2}\right)\hat{z} + \frac{B_0 z r}{L^2}\hat{r}$$

$$B^2 \simeq B_0^2\left(1 - \frac{r^2}{L^2} + \frac{2z^2}{L^2}\left(1 + \frac{1}{2}\frac{r^2}{L^2}\right)\right)$$

↑ decreasing mod B

By adding quadrupole fields (Ioffe) and breaking symmetry, it is possible to create

$\underset{\sim}{\nabla} \cdot \psi \cdot \underset{\sim}{\nabla} \bmod B > 0$



Mathematically add a quadrupole-like magnetic potential

$$\Phi = \frac{\lambda B_0 r^2}{2L} \sin 2\theta \qquad (\nabla^2 \Phi = 0)$$

$$\underset{\sim}{B}_1 = \frac{\lambda r B_0}{L} \left[\hat{r} \sin 2\theta + \hat{\theta} \cos 2\theta \right]$$

$$B^2 (z = 0) = B_0^2 \left[1 - \frac{r^2}{L^2} + \frac{\lambda r^2}{L^2} \right]$$

If $\lambda > 1$ we have ideal magnetic well

 : Flux tube