Maxwell's Equations

Lecture # 20

Checkup 32.1

During a "brownout," the oscillating emf supplied by power companies can drop significantly; suppose that the amplitude is $E_{max} = 141$ V. What average power does this emf supply to a 100-Ω resistor?

a. 100 W
b. 141 W
c. 200 W
d. 282 W

a

Checkup 32.2

Suppose that a capacitor is at first plugged into a 60-Hz, 115-V outlet. If you move the capacitor to a 60-Hz, 230-V outlet, by what factor is the new current amplitude related to the old one? By what factor is the new maximum power related to the old?

a. 1, 4

b. 1, 2

c. 2, 4

d. 2, 2

e. 2, 1

If capacitor is now moved to a 115-V, 120 Hz outlet, how is the newest current amplitude and maximum power output related to the oldest one?

Checkup 32.5

A series *RLC* circuit is connected to an emf of amplitude 2.0 V. At the resonant frequency, the amplitude of the voltage across the inductor is 50 V. What is the amplitude of the voltage across the resistor?

a. 102 V
b. 98 V
c. 52 V
d. 48 V
e. 2.0 V

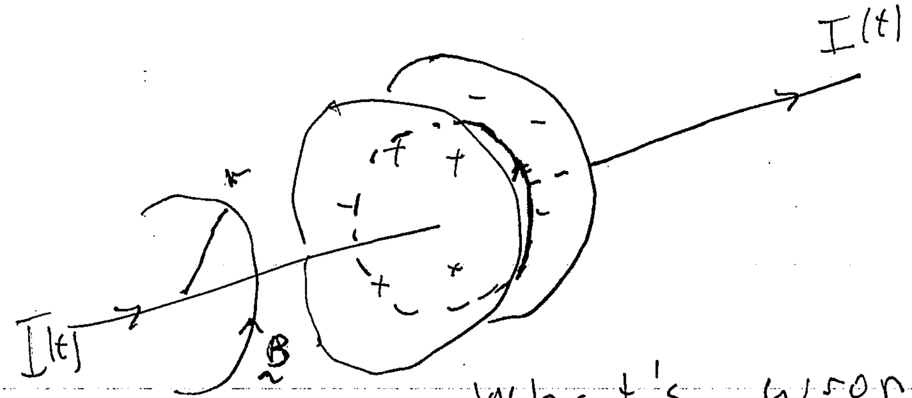
c

Checkup 32.6

Suppose that you have two transformers, one with a turns ratio of 2 and one with a turns ratio of 5. Each may be used to step up or to step down a voltage. The two are to be connected one after the other. What are possible values for the ratio of input to output voltages for the pair?

a. 10, 0.1
b. 10, 5, 2.5
c. 25, 4
d. 10, 5, 2, 0.1
e. 10, 2.5, 0.4, 0.1

d



What's wrong with Amperes Law?

Time varying current going into capacitor produces magnetic field

$$\oint \tilde{B}(t) \cdot d\tilde{l} = \mu_0 I(t)$$

$$2\pi r\, B(t) = \mu_0 I(t)$$

$$\boxed{B(t) = \frac{\mu_0 I(t)}{2\pi r}}$$

But around red path no current penetrates cross section of enclosed plane, and experimentally magnetic fields are measured inside capacitor.

Electric Field in capacitor

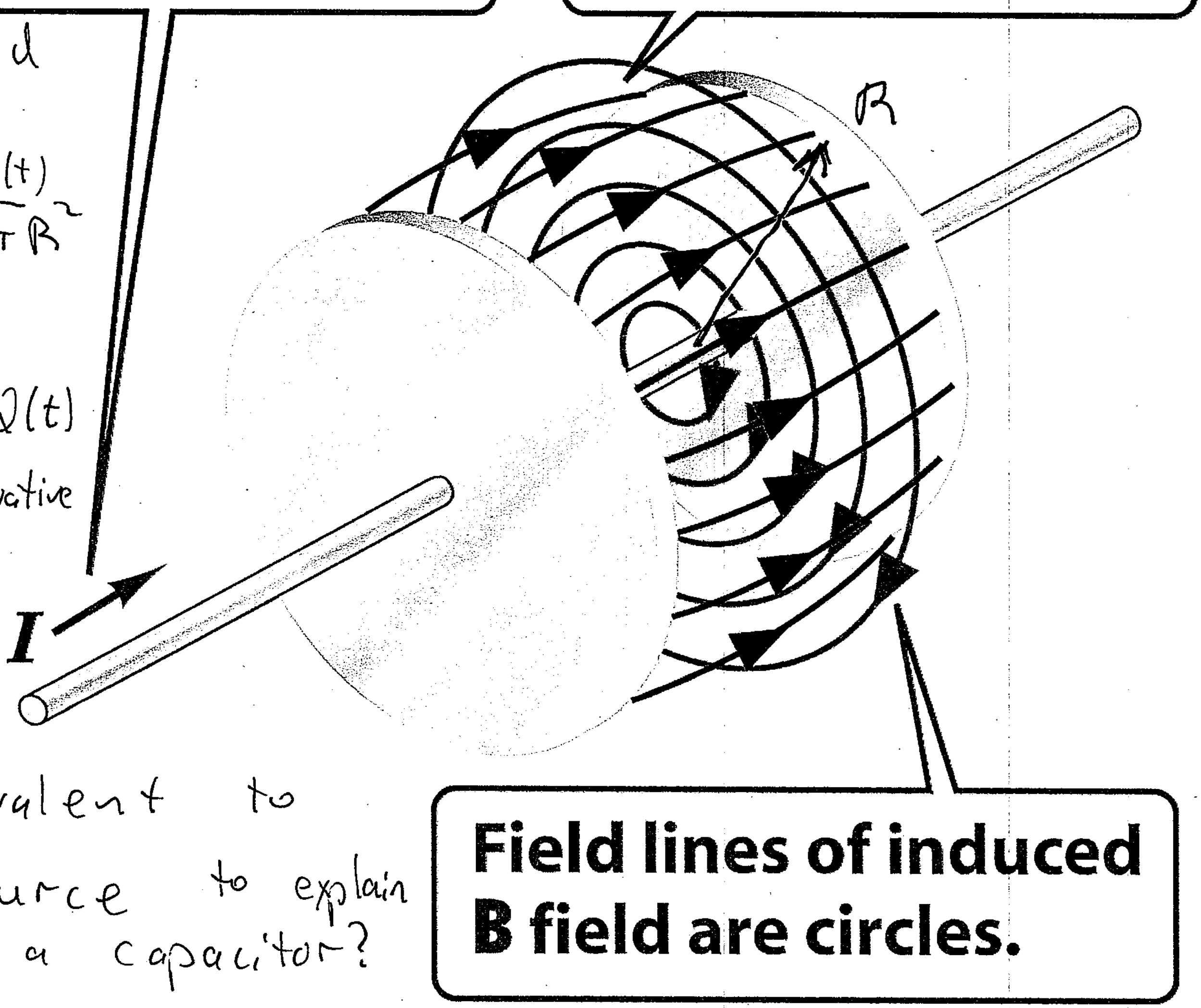
$E(t) = \frac{Q(t)}{\epsilon_0 A} = \frac{Q(t)}{\epsilon_0 \pi R^2}$

$\therefore$

$\epsilon_0 E(t) A = \epsilon_0 \Phi_E(t) = Q(t)$

Take time derivative

$\epsilon_0 \frac{d\Phi_E}{dt} = \frac{dQ(t)}{dt} = I(t)$



Could $\epsilon_0 \frac{d\Phi_E}{dt}$ be equivalent to a current source to explain B-fields in a capacitor?

Figure 33-3 Physics for Engineers and Scientists 3/e

$\boxed{\text{Maxwell says YES!}}$

Complete E & M equations Have a Symmetry

Faraday's Law

$$\oint \vec{E}\cdot d\vec{s} = -\frac{\partial \Phi_B}{\partial t}$$

Ampere-Maxwell Law

$$\oint \vec{B}\cdot d\vec{s} = \mu_0\left[I + \epsilon_0 \frac{\partial \Phi_E}{\partial t}\right]$$

displacement current

B-Flux Law

Total Magnetic Flux, Φ_B, through a closed surface is zero

$$\Phi_B \equiv \oint \vec{B}\cdot d\vec{A} = 0$$

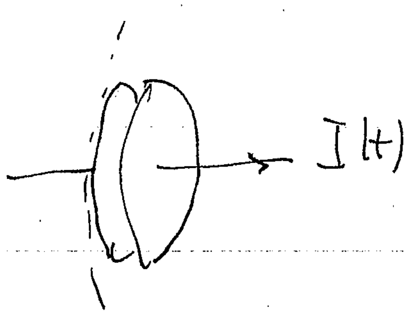
Gauss Law

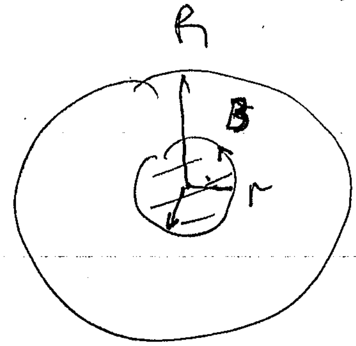
Total Electric flux, Φ_E, through a closed surface is the total enclose charge (divided by ϵ_0)

$$\Phi_E \equiv \oint \vec{E}\cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

If magnetic charges existed in this universe, Maxwell's equations would by complete symmetric.

Magnetic Field in a Circular Plate capacitor





Within the capacitor no charge current flows, but there is displacement current.

$$E(t) = \frac{Q}{A\epsilon_0}$$

$$\frac{\partial E(t)}{\partial t} = \frac{\dot{Q}}{A\epsilon_0} = \frac{I(t)}{A\epsilon_0} \quad ; \quad A = \pi R^2$$

$$\frac{\partial}{\partial t}\Phi_E(r) = \pi r^2 \frac{\partial E(t)}{\partial t} = I(t)\frac{\pi r^2}{\pi R^2 \epsilon_0}$$

$$\therefore \text{ Now } \oint B \cdot dl = 2\pi r B = \mu_0\left(\epsilon_0 \frac{\partial \Phi_E(r)}{\partial t}\right) = \frac{I(t)\, r^2 \mu_0}{R^2}$$

$$\therefore \quad \boxed{B(r) = \frac{\mu_0 I(t)}{2\pi}\frac{r}{R^2}} \quad r < R$$

At edge of capacitor

$$B(R) = \frac{\mu_0 I(t)}{2\pi R} \quad \left(\text{just like from a current}\right)$$

also $R > r$ $\quad B(r) = \frac{\mu_0 I(t)}{2\pi r}$

Waves

Recall wave theory of oscillatory wave

$$\Phi = \Phi_0 \cos\left(2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) - \phi\right)$$

$\Phi_0 \equiv$ amplitude

$\phi \equiv$ phase

$T \equiv$ wave period, $f = \frac{1}{T} \equiv$ frequency

$\lambda \equiv$ wave length

$V_w \equiv$ wave velocity $\equiv f\lambda$

More convenient is to eliminate 2π in trigonometric argument

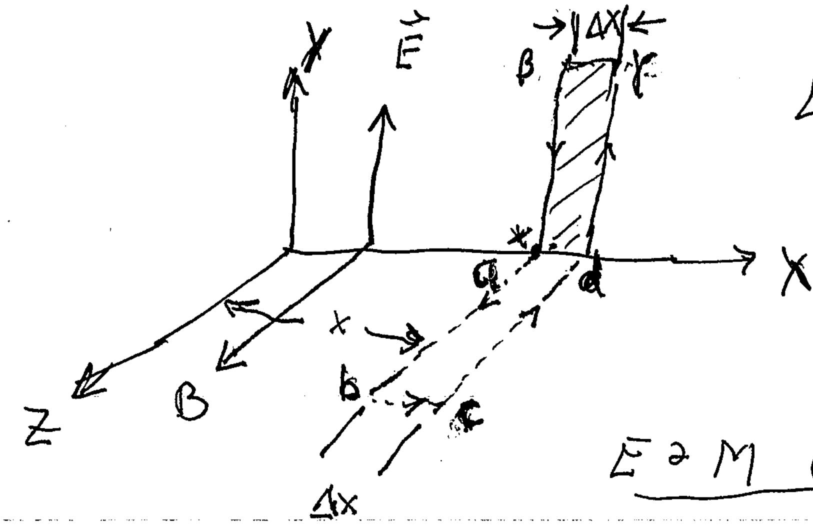
$$\omega \equiv \frac{2\pi}{T} = 2\pi f \; ; \quad k \equiv \frac{2\pi}{\lambda} \equiv \text{wave number}$$

$$V_w = \lambda f = 2\pi f \left(\lambda/2\pi\right) = \frac{\omega}{k}$$

$$\Phi = \Phi_0 \cos\left[\omega t - kx - \phi\right] = \Phi_0 \cos\left[\omega\left(t - \phi - \frac{x}{V_w}\right)\right]$$

$$= \Phi_0 \cos\left[k\left(x - V_w t\right) + \phi\right]$$

Wave Travelling Wave $\Phi = \Phi(x - v_w t)$



E & M waves

$$\mu_0 \epsilon_0 \equiv \frac{1}{c^2} = \frac{1}{(3\times 10^8)^2} \frac{s^2}{m^2}$$

E & M equations

Faraday's Law $\quad \dfrac{\partial \Phi_B}{\partial t} = - \oint \vec{E}\cdot d\vec{l} \quad$ (a)

and

Maxwell's Law $\quad \mu_0 \epsilon_0 \dfrac{\partial \Phi_E}{\partial t} = \oint \vec{B}\cdot d\vec{l} \quad$ (b)

imply waves exist in free space

Assume plane waves exist in the form

$$\vec{E}(x,t) = \hat{y} E_0 \cos(\omega t - kx)$$
$$\vec{B}(x,t) = \hat{z} B_0 \cos(\omega t - kx)$$

Can such solution satisfy E & M equations

First consider Faraday's Law around surface contained inside loop, $\overrightarrow{d\gamma\beta a}$

$$\Phi_B = B_z(x,t) A = B_0 \cos(\omega t - kx)\, y\Delta x$$

$$\frac{\partial \Phi_B}{\partial t} = -\omega B_0 \sin(\omega t - kx)\, y\Delta x$$

$$\oint \vec{E}\cdot d\vec{l} = (\text{loop } \overrightarrow{d\gamma\beta a}) = E(x+\Delta x, t)y - E(x,t)y \approx \frac{\partial E(x,t)}{\partial x}\Delta x y$$

$$= k E_0 \sin(kx - \omega t)\Delta x y$$

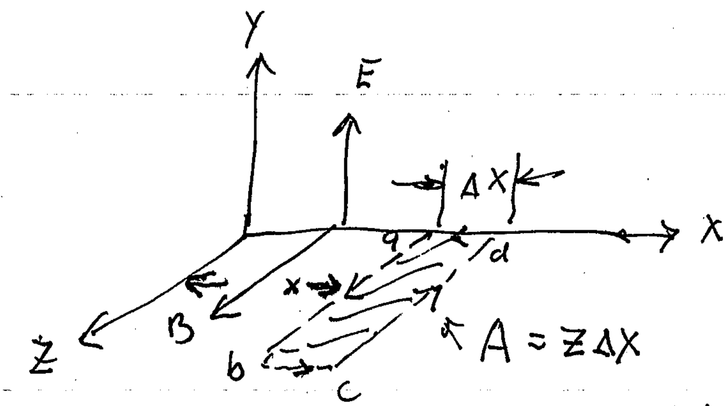
now $-\dfrac{\partial \Phi_B}{\partial t} = \omega B_0 \sin(\omega t - kx)\, y\Delta x = \oint \vec{E}\cdot d\vec{l} = k E_0 \sin(kx - \omega t)\, y\Delta x$

$$E_0 / B_0 = \omega / k$$

Thus from Faraday Law this wave solution must satisfy

$$\frac{E_0}{B_0} = \frac{\omega}{k}$$

$$\vec{E} = \hat{y} E_0 \cos(\omega t - kx)$$
$$\vec{B} = \hat{z} B_0 \cos(\omega t - kx)$$



Now apply Maxwell's Equation around path a, b, c, d

$$\frac{1}{c^2}\Phi_E(t) = A E(x,t) = E_0 \cos(\omega t - kx)\, z\Delta x$$

$$\frac{1}{c^2}\frac{\partial \Phi_E}{\partial t} = \frac{A}{c^2}\frac{\partial E(x,t)}{\partial t} = -\frac{\omega}{c^2} E_0 \sin(\omega t - kx)\, z\Delta x$$

$\oint \vec{B}\cdot d\vec{l}$ about path $\overrightarrow{cdab}$

$$\oint \vec{B}\cdot d\vec{l} = -zB_0(x+\Delta x) + zB(x,t) = -z B_0 \left[\cos(\omega t - k(x+\Delta x)) - \cos(\omega t - kx) \right]$$

$$\simeq -k z \Delta x\, B_0 \sin(\omega t - kx)$$

Thus: $\frac{1}{c^2}\frac{\partial \Phi_E(t)}{\partial t} = \oint \vec{B}\cdot d\vec{l}$

$$-\frac{\omega}{c^2} E_0 \sin(\omega t - kx)\, z\Delta x = -k B_0 \sin(\omega t - kx)\, z\Delta x$$

cancelling common factors } $\boxed{\frac{\omega}{c^2} E_0 = B_0 k}$; and from above $\boxed{\frac{\omega}{k} = \frac{E_0}{B_0}}$

From Faraday Law $V_W = \frac{\omega}{k} = \frac{E_0}{B_0}$

From Maxwell's Law $V_W = \frac{\omega}{k} = c^2 \frac{B_0}{E_0}$

multiply both terms

$$\frac{\omega^2}{k^2} = \frac{E_0}{B_0} \cdot c^2 \frac{B_0}{E_0} = c^2$$

$$\boxed{\frac{\omega^2}{k^2} = c^2} = V_W^2$$

$$\frac{\omega}{k} = c = 3 \times 10^8 \text{ m/s}$$

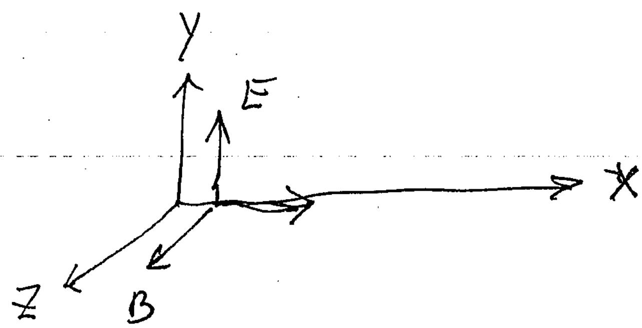
This is the same speed as speed of light

In fact visible light is a special case of an electromagnetic wave at frequencies

$(4.0 - 7.5) \times 10^{14}$ Hz

Polarization of E & M waves.

We saw that wave is transverse:



$\vec{E} \perp \hat{x}$ - direction of wave propagation

$\vec{B} \perp \hat{x}, \vec{E}$

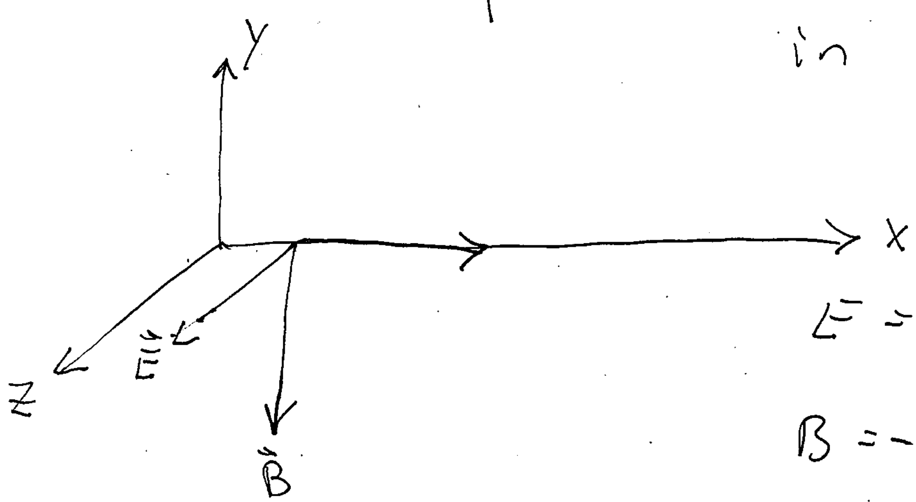
$\vec{E} \perp \vec{B}$

Also note

$\vec{E} \times \vec{B}$ is in direction of wave propagation

& recall $\boxed{E = cB}$ for E&M wave

There is a second independent polarization for propagation in $\hat{x}$-direction



$E = \hat{z} E_0 \cos(\omega t - \frac{\omega}{c} x)$

$B = -\hat{y} E_0 \cos(\omega t - \frac{\omega}{c} x)$

Energy Density of E & M wave

Electric Energy Density

$$u_E = \frac{\epsilon_0 E^2}{2}$$

Magnetic Energy Density

$$u_B = \frac{B^2}{2\mu_0}$$

And E & M wave has both (like an LC circuit)

$$u_E = \frac{\epsilon_0 E^2}{2} \; ; \; u_B = \frac{B^2}{2\mu_0} = \frac{1}{2\mu_0}\left(\frac{E}{c}\right)^2 = \frac{E^2}{2c^2\mu_0}$$

$$= \frac{\mu_0 \epsilon_0}{2\mu_0} E^2 = \frac{\epsilon_0 E^2}{2}$$

$$= u_E$$

E & M wave stores an equal amount of electric and magnetic energy!

$$u_{tot} = \epsilon_0 E^2$$

rms stored energy

$$\overline{u_{tot}} = \epsilon_0 \overline{E^2} = \overline{\epsilon_0 E_0^2 \cos^2(\omega t - kx - \phi)}$$

$$= \frac{\epsilon_0 E_0^2}{2}$$

$\hat{c} \equiv$ velocity of E+M in arbitrary direction

Energy Flux

$$\underset{\sim}{S} = \vec{c}\, U_{tot}$$

$$= \hat{c}\, c\, \epsilon_0 |E|^2 = \hat{c}\, c^2 |\vec{E}| \left|\frac{\vec{E}}{c}\right| \epsilon_0$$

$$= \frac{\hat{c}}{\mu_0 \epsilon_0} |\vec{E}|\,|\vec{B}|\, \epsilon_0$$

$$= \frac{\vec{E} \times \vec{B}}{\mu_0} = \vec{S}$$

$$\boxed{\text{Poynting Flux} \equiv \vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}}$$

$\equiv \vec{E}+\vec{M}$ energy impinging on a unit Area ($1\text{-}m^2$) / sec

$$\equiv \frac{\text{Power}}{\text{Area}} = \frac{\text{Watts}}{m^2}$$

(11) Intensity $\equiv$ Time Average of Poynting Flux

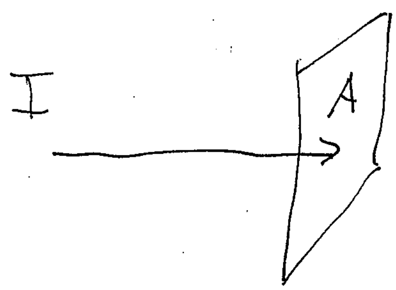
$$I\hat{c} = \overline{\frac{\vec{E}\times\vec{B}}{\mu_0}} = \vec{c}\,\epsilon_0\underbrace{\frac{|E_0|^2}{2}}_{\text{comes from time average}}$$

where $\vec{E} = \hat{y}E_0\cos(\omega t - kx - \phi)$

If we use RMS amplitude

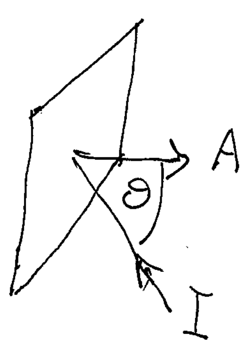
$$I\hat{c} = c\,|E_{RMS}|^2\epsilon_0, \text{ wher } E_{RMS} = \frac{E_0}{\sqrt{2}}$$

through



Power passing area A

$$P = \vec{I}\cdot\vec{A} \equiv \text{Intensity} \times \text{Area}$$

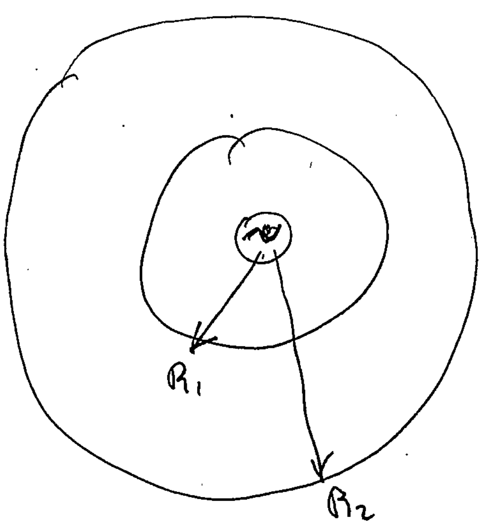


$$P = IA\cos\theta$$

E&M Radiation is produced by accelerating charges (typically on an antenna, but also vibrating atoms)

Beautiful (but detailed) example given in text for why accelerated charge will produce a tranverse E&M wave far from its position.

Consider two spheres of radii R_1 and R_2 around a sinusoidally accelarated charge



Total Power escaping at Radius R_1 must equal total power escaping at R_2

$$\bar{P}_t = I(R_1)\,4\pi R_1^2 = I(R_2)\,4\pi R_2^2$$

Thus

$R_1^2 I(R_1) \equiv$ constant (from identical source)

$I(R) \propto \frac{1}{R^2} \equiv$ Intensity Falls off as $\frac{1}{R^2}$ from antenae IR

$\underset{\sim}{E}$ & $\underset{\sim}{M}$ waves, like a gas of moving particles, also have a momentum density

$$\text{Momentum Density} \equiv \frac{\underset{\sim}{\vec{S}}}{c^2} = \frac{\text{Poynting Flux}}{c^2}$$

$$= \frac{1}{c^2} \frac{\vec{E} \times \vec{B}}{\mu_0} = \epsilon_0 \mu_0 \frac{\vec{E} \times \vec{B}}{\mu_0} = \epsilon_0 \vec{E} \times \vec{B}$$

Momentum density produces pressure, that though slight, has various applications (a space craft sail)