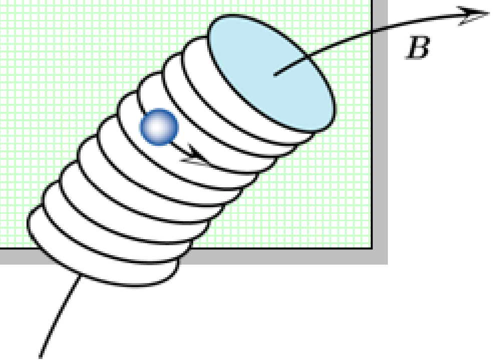
Variational Symplectic Integrator of the Guiding Center Motion



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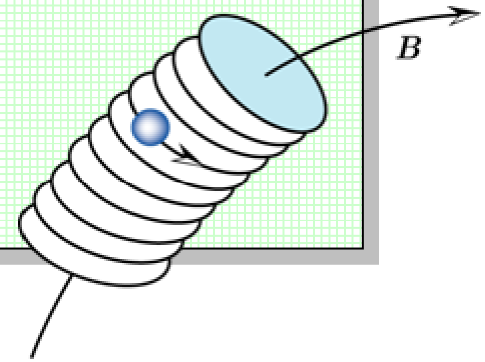
Princeton Plasma Physics Laboratory, Princeton University
Center for Magnetic Fusion Theory, Chinese Academy of Sciences

US-Japan Workshop on Progress of Multi-Scale Simulation Models
Nov. 20, 2008, Dallas, USA

www.pppl.gov/~hongqin/Gyrokinetics.php

How to Calculate the Guiding Center Motion?



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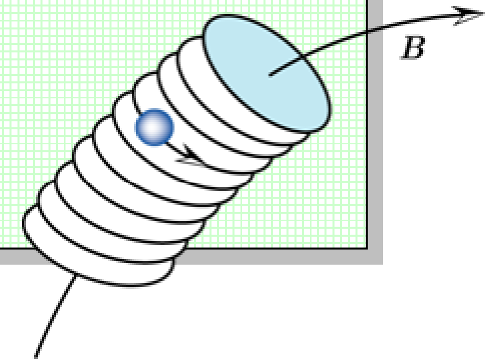
You Don't How to Calculate the Guiding Center Motion!



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Gyrocenter dynamics and algorithms

$$\frac{d\boldsymbol{X}}{dt} = u\boldsymbol{b} + \frac{\mu\boldsymbol{\nabla}B\times\boldsymbol{b}}{B} + \frac{\boldsymbol{E}\times\boldsymbol{b}}{B} + \frac{u^2\boldsymbol{b}\times(\boldsymbol{b}\cdot\boldsymbol{\nabla}b)}{B}$$

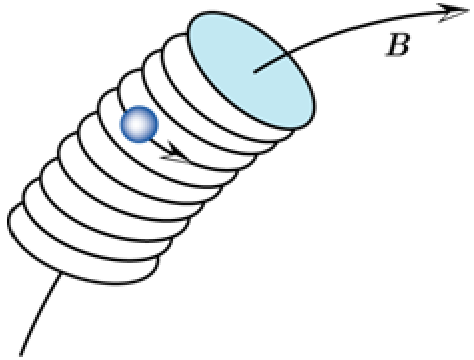
$$\frac{du}{dt} = -\frac{\mu\boldsymbol{b}\cdot\boldsymbol{\nabla}B}{B} + \boldsymbol{b}\cdot\boldsymbol{E}$$

Carl Runge

(1856-1927)

Martin Kutta

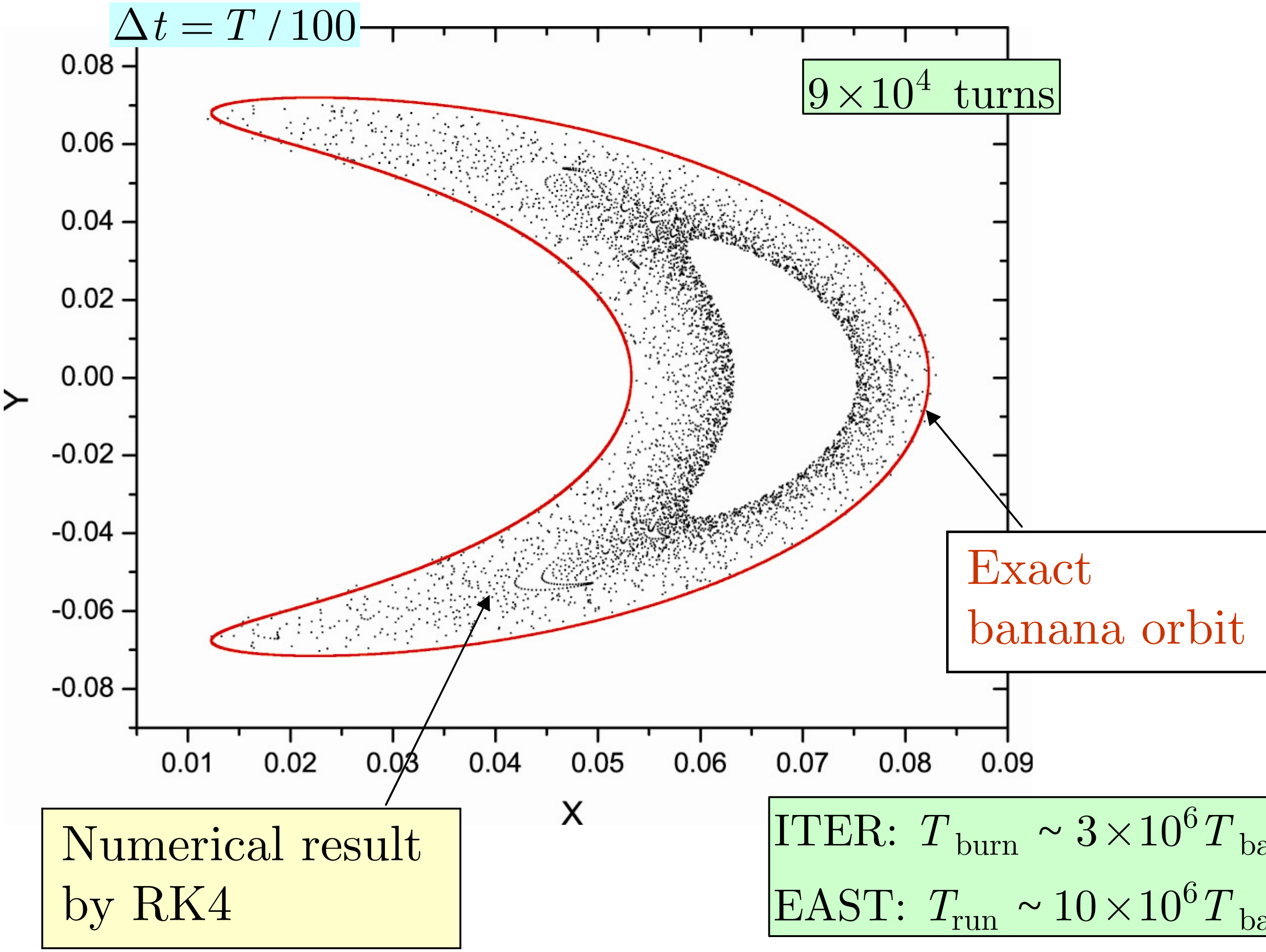
(1867-1944)



nth (4^{th}) order Runge-Kutta methods

- Long time non-conservation.
- Errors add up coherently.

Example – banana orbit go bananas

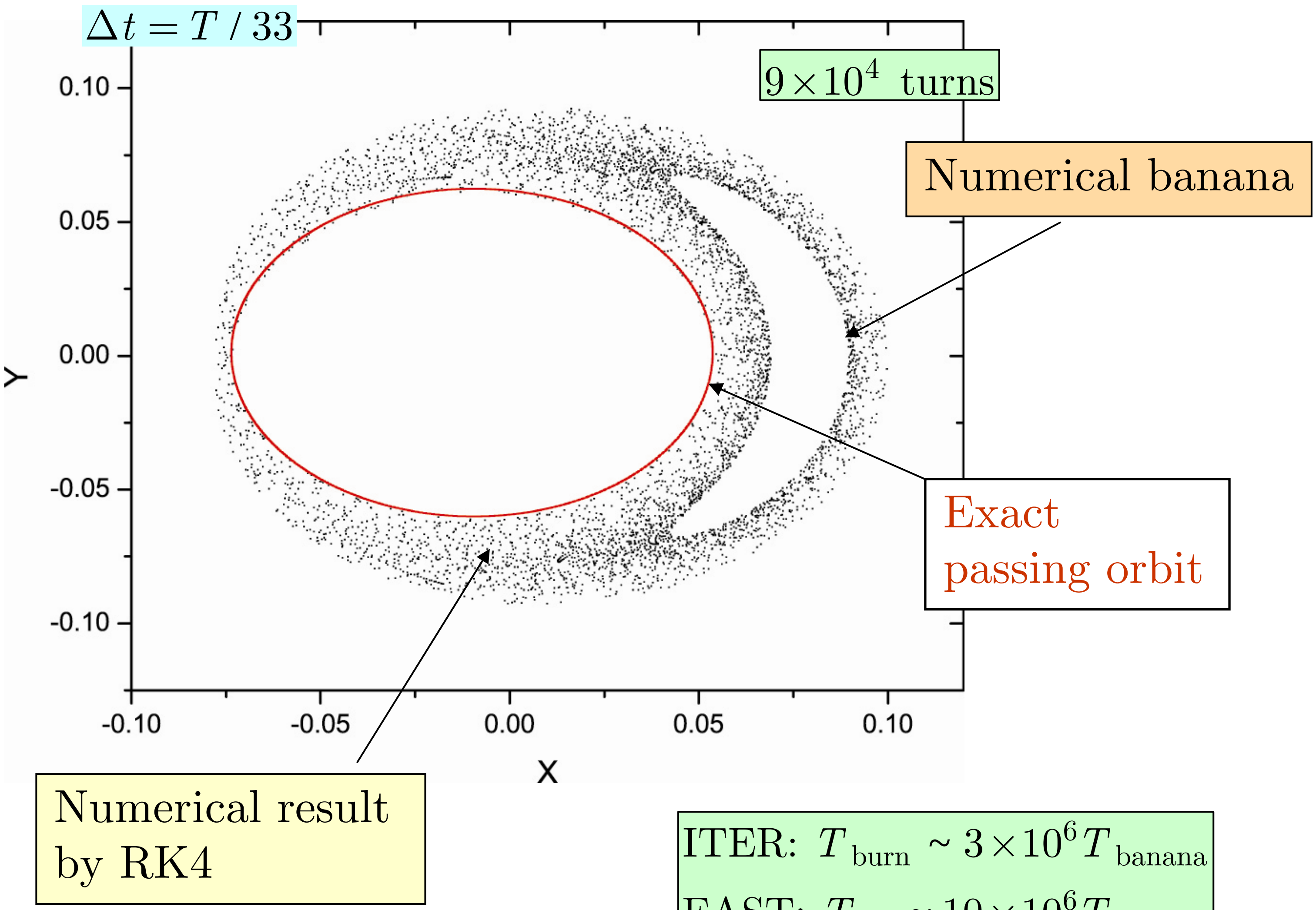


ITER: $T_{\text{burn}} \sim 3 \times 10^6 T_{\text{banana}}$

EAST: $T_{\text{run}} \sim 10 \times 10^6 T_{\text{banana}}$

H. Qin and X. Guan, PRL **100**, 035006 (2008).
X. Xiao and S. Wang, submitted to PoP.

Example – passing orbit go bananas too

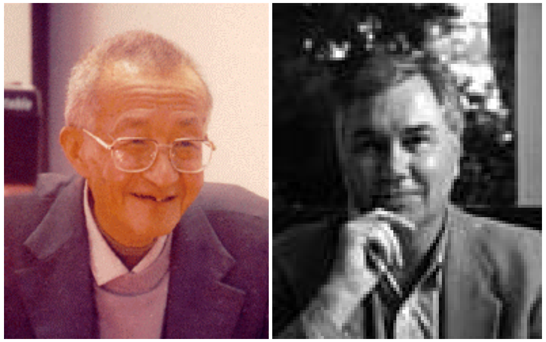


ITER: $T_{\text{burn}} \sim 3 \times 10^6 T_{\text{banana}}$

EAST: $T_{\text{run}} \sim 10 \times 10^6 T_{\text{banana}}$

60 million years ago ……

Can we do better than RK4? -- symplectic integrator



Feng (1983) Ruth (1984)

Conserves symplectic structure;
Bound energy error globally

Symplectic integrator ?

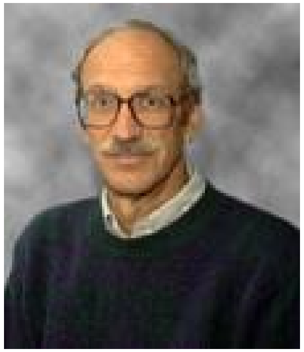
Requires canonical Hamiltonian structure

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{,q} \\ H_{,p} \end{pmatrix}$$

Application areas:

- Accelerator physics (everybody)
- Planetary dynamics (S. Tremaine)
- Nonlinear dynamics (everybody)
- **Plasma physics (J. Cary 89')**

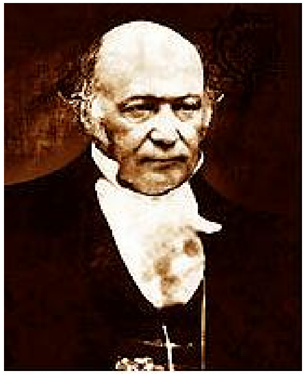
What is the canonical structure for gyrocenter dynamics?



R. White

Gyrocenter dynamics does not have a global canonical structure

Geometry of gyrocenter



Hamilton

Exterior derivative

Inner product

$$i_T d\gamma = 0$$

Worldline

$$\gamma = \omega - \left(\tfrac{1}{2}u^2 + \mu B + \phi\right)dt$$

$$\omega = (A + u\boldsymbol{b}) \cdot d\boldsymbol{X}$$

Only has a non-canonical symplectic structure $\Omega \equiv d\omega$

Gyrocenter dynamics

R. Littlejohn

$$\boxed{\gamma = (\boldsymbol{A} + u\boldsymbol{b})\cdot d\boldsymbol{X} + \mu d\theta - \left(\tfrac{1}{2}u^2 + \mu B + \phi\right)dt}$$

$$\approx$$

$$\boxed{L = (\boldsymbol{A} + u\boldsymbol{b})\cdot \dot{\boldsymbol{X}} + \mu\dot{\theta} - \left(\tfrac{1}{2}u^2 + \mu B + \phi\right)}$$

$$\boxed{i_\tau d\gamma = 0}$$

E-L Eq.

$\boldsymbol{E}\times\boldsymbol{B}$, ∇B drift

$$\frac{d\boldsymbol{X}}{dt} = \frac{\boldsymbol{B}^\dagger}{B_\parallel^\dagger}\left(u + \frac{\mu}{2}\boldsymbol{b}\cdot\boldsymbol{\nabla}\times\boldsymbol{b}\right) - \frac{\boldsymbol{b}\times\boldsymbol{E}^\dagger}{B_\parallel^\dagger}$$

$$\frac{du}{dt} = \frac{\boldsymbol{B}^\dagger\cdot\boldsymbol{E}^\dagger}{B_\parallel^\dagger}$$

$$\frac{d\theta}{dt} = B, \quad \frac{d\mu}{dt} = 0$$

curvature drift

Banos drift (1967)

$$\boldsymbol{B}^\dagger \equiv \boldsymbol{\nabla}\times\boldsymbol{A}^\dagger, \quad \boldsymbol{A}^\dagger \equiv \boldsymbol{A} + u\boldsymbol{b}$$

$$B_\parallel^\dagger = \boldsymbol{B}^\dagger\cdot\boldsymbol{b}, \quad \boldsymbol{E}^\dagger \equiv \boldsymbol{E} - \mu\boldsymbol{\nabla}B$$

Darboux Theorem

Jean Gaston Darboux
(1842-1917)

<u>Darboux Theorem (1882):</u>

Every symplectic structure is **locally** canonical.

↓

Gyrocenter dynamics can be canonical **locally**.

↓

No symplectic integrator for gyrocenter?

Variational symplectic integrator

J. Marsden (2001)

$\boxed{i_\tau d\gamma = 0}$ $\approx$

$$A = \int_0^{t_1} L\,dt$$

$$L = \left(\boldsymbol{A} + u\boldsymbol{b}\right)\cdot\dot{\boldsymbol{X}} - \left(\tfrac{1}{2}u^2 + \mu B + \phi\right)$$

discretize on

$t = [0, h, 2h, \ldots, (N-1)h]$

$$A \approx A_d = \sum_{k=0}^{N-1} h L_d\left(k, k+1\right)$$

$$L_d\left(k, k+1\right) \equiv L_d\left(\boldsymbol{x}_k, u_k, \boldsymbol{x}_{k+1}, u_{k+1}\right)$$

minimize w.r.t. $\left(\mathbf{x}_k, u_k\right)$

$$\frac{\partial}{\partial x_k^j}\left[L_d\left(k-1, k\right) + L_d\left(k, k+1\right)\right] = 0, (j = 1, 2, 3)$$

$$\frac{\partial}{\partial u_k}\left[L_d\left(k-1, k\right) + L_d\left(k, k+1\right)\right] = 0$$

discretized Euler-Lagrangian Eq.

$\left[\left(\mathbf{x}_{k-1}, u_{k-1}\right), \left(\mathbf{x}_k, u_k\right)\right]$

$\rightarrow \left(\mathbf{x}_{k+1}, u_{k+1}\right)$

Conserved symplectic structure

$$\theta^{+}(k,k+1) \equiv \frac{\partial}{\partial \boldsymbol{x}_{k+1}} L_d(k,k+1)\cdot d\boldsymbol{x}_{k+1} + \frac{\partial}{\partial u_{k+1}} L_d(k,k+1) du_{k+1}$$

$$\theta^{-}(k,k+1) \equiv -\frac{\partial}{\partial \boldsymbol{x}_{k}} L_d(k,k+1)\cdot d\boldsymbol{x}_{k} - \frac{\partial}{\partial u_{k}} L_d(k,k+1) du_{k}$$

$$dL_d(k,k+1) = \theta^{+}(k,k+1) - \theta^{-}(k,k+1)$$

minimize w.r.t. $(\mathbf{x}_k, u_k)$

$$\Omega_d(k,k+1) \equiv d\theta^{+} = d\theta^{-}$$

$$dA_d = \theta^{+}(0,1) - \theta^{-}(N-1,N)$$

$$\Omega_d(0,1) = \Omega_d(N-1,N)$$

1^{st} order variational symplectic integrator

$$L_d(k,k+1) \equiv \frac{\left[\boldsymbol{A}^\dagger(k+1)+\boldsymbol{A}^\dagger(k)\right]}{2}\cdot\frac{\left[\boldsymbol{x}_{k+1}-\boldsymbol{x}_k\right]}{h}-\frac{u_k u_{k+1}}{2}-\mu B(k)-\varphi(k)$$

$$\downarrow$$

$$\frac{1}{2h}A^{\dagger i}_{,j}(k)\left(x^i_{k+1}-x^i_{k-1}\right)-\frac{1}{2h}\left[A^{\dagger j}(k+1)-A^{\dagger j}(k-1)\right]=\mu B_{,j}(k)+\varphi_{,j}(k)$$

$$\frac{1}{2h}b^i(k)\left(x^i_{k+1}-x^i_{k-1}\right)=\frac{u_{k+1}+u_{k-1}}{2}$$

$$\left[\left(\mathbf{x}_{k-1},u_{k-1}\right),\left(\mathbf{x}_k,u_k\right)\right]$$
$$\to\left(\mathbf{x}_{k+1},u_{k+1}\right)$$

implicit

H. Qin and X. Guan, PRL **100**, 035006 (2008).

Semi-explicit Newton's method

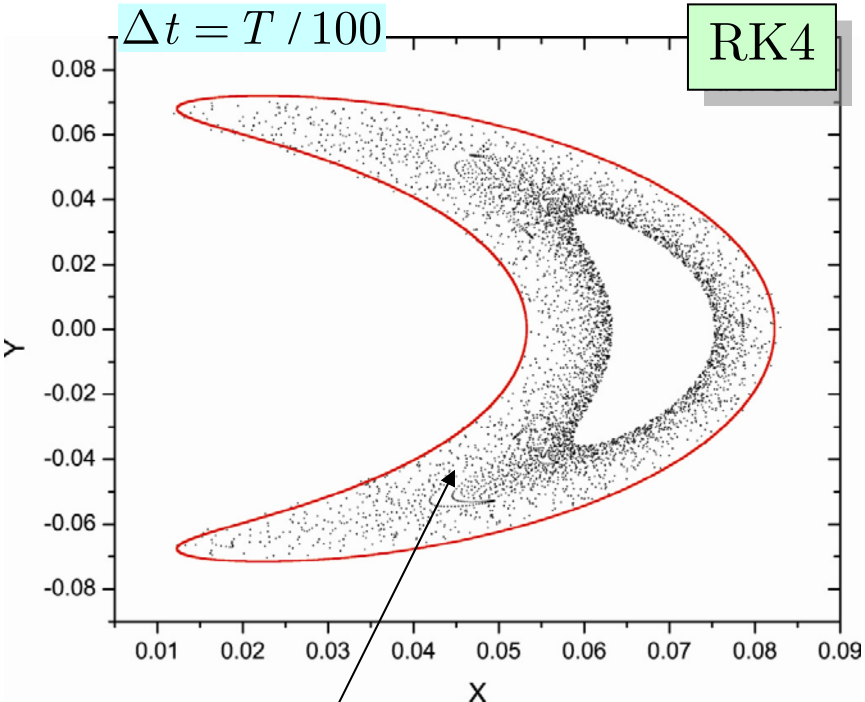
$$\left[A^{\dagger j}(k+1) - A^{\dagger j}(k-1) \right] \approx A^{\dagger j}_{,i}(k)\left(x^i_{k+1} - x^i_{k-1} \right) + b^j(k)\left(u_{k+1} - u_{k-1} \right)$$

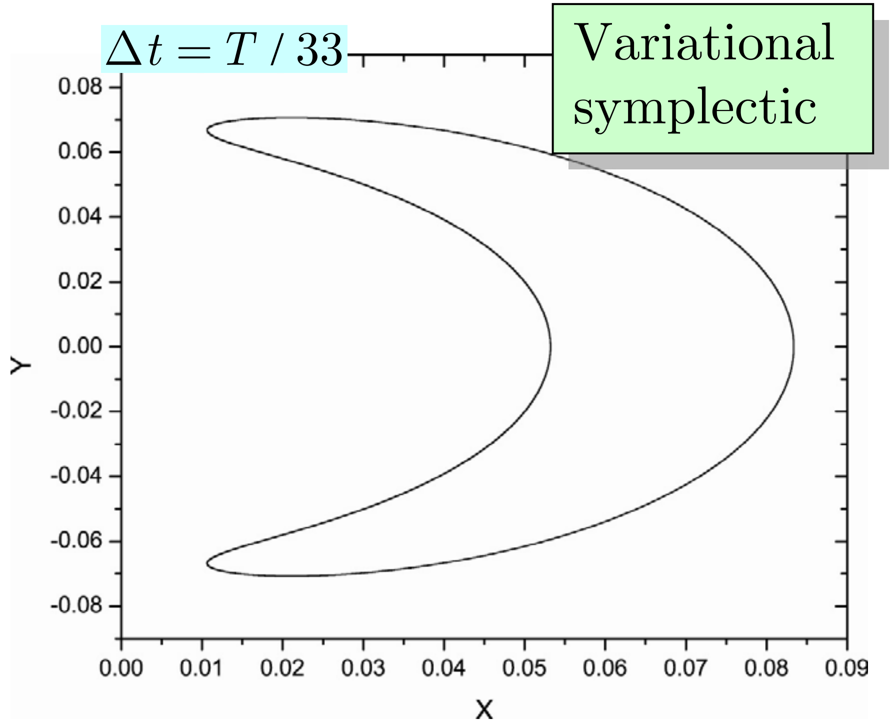
$$\frac{1}{2h}\left[A^{\dagger i}_{,j}(k) - A^{\dagger j}_{,i}(k) \right]\left(x^i_{k+1} - x^i_{k-1} \right) - \frac{b^j(k)}{2h}\left[2u_{k-1} - \frac{b^i(k)}{h}\left(x^i_{k+1} - x^i_{k-1} \right) \right]$$
$$= \mu B_{,j}(k) + \varphi_{,j}(k)$$

$$\frac{1}{2h} b^i(k)\left(x^i_{k+1} - x^i_{k-1} \right) = \frac{u_{k+1} + u_{k-1}}{2}$$

explicit, initial guess for the Newton's method

Variational symplectic integrator, banana orbits



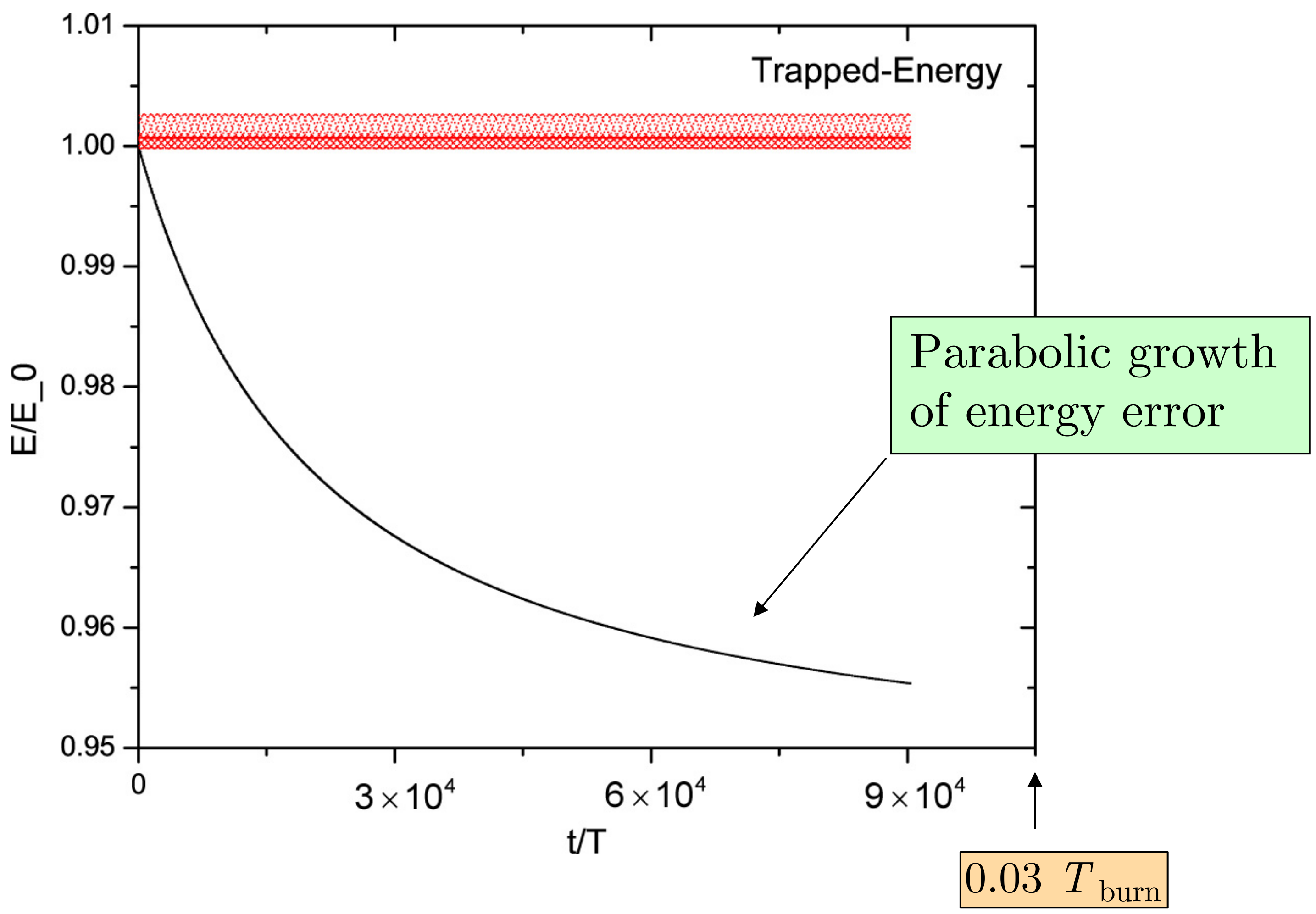


Transport reduction by integration errors

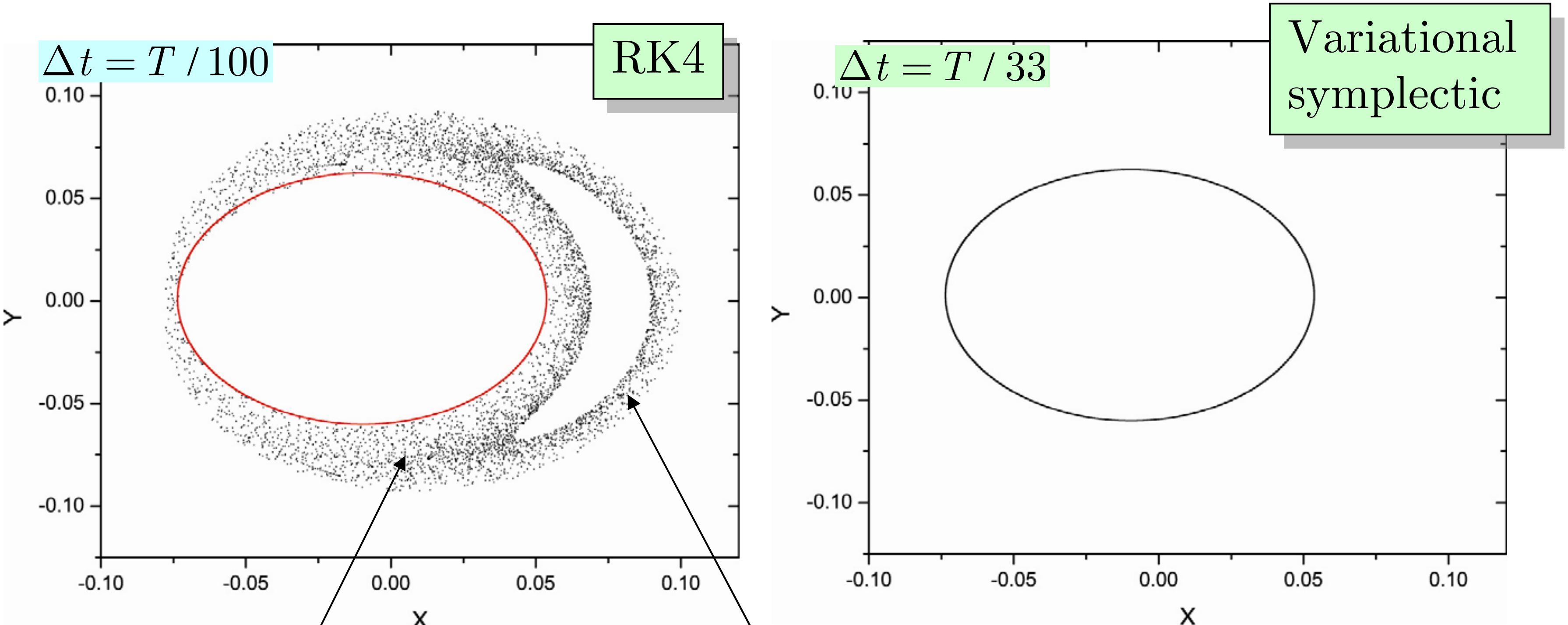
9×10^4 turns

H. Qin and X. Guan, PRL **100**, 035006 (2008).

Variational symplectic integrator bounds energy error globally



Variational symplectic integrator, no numerical banana

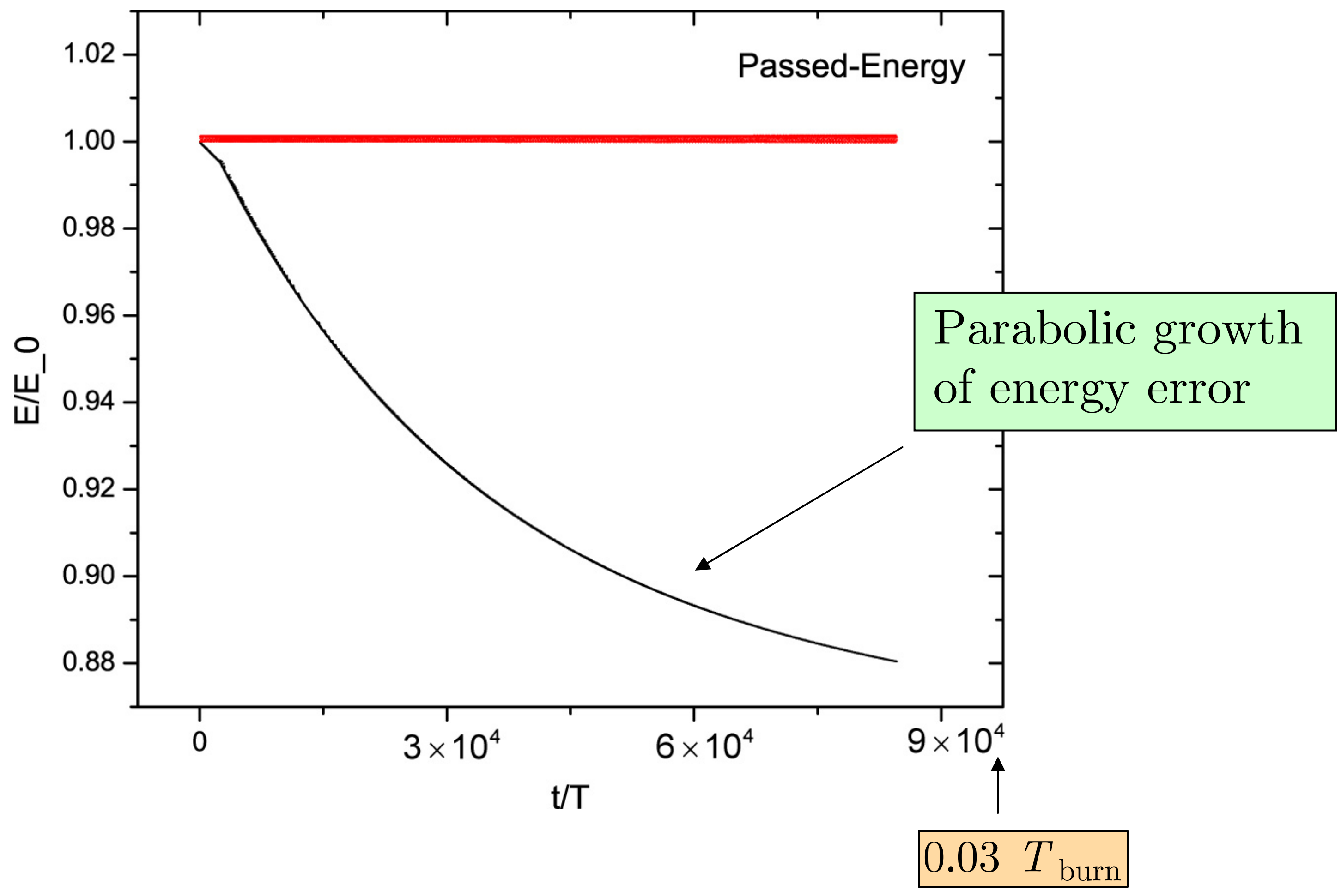


Transport reduction by integration errors

Numerical banana

9×10^4 turns

Variational symplectic integrator bounds energy error globally



Conclusions

How to calculate guiding center motion?

Physics is geometry. So is algorithm.

Symplectic bounds error **globally**, others do not.

Multi-scale dynamics needs global algorithms

"Can Billy come out and compete in the global economy?"

How about collisions?

Frequent heartburns do not cure stomach cancer.

Collisions do not remove the energy errors of RK4.

Global energy errors of RK4 are numerical noises to collisional physics.

How about implicitness?

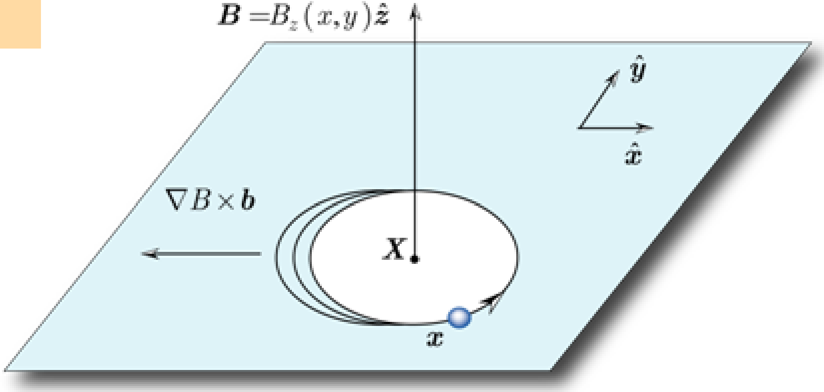
Implicit root search does destroy the symplectic structure.

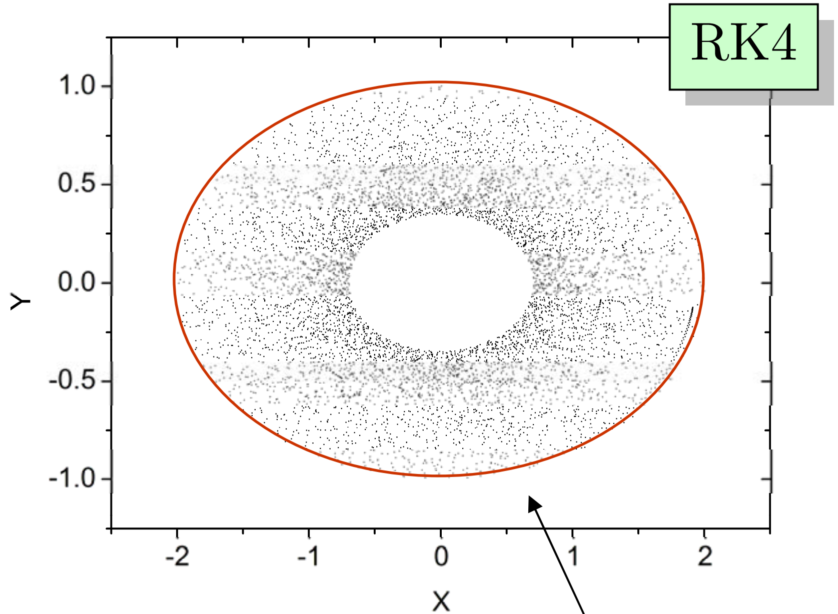
Approximate solution is much better than no solution.

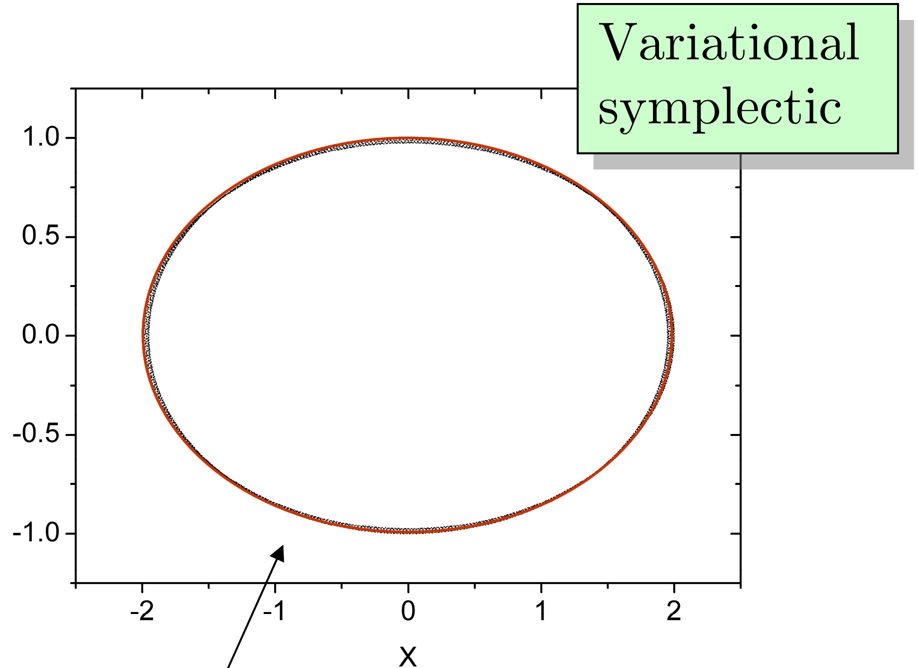
Example – gradient drift

$$\boldsymbol{B} = B(x,y)\hat{\boldsymbol{z}}$$

$$B(x,y) = 1 + 0.05\left(\frac{x^2}{4} + y^2\right)$$





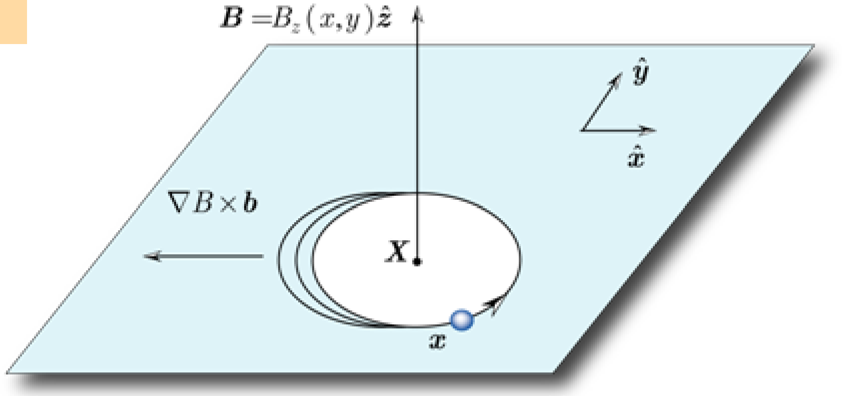


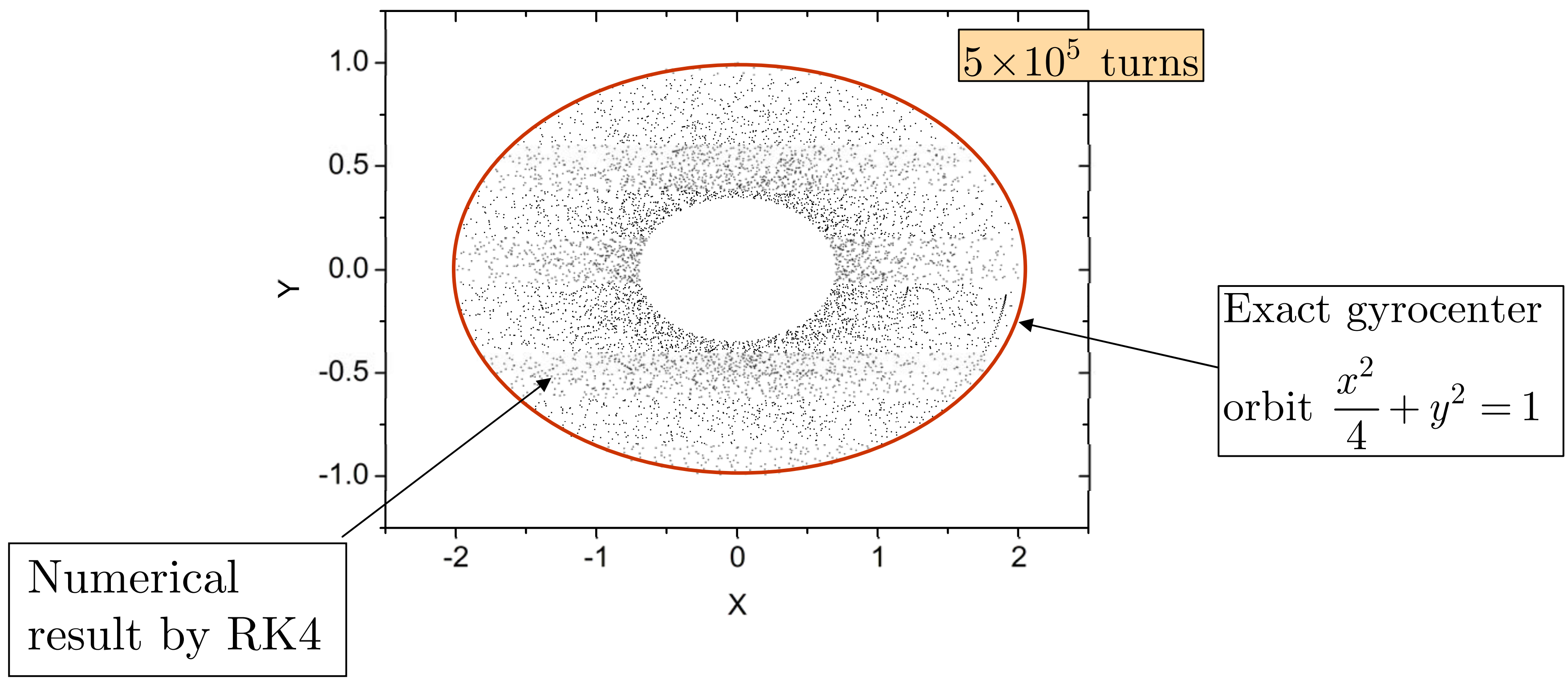
Exact gyrocenter orbit $\frac{x^2}{4} + y^2 = 1$

5×10^5 turns

Example 1 – gradient drift

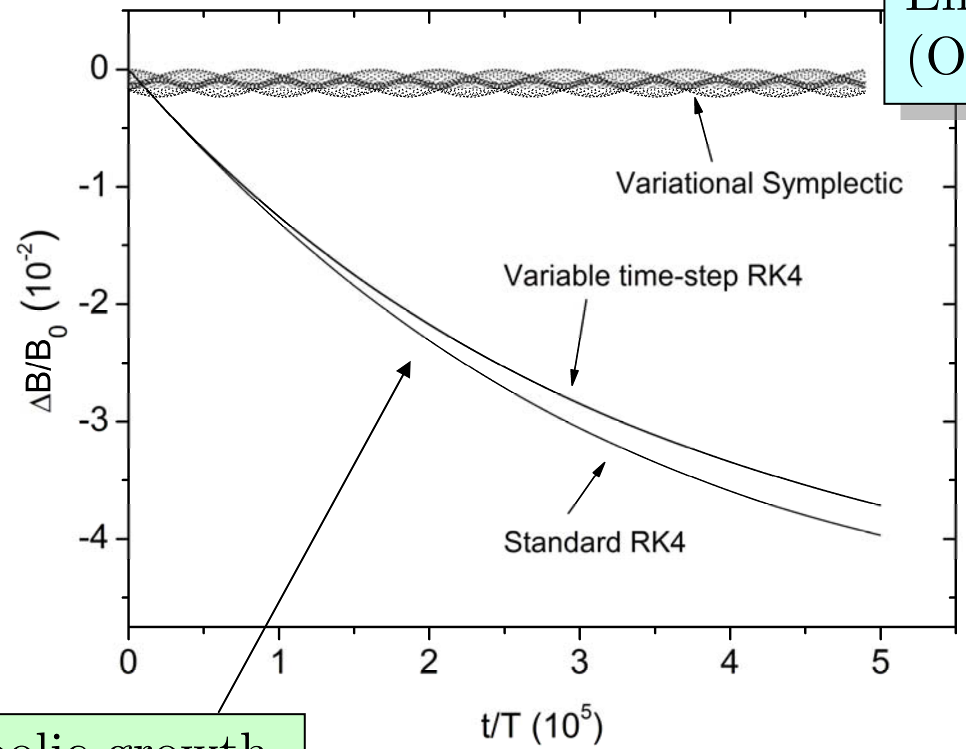
$$B_z(x,y) = 1 + 0.05\left(\frac{x^2}{4} + y^2\right)$$





Variational symplectic globally bounds energy error

Energy error (Orbit deviation)



Parabolic growth of energy error