Studies of Nonlinear Two Fluid Tearing Modes in Cylindrical Reversed Field Pinches

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The Madison Symmetric Torus RFP experiment

$R= 1.5 \text{ m}, a = 0.5 \text{ m}, I \sim 0.5 \text{ MA}, T \leq 1 \text{ keV}, n \leq 10^{13}\text{cm}^{-3}$
Magnetic configuration of Reversed Field Pinch

- Toroidal field $B_T$ is $\sim 5$ times smaller than same-current tokamak

- Equilibrium substantially determined by self-generated plasma currents
RFP equilibrium - resonant unstable modes exist

\[ \tilde{B} = \sum_{n,m} \tilde{B}_{mn}(r) \exp(i m \theta + i n z / R) \]

\[ \nabla \cdot \mathbf{B} = m B_p / r - n B_T / R = 0 \rightarrow m/n = q(r) = r B_T / R B_p \]

\[ \text{Stability depends on } \lambda(r) = \frac{J}{B} \]

MST Reversed Field Pinch

~0.2

\( q(r) \)

resonant modes (tearing / resistive kink)

(1,6)

(1,7)

m=1, n = 6, 7...
core modes

m=0

minor radius

r

0
3D structure of core and edge modes

3D magnetic perturbations (m=1, n=6 core mode)

magnetic surfaces for m=0, n=1 edge mode

two unstable core modes are nonlinearly coupled to the edge mode

\[ k_{1,7} + k_{-1,-6} = k_{0,1} \]
Large magnetic energy only occurs with both core and edge reconnection.
Multi-scales of two fluid MHD: key findings

resistive tearing layer
\[ w_T \sim a/S^{2/5} \]

ion-sound gyroradius
\[ \rho_s \sim (1/\omega_{ci}) [(T_e + T_i) / m]^{1/2} \]

minor radius \( a \)

Linear tearing mode

- Two-fluid physics + cylindrical field line curvature

complex growth rate (mode rotation) without \( \omega_{*i,e} \) effects

Nonlinear NIMROD simulations

- Small modification of the mean current is enough for nonlinear mode saturation

- Hall dynamo is broadened in cylindrical geometry contrary to the fine structure along the separatrix observed in 2D slab simulations
Cylindrical RFP configuration brings new effects into play

- Radial component of the induction equation
  \[
  \frac{\partial B_r}{\partial t} - \frac{\eta c^2}{4\pi} \nabla^2 B_r = ik_B B v_r + \frac{i k_B}{n_e} (j^{(1)} \times B + j \times B^{(1)}) |_\perp = -\frac{e k_B k_\perp}{n_e} p^{(1)}
  \]

- Parallel component with curvature and current gradient effects
  \[
  \frac{\partial B_\parallel}{\partial t} + [B \nabla \cdot v_\perp] - \frac{2}{r B} \frac{2 B_\theta}{r B} - \frac{\eta c^2}{4\pi} \nabla^2 B_\parallel = -\frac{1}{n_e} \mathbf{b} \cdot \nabla \times (j^{(1)} \times B + j \times B^{(1)})
  \]

- Force balance in cross-field \( \mathbf{b} \times \mathbf{e}_r \) direction
  \[
  p^{(1)} = -\frac{k_\|^2}{k_\perp^2} B B_\parallel + \frac{j}{i k_\perp c} B r + \frac{i k_B}{4\pi k_\perp r} \frac{\partial}{\partial r} (r B_r)
  \]

  - Two fluid tearing instability is driven mainly by the electrons.
  - Ion motion enters the equations through the plasma compressibility.
  - Equilibrium diamagnetic flows are ignored (force free equilibrium).
Tearing equations are simplified on short scales

- Coupled equations for two fluid cylindrical tearing mode

\[
\left[ 1 + B^2 \left( \frac{1}{\beta} + \frac{k_\parallel^2}{\gamma^2} \right) + \frac{2id_i k_0 B_\parallel^2}{\gamma r B} \right] \dot{B}_\parallel = \delta^2 \frac{d^2 B_r}{dr^2} - \frac{d_i B}{\gamma} \left( \frac{k_\parallel}{k_0} \frac{\partial^2 B_r}{\partial r^2} - \frac{d\lambda}{dr} B_r \right), \quad \lambda = \frac{4\pi j a}{c B_0 B}
\]

\[
B_r - \delta^2 \frac{d^2 B_r}{dr^2} = \frac{d_i k_\parallel k_0 B}{\gamma} \dot{B}_\parallel, \quad \beta = \frac{4\pi p^{(0)}(0) \Gamma}{B_0^2}, \quad \Gamma = 5/3, \quad \gamma \rightarrow \gamma_A, \quad \delta^2 = \frac{1}{\gamma S}
\]

- Renormalized function \( B_\parallel \)

\[
B_\parallel = \frac{i\lambda}{k_0} B_r
\]

- Quasi-linear approach to mean field characteristics

\[
\frac{1}{en^{(0)}(c)} \left[ j^{(1)\times B^{(1)}} \right]_\parallel - \frac{1}{c} j^{(1)\times B^{(1)}}_\parallel = \langle E \rangle_\parallel - \eta \langle j \rangle_\parallel,
\]

H\( \text{all dynamo} \)  M\( \text{HD dynamo} \)

- Cross-phase between \( B_r \) and \( B_\parallel \) determines the Hall dynamo

\[
e^{(H)}_\parallel = c_0 d_i < b^{(0)}[\text{Re} j^{(1)\times} \text{Re} B^{(1)}] >= \frac{e_0 d_i}{2r} \text{Re} \frac{\partial}{\partial r} \left( r B_r B_\parallel^* \right), \quad c_0 = B_0 \frac{v_A}{c}
\]
Plasma flow velocities are reduced and broaden to large $\rho_s$ scale

- Flow profiles in two limiting cases:
  (a) $\rho_s S^{2/5} = 0.1$ (MHD)
  (b) $\rho_s S^{2/5} = 10$ (two-fluid)

- Amplitudes of the flow velocities in 2F and 1F cases

\[
\frac{u_{\text{max}}^{(MHD)}}{u_{\text{max}}^{(2F)}} \sim (\rho_s S^{2/5})^{1/3}
\]
NIMROD is an initial value solver that capable of modeling the two fluid tearing instability.

\[
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \left( \eta \mathbf{J} - \mathbf{V} \times \mathbf{B} + \frac{1}{ne} \mathbf{J} \times \mathbf{B} - \frac{1}{ne} \nabla p_e \right)
\]
Faraday’s / Ohm’s law

\[\mu_0 \mathbf{J} = \nabla \times \mathbf{B}\]
Ampere’s law

\[\nabla \cdot \mathbf{B} = 0\]
divergence constraint

\[
\rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = \mathbf{J} \times \mathbf{B} - \nabla p + \nabla \cdot \nu \rho \mathbf{W}
\]
flow evolution

\[\mathbf{W} \equiv \nabla \mathbf{V} + \nabla \mathbf{V}^T - \frac{2}{3} \mathbf{I} \nabla \cdot \mathbf{V}\]
particle continuity

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{V}) = \nabla \cdot D \nabla n
\]
with artificial diffusivity

\[
\frac{n}{\gamma - 1} \left( \frac{\partial T_\alpha}{\partial t} + \mathbf{V}_\alpha \cdot \nabla T_\alpha \right) = -p_\alpha \nabla \cdot \mathbf{V}_\alpha + \nabla \cdot n \chi \nabla T_\alpha
\]
temperature evolution

- We set \( D, \chi, \) and \( \nu \ll \eta / \mu_0. \)
Equilibrium magnetic configuration of the paramagnetic pinch

- Force-free equilibrium
  \[ j = \lambda(r)B \]

- External electric field \( E_z \) determines \( \lambda \)
  \[ E + \frac{1}{c} \mathbf{v} \times \mathbf{B} = \eta j \quad \rightarrow \quad \lambda = \frac{E_z B_z(r)}{\eta B^2(r)} \]

- Ampere’s law \( \rightarrow \) magnetic configuration is determined by one dimensionless parameter \( \lambda_0 \)
  \[ \nabla \times \mathbf{B} = \lambda_0 \frac{\mathbf{B} \cdot \mathbf{B}_z}{B^2}, \quad \lambda_0 = \frac{4\pi E_z a}{\eta c B(0)}, \quad \mathbf{B} = B/B(0), \quad r \rightarrow r/a \]

(2) \( \lambda_0 = 3.3, \ S = 5 \times 10^3 \)
Robinson's study of paramagnetic equilibrium provides a map of the linear stability. The mode is growing if the stability factor $\Delta' > 0$.

$$\Delta' = \frac{1}{B_r} \left[ \frac{\partial B_r}{\partial r}(+0) - \frac{\partial B_r}{\partial r}(-0) \right]$$

$k / \lambda_0 = 0.6$ with $\lambda_0 = 2.7$ are chosen initially to provide a mode close to marginal stability with $\Delta' = 2$.

More unstable modes with $\Delta' = 16$ are studied at $\lambda_0 = 3.3$ ($k/\lambda_0$ remains constant to keep $q_0$ constant).

D.C. Robinson, Nucl. Fusion (1978)
The growth rate and mode rotation scales with \( \rho_s \)

- \( \lambda_0 = 3.3 \)
- \( S = 5 \times 10^3 \)
- \( \beta = 0.1 \)
- \( P_m = 0.1 \)
- \( d_e = 1.8 \times 10^{-2} \)

\[ \delta = 0.1 \]

**Electron skin-depth**

\[ \delta^2 = d_e^2 + \frac{1}{\gamma \tau_a S} \]

 Collisionless part \( d_e \) is small compared to the resistive term

**Some examples of plasma parameters:**

- \( B_0 = 0.4 \, \text{T} \)
  - \( n = 1 \times 10^{19} \, \text{m}^{-3} \)
  - \( \beta = 0.1 \)
  - \( a = 0.51 \, \text{m} \)
  - \( \rho_s/a = 5.8 \times 10^{-2} \)

- \( B_0 = 0.2 \, \text{T} \)
  - \( n = 1 \times 10^{18} \, \text{m}^{-3} \)
  - \( \beta = 0.1 \)
  - \( a = 0.51 \, \text{m} \)
  - \( \rho_s/a = 1.8 \times 10^{-1} \)
2F dynamics and curvature effects result in mode rotation without diamagnetic flows.
The quasilinear MHD dynamo is broadened at large $\rho_s$

The amplitude of $\langle v^{(1)} \times B^{(1)} \rangle$ decreases in 2F case (consistent with the theory)

![Graphs showing flow velocities and MHD dynamo for single fluid and $\rho_s = 0.1$.](image)
The magnitude of the quasilinear Hall dynamo varies with $\rho_s$ and is localized at the rational surface. The Hall dynamo is determined by $\text{Re} J_\perp$, which is zero in the single fluid limit.
Mean field modification takes place after the mode saturation

- small equilibrium field reconstruction is enough for nonlinear mode saturation

\[ \ln E_{\text{mag}} \]

\[ \rho_s = 0.10 \]

\[ \theta = \langle B_z \rangle \]

\[ \rho_s = 0.02 \]

\[ \rho_s = 0.10 \]

\[ \rho_s = 0.50 \]
The saturated magnetic island is wider than the resistive layer

**Nonlinear saturation**

- Single fluid
  - $\rho_s = 0.10$
  - Unlike the linear cases, the out-of-phase nonlinear solutions are small
  - Island width is consistent with the theory

**Linear stage**

- $\rho_s = 0.10$

- $w = 4r_s \sqrt{\frac{B_r}{mB_\theta} \left( \frac{q}{rq'} \right)}$
Both Hall and MHD dynamos are small in saturated state.

Nonlinear saturation

Single fluid with $\rho_s = 0.10$

Linear stage with $\rho_s = 0.10$
The overall shape of the parallel current modification is not sensitive to $\rho_s$

- Small modification of the equilibrium current $\Delta \lambda \sim 1\%$ is needed for saturation

$$\epsilon = \frac{1}{c} \langle \mathbf{v} \times \mathbf{B} \rangle_{||} - \frac{1}{en c} \langle \mathbf{j} \times \mathbf{B} \rangle_{||} = \frac{1}{c} \langle \mathbf{v}_e \times \mathbf{B} \rangle_{||}$$

The saturated island reduces the parallel current inside the island, and enhances the parallel current on the outboard side of the island.
Fine structure of the slab model is not seen in cylindrical geometry

A characteristic of the fine structure is large $m,n \gg 1$ harmonics – this is not seen in the current calculations – the $m=2$ mode is small.
Summary

- Electron-ion decoupling on short scales affects the phases of the eigenfunctions creating a Hall dynamo absent in single fluid MHD.
- Two fluid effects and field line curvature cause a mode rotation even without diamagnetic flows.
- Quasilinear analytic predictions are confirmed by NIMROD simulations.
- Small modification of the mean current is enough for the nonlinear mode saturation.
- The magnitude of the saturated dynamo is small.
- In saturated state, the radial profile of the total (MHD + Hall) dynamo is much less sensitive to $\rho_s$ than during the linear stage.
- Fine structure of the reconnection layer observed in slab simulations is not seen in the cylindrical case.