Multi-scale aspects of magnetic nozzle modeling

Alexey Arefiev, Mikhail Tushentsov, and Boris Breizman

Institute for Fusion Studies, UT Austin

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Plasma-based propulsion systems generate thrust by ejecting directed plasma flow.

A strong magnetic field is used to guide the plasma towards the nozzle exit.

The ejected plasma must break free from the nozzle to produce thrust.

**MHD detachment scenario:**
A super-Alfvénic flow detaches together with the magnetic field by stretching the field lines and changing their configuration.

MULTI-SCALE ASPECTS OF THE PROBLEM

Key time scales are:
- ion time of flight through the nozzle \( \tau_i = L / V \)
- electron travel time through the nozzle \( \tau_e = L / v_e \)
- propagation time of magneto-acoustic perturbations across the flow \( \tau_A = R / V_A \)
- propagation time of sonic waves along the flow \( \tau_S = L / C_S \)

There is a significant change of parameters along the flow:
- sonic Mach number \( M_S \) changes from \( M_S << 1 \) to \( M_S >> 1 \)
- Alfvénic Mach number \( M_A \) changes from \( M_A << 1 \) to \( M_A > 1 \)

Practical applications require simulation times much longer than \( \tau_i \).

Computational challenges:
- multiple time scales need to be resolved in a dynamical simulation
- increasing computational domain due to the flow expansion
- dramatic change of key physics parameters along the flow
A brute-force approach is to employ an existing comprehensive code.

This option is not practical for an optimization tool that should be able to scan the parameter space. The specific drawbacks are
- a dynamical MHD simulation requires huge computational resources
- running time for a single set of parameters is too long to scan the parameter space

We use the following method to tackle the problem
- split the problem into components that can be treated independently
- use the multi-scale aspects to treat some components analytically
- assemble the components into a multi-scale model tailored to the specific problem

The sub-problems are
- electron kinetics
- ion dynamics along the field lines and the self-consistent magnetic field distortion
A magnetic mirror is needed to convert an incoming subsonic plasma flow into a supersonic outgoing flow.

Electron pressure produces an ambipolar electric field that accelerates plasma ions downstream.

Collisionless plasma expansion through the mirror distorts an incoming Maxwellian electron distribution.

Electrons require a fully kinetic description downstream from the mirror.

The flow expansion is adiabatic, because the rate of the expansion is determined by the ion speed.

We can then use the conservation of the electron magnetic moment and longitudinal adiabatic invariant to treat a part of the problem analytically.
The expanding plume is time-dependent and it consists of two parts:
- a steady-state supersonic flow adjacent to the mirror
- a rarefaction wave at the plasma edge

Most of the flow is in a steady-state, but the rarefaction wave affects the flow globally through collisionless electrons.
Electron dynamics is controlled by the ambipolar potential $\phi(z,t)$ and the guiding magnetic field $B(z)$.

Electron motion is characterized by $U_{\text{eff}}(z,\mu,t) = \mu B(z) - |e| \phi(z,t)$, where $\mu$ is the conserved magnetic moment.
The total energy $\varepsilon$ and magnetic moment $\mu$ of each passing electron are conserved.

The longitudinal adiabatic invariant $I$ and magnetic moment $\mu$ of each trapped electron are conserved.

The electron distribution is nearly symmetric with respect to the axial velocity everywhere downstream.
The dynamical behavior of quasineutral flow is governed by the following equations:

\[
\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial z} = -\frac{|e|}{m_e} \frac{\partial \phi}{\partial z},
\]

\[
\frac{\partial n}{\partial t} + B \frac{\partial}{\partial z} \left( \frac{nV}{B} \right) = 0,
\]

\[
n = \frac{2^{3/2} \pi}{m_e^{3/2}} \frac{2}{3} \left[ \int_{|\phi|}^{\infty} f_0'(\varepsilon) \left[ \varepsilon + |e| \phi - \mu_e(\varepsilon) B \right]^{3/2} d\varepsilon - \int_{-|\phi|}^{\infty} f_0'(\varepsilon) \left[ \varepsilon + |e| \phi \right]^{3/2} d\varepsilon \right].
\]

The function \(\mu_e(\varepsilon)\) is determined by the conditions

\[
\varepsilon = |e| [B/B' - \phi],
\]

\[
\mu_e = |e| / B'.
\]

In the steady-state portion of the flow, the system reduces to an integral equation that relates \(B\) and \(\phi\).
The phase space below the separatrix (red line) is under populated during the collisionless expansion, compared to the collisional regime.
The potential drop in the steady-state part is only $\Delta |e| \varphi \approx -2T$.

A rarefaction wave at the plasma edge reflects energetic electrons back.

The distortion of the Maxwellian distribution limits the ion energy gain and makes it finite.
A supersonic flow downstream from the mirror consists of a steady-state part and a rarefaction wave at the leading edge of the expanding plasma.

The magnetic mirror, together with the rarefaction wave, produce a population of trapped electrons.

The model allows us to solve the problem quantitatively for a given distribution of incoming electrons.

The model for the electrons allows us to eliminate the electron time scale from consideration.
Conserved quantities:

- Magnetic flux \( (BS) = \text{const} \)
- Flow velocity \( (V_\parallel) = \text{const} \)
- Plasma flux \( (n_i V_\parallel S) = \text{const} \)

- Magnetic energy decreases downstream faster than plasma kinetic energy.
- An initially sub-Alfvénic flow becomes super-Alfvénic downstream.
- Plasma flow can stretch the magnetic field lines after \( B^2/8\pi \) drops below \( m_i n_i V_\parallel^2/2 \).

The nozzle is required to be **weakly diverging** to produce a well-directed plume.

The flow adjacent to the nozzle reaches a steady-state regime.

The expanding plasma edge is highly super-Alfvénic, so that magnetic field perturbations are unable to reach the nozzle.
The steady-state flow is described by MHD-like equations

\[
\left( \frac{\partial}{\partial z} + \frac{B_r}{B_z} \frac{\partial}{\partial r} \right) B_z = -B_z \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{B_r}{B_z} \right)
\]

\[
\left( \frac{\partial}{\partial z} + \frac{B_r}{B_z} \frac{\partial}{\partial r} \right) B_r = -\frac{1}{\Pi_\parallel} \frac{\partial}{\partial r} \left( \Pi_\perp + \frac{B_z^2}{8\pi} \right)
\]

\[
\left( \frac{\partial}{\partial z} + \frac{B_r}{B_z} \frac{\partial}{\partial r} \right) \Pi_\parallel = -\left( \Pi_\parallel - \Pi_\perp \right) \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{B_r}{B_z} \right)
\]

\[
\Pi_\perp = \int \langle f_i \rangle \langle a, \mu, \varepsilon \rangle \frac{4\pi\mu B_z^2}{m_i^2 v_{\parallel}} \sqrt{\frac{m_i}{2(\varepsilon - \mu B_z^2)}} d\mu d\varepsilon
\]

\[\langle f_i \rangle = \text{given input function}\]

\[a = \text{magnetic flux surface label}\]

\[\varepsilon = \text{total ion energy}\]

\[\mu = \text{magnetic moment}\]

The flow is assumed to be axisymmetric and paraxial.

We neglect ion ambipolar acceleration, by assuming that electrons are cold.
The steady-state problem is a boundary value problem in \( r \) and \( z \) that involves the vacuum field.

A straightforward approach is challenging, so we use the difference in scales to our advantage to reformulate the problem.

We solve the equations as an initial value problem with \( z \) playing the role of time.

The vacuum field boundary condition is calculated prior to solving the equations and can then be corrected iteratively.

The reason why this method works is explained after the following numerical example for a “conical” magnetic nozzle.
Incoming flow parameters

- Ion energy: $\varepsilon_\parallel = 250$ eV
- Plasma density: $n = 5.0 \cdot 10^{14}$ cm$^{-3}$

The code accurately reproduces the analytic solution.

WHY DOES THE METHOD WORK?

- The weakly diverging nozzle design produces
  - a slowly changing vacuum field inside the nozzle
  - a sharply decreasing vacuum field outside of the nozzle

- The weakly diverging plasma flow can be treated as an ideally conducting cylinder to find the vacuum field prior to solving the flow equations.

- Inside the nozzle, magneto-acoustic waves make the plasma field almost uniform in the flow cross-section.

- What matters is the transverse propagation and that is why our method recovers the solution correctly.

- The flow is super-Alfvénic at the nozzle end, such that magnetic field perturbations propagate only downstream.

- In a super-Alfvénic flow, the equations should be solved in the downstream direction, as done in our method.
SUB- TO SUPER-ALFVÉNIC TRANSITION

Vacuum field lines (solenoid)

Alfvén Mach number

Incoming flow parameters

Ion energy: $\epsilon_|| = 10 \text{ eV}$

Plasma density: $n = 5.0 \cdot 10^{14} \text{ cm}^{-3}$

Thrust = 114 N

Power = 400 kW (Ar)
**PLASMA FLOW WITH ION GYROMOVEMENT**

**Vacuum field lines**

**Incoming flow parameters**

- Ion gyroenergy: $\varepsilon_\perp = 100 \text{ eV}$
- Axial energy: $\varepsilon_\parallel = 10 \text{ eV}$
- Plasma density: $n = 5.0 \cdot 10^{13} \text{ cm}^{-3}$
- Plasma radius: $R_p = 10 \text{ cm}$

**Equations**

- Thrust: $16 \text{ N}$
- Power: $193 \text{ kW (Ar)}$

**Graphs**

- $B_z [\text{ Gauss}]$
- Transverse / Longitudinal energy

**Notes**

- $2\mu B/m_i V_\parallel^2 = 1$
A properly shaped paraxial magnetic nozzle generates a well-directed detached super-Alfvénic plume.

The detachment problem reduces to a steady-state problem for applications with continuous operation.

The developed steady-state Lagrangian code enables broad parameter scans in detachment modeling with modest computational requirements (single work station).

The next step is to combine the electron and ion modules into an integrated model.