Introduction to the Science of Control

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4th ITER International Summer School
May 31 – June 4, 2010, Austin, Texas, USA
Objectives of Talk

• Learn some control terminology

• Develop some intuition about control concepts
  – Details occasionally (and intentionally) omitted

• Understand the multiple objectives of control
Model-Based Control Design Process

1. Make system model
2. Verify model predicts behavior of system
3. Design controller
4. Test using models in closed-loop simulation
5. Implement and test implementation
6. Deploy in operation

• Using only 5-6 is feasible and often successful – why do steps 1-4?
  – Requires empirical tuning, cost = $50,000 - $100,000 per day on present devices
  – Performance:
    – Large systems (many inputs / outputs) difficult to tune properly for best control
    – Nonlinear systems require retuning over many equilibrium states.
  – Even if Steps 5-6 is chosen approach, studying models is useful to understand how control affects system
  – Next Generation devices (e.g. ITER) will not allow empirical tuning
Introduction to System Representation - Block Diagrams

- A Block Diagram consists of two parts:
  - Signals (arrows in diagram)
  - Operations (blocks in diagram)

- Example (poloidal field system producing plasma shape)

- Equivalently, hiding all details:
**State Space Models**

- General (x is "state"): 
  \[ \dot{x} = f(x,u,t) \]
  \[ y = g(x,u,t) \]

- Note **ordinary differential equation (ODE)** is 1\textsuperscript{st} order

- Linear, time-invariant (LTI) system:
  \[ \dot{x}(t) = Ax(t) + Bu(t) \]
  \[ y(t) = Cx(t) + Du(t) \]

- Example (plasma + conductors):
  \[ M_* \frac{d(\delta I)}{dt} + R\delta I = U\delta v \]
  \[ y = CI \]
  \[ \Rightarrow \frac{d(\delta I)}{dt} = A\delta I + Bv \]
  \[ y = CI \]
  \[ (A = -M_*^{-1}R, \ B = M_*^{-1}U) \]

- \( I(t) \) = toroidal conductor currents (perturbations \( \delta I \) from equilibrium \( \Leftrightarrow \) states \( x \)); 
  \( M_* \) = mutual inductance matrix (modified by plasma response), \( R \) = resistance matrix

- \( y(t) \) = coil currents, flux and field in vacuum region; \( C \) = green functions

- \( v(t) \) = input voltage from power supplies (\( \delta v \) from equilibrium); \( U = \text{ones for coils, zeros for vessel conductors} \)
System Representation – Laplace Transform

• Definition: For a given function \( f(t) \) with \( f(0)=0 \), Laplace transform of \( f \) is:

\[
F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt, \quad s = \sigma + j\omega
\]

• Nice properties:

\[
\mathcal{L}\left\{ \frac{df}{dt} \right\} = sF(s), \quad \mathcal{L}\left\{ \frac{d^2f}{dt^2} \right\} = s^2F(s), \quad \ldots \text{etc} \ldots
\]

\[
\mathcal{L}\left\{ \int_0^t f(\tau) d\tau \right\} = \frac{1}{s} F(s)
\]

• For an example of how it’s used, apply to:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \quad \Rightarrow \quad sX(s) = AX(s) + BU(s) \\
y(t) &=Cx(t) + Du(t) \quad \Rightarrow \quad Y(s) = CX(s) + DU(s)
\end{align*}
\]

\[
sX(s) = AX(s) + BU(s) \Rightarrow (sI - A)X(s) = BU(s) \Rightarrow X(s) = (sI - A)^{-1}BU(s)
\]

\[
Y(s) = CX(s) + DU(s)
\]

\[
= C(sI - A)^{-1}BU(s) + DU(s) \quad \Rightarrow \quad Y(s) = \left(C(sI - A)^{-1}B + D\right)U(s)
\]
System Representation - Transfer Functions

- Transfer Function = ratio of Laplace Transforms of (scalar) output and input signals: \( \frac{Y(s)}{U(s)} \)

- Example (simple mechanical system; \( x \) is displacement):
  \[
m\ddot{x}(t) + d\dot{x}(t) + kx(t) = u(t) \Rightarrow (ms^2 + ds + k)X(s) = U(s) \Rightarrow \frac{X(s)}{U(s)} = \frac{1}{(ms^2 + ds + k)}
  \]

- Example (lowpass RC filter):
  \[
  \begin{align*}
  &\text{Vin} \quad R \quad C \quad \text{Vout} \\
  \end{align*}
  \Rightarrow \quad \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{RCs + 1}
  \]

- General LTI case, from previous page: \( Y(s) = \left(C(sI - A)^{-1}B + D\right)U(s) \)

- If \( Y, U \) are scalars: \( \frac{Y(s)}{U(s)} = \left(C(sI - A)^{-1}B + D\right) \) (Single-Input-Single Output (SISO) system)

- If Multi-Input-Multi-Output (MIMO) system, each element in matrix \( C(sI - A)^{-1}B + D \) is a scalar transfer function, so still called "transfer function"
System Representation - Equivalent Representations

- Block Diagram

\[
\begin{align*}
\text{force } & \rightarrow \text{mechanical system} \\
& \rightarrow \text{displacement } x \\
m\ddot{x} + d\dot{x} + kx &= u
\end{align*}
\]

State Space (1st order ODE)

\[
\begin{align*}
\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ x \end{bmatrix} + \begin{bmatrix} d & k \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} &= \begin{bmatrix} u \\ 0 \end{bmatrix} \\
y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix}
\end{align*}
\]

Transfer Function

\[
\begin{align*}
\frac{Y(s)}{U(s)} &= \frac{1}{(ms^2 + ds + k)} \\
\frac{V_{out}(s)}{V_{in}(s)} &= \frac{1}{RCs + 1}
\end{align*}
\]

\[
\begin{align*}
\dot{V}_{out}(t) &= -\frac{1}{RC} V_{out}(t) + \frac{1}{RC} V_{in}(t) \\
y(t) &= V_{out}(t)
\end{align*}
\]

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

\[
Y(s) = \left(C(sI - A)^{-1} B + D\right)U(s)
\]
System Representation – Feedforward/Feedback

**Open-Loop Control**

- Shape request → Feedforward Controller → Power Supply → Plasma / conductors → Control parameter calculation → shape params

**Closed-Loop Control**

- Shape request → Feedback Controller → Power Supply → Plasma / conductors → Control parameter calculation → shape params

**Combined**

- Shape request → Feedback Controller → Power Supply → Plasma / conductors → Control parameter calculation → shape params
What is undriven "natural" behavior of system?

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

Defined by the eigenvalues \( \lambda \):

\[ \lambda x = Ax \]

An arbitrary vector \( v \) can be expressed as sum of eigenvectors:

\[ v = \sum_{k=1}^{n} \alpha_k x_k \]

Then:

\[ Av = \sum_{k=1}^{n} \alpha_k Ax_k = \sum_{k=1}^{n} \alpha_k \lambda_k x_k \quad \Rightarrow \quad \dot{x} = \sum_{k=1}^{n} \alpha_k \dot{x}_k = \sum_{k=1}^{n} \alpha_k \lambda_k x_k \]

That is, we can analyze as n scalar ODE's:

\[ \dot{x}_k = \lambda_k x_k \quad \Rightarrow \quad x_k(t) = e^{\lambda_k t} x_k(0) \]

To determine stability of the system:

\[ \sigma_k = \text{real}(\lambda_k) < 0 \quad \Rightarrow \quad x_k(t) \to 0, \quad t \to \infty \quad \text{(stable)} \]

\[ \sigma_k = \text{real}(\lambda_k) > 0 \quad \Rightarrow \quad x_k(t) \to \infty, \quad t \to \infty \quad \text{(unstable)} \]

If ANY eigenvalue has \( \text{Re}(\lambda) > 0 \) => system is UNSTABLE.

Otherwise, system is STABLE.
Analysis of Dynamics (Laplace Domain)

- **Poles and zeros of Transfer Functions:**
  - Complex function theory terminology:
    - Roots of denominator polynomial $a(s) =$ **poles**
    - Roots of numerator polynomial $b(s) =$ **zeros**
  - If ANY poles have $\sigma = \text{Re}(s) > 0$, system is **UNSTABLE**, otherwise, **STABLE**. (Explanation in a moment.)
  - Examples:
    - \[ \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{RCs + 1} \]
      has 1 poles (in LHP) and no zeros \(\Rightarrow\) **STABLE**
    - \[ \frac{Y(s)}{U(s)} = \frac{1}{ms^2 + ds + k} \]
      has 2 poles (in LHP) and no zeros \(\Rightarrow\) **STABLE**
    - \[ \frac{V_{out}(s)}{V_{in}(s)} = \frac{RCs}{RCs + 1} \]
      (high-pass filter) has 1 pole (in LHP) and 1 zero (at 0) \(\Rightarrow\) **STABLE**

\[ s = \sigma + j\omega \]

**LHP/RHP = Left/Right Half Plane**
Analysis of Dynamics (Time vs. Laplace Domains)

- **Eigenvalue is a complex number** $\lambda$ **satisfying:**
  - $(\lambda I - A)x = 0$ for some $x \neq 0$
  - $\Leftrightarrow (\lambda I - A)^{-1}$ does not exist
  - $\Leftrightarrow$ determinant $|\lambda I - A| = 0$

- **Note similarity to portion of Transfer Function:**
  
  \[
  Y(s) = \left( C(sI - A)^{-1}B + D \right)U(s)
  \]

- **In fact,**
  
  \[
  (sI - A)^{-1} = \frac{1}{|sI - A|} \text{Adj}(sI - A)
  \]

  - **where:** $|X|$ = determinant of $X$
  - $\text{Adj}(X) = \text{adjugate of } X$ (matrix of cofactors)

- **A common situation is** $D=0$, **so that the transfer function is:**

  \[
  \frac{1}{sI - A} \left( C\text{Adj}(sI - A)B \right)
  \]

  - matrix of polynomials in $s$
  - polynomial in $s$

- **That is, the POLES of the transfer function** $=$ roots of determinant of $(sI - A)$
  $=$ **EIGENVALUES of A**
Understanding System Response – Correspondence Between Eigenvalue (Pole) Location and Time Response

\[
\dot{x}_k = \lambda_k x_k \quad \Rightarrow \quad x_k(t) = e^{\lambda_k t} x_k(0)
\]

\[\dot{x} = \lambda x \quad \Rightarrow \quad x(t) = e^{\lambda t} x(0)\]

\[\lambda = s = \sigma + j\omega\]
Understanding System Response – Frequency Response

- Recall Laplace Transform definition:
  \[ F(s) = \mathcal{L}\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t) \, dt \]

- **Restrict to jω axis** obtains Fourier Transform if \( f(t < 0) = 0 \):
  \[ F(j\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} e^{-j\omega t} f(t) \, dt \]

- For a system with transfer function \( \frac{Y(s)}{U(s)} \),
  \[ \frac{Y(j\omega)}{U(j\omega)} = \frac{|Y(j\omega)|e^{j\text{phase}(Y(j\omega))}}{|U(j\omega)|e^{j\text{phase}(U(j\omega))}} \]

- System Gain is defined to be \( |\frac{Y(j\omega)}{U(j\omega)}| \)

- **System Delays:** Two types:
  - Phase lag = frequency dependent time delay
    \[ \text{lag} = \text{phase}(Y(j\omega)) - \text{phase}(U(j\omega)) \]
    low frequency = small delay, high frequency = large delay
  - Pure delay = frequency independent time delay
Understanding System Response – Bode Plots of Frequency Response

- **Examples:**
  
  \[
  \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{RCs + 1}
  \]
  
  **Lowpass filter**

  \[
  \frac{V_{out}(s)}{V_{in}(s)} = \frac{RCs}{RCs + 1}
  \]
  
  **Highpass filter**

  - **Gain:** \( s=0 \) = 1
  - **Gain "roll-off"**
  - **Phase lag**
  - **Phase lead**

  **NOTE:** Bode gain plot is ratio of powers \((20\log_{10}(\text{amplitude ratio}))\).
Objectives of Control – Tracking and Regulation

- **Control plasma major radius:**
  - Assume plasma current ($I_p$) is positive
  - Radial hoop force $F_R$ pushes plasma outward
  - Vertical field ($B_z$) produced by outer coils holds it in desired location (regulation) ...
  - ... or moves plasma in/out to match a time-dependent request (tracking)

**tokamak positive current sign convention (viewed from above)**
Objectives of Control – Tracking and Regulation

• **Control plasma elongation:**
  - Increasing elongation (κ) has been shown to improve performance, so we want to control:
    \[ \kappa = \frac{b}{a} \]
  - Control accomplished by "pulling" on top and bottom of plasma
  - However, elongating plasma introduces destabilizing field curvature (explained in a moment)
Objectives of Control – Tracking and Regulation

• Derivation of Closed-Loop Transfer Function:

\[
p(s) = G(s)K(s)(r(s) - p(s))
\]

\[
(I + G(s)K(s))p(s) = G(s)K(s)r(s)
\]

\[
\Rightarrow \frac{p(s)}{r(s)} = \frac{G(s)K(s)}{(I + G(s)K(s))}
\]

• What we want:

Open-Loop Transfer Function \( G(s)K(s) \)

Closed-Loop Transfer Function \( \frac{G(s)K(s)}{(I + G(s)K(s))} \)

~1 in control band

~0 at high frequencies

large at low frequencies

small at high frequencies

26.0
Objectives of Control - Stabilization

- **Open-loop instability:**
  - Plasma vertical instability (caused by destabilizing curvature):

  Anti-symmetric coils provide radial field to apply force that opposes plasma vertical motion.
Objectives of Control – Avoid Closed Loop Instability

- Gain cannot be considered independently from phase.
- If gain > 1 ....
- ... when phase = -180 (opposite sign)
- => positive-feedback at that frequency and result is control-driven instability...

... and closed-loop transfer function has pair of poles in RHP
Objectives of Control – Closed Loop Stability

• Need to consider both gain AND phase:

- Large at low frequencies
- Small at high frequencies

Pay attention to stability (phase) in the middle

• Need gain << 1 for phase = -180°
• Need phase lag << 180° for gain > 1

(Gain/Phase margins are one example of stability margins.)
Objectives of Control – Disturbance & Noise Rejection

- Disturbance rejection means ratio of norms of errors to input is small:
  \[ \frac{\| e(s) \|}{\| d_V(s) \|} \ll 1, \quad \frac{\| e(s) \|}{\| d_B(s) \|} \ll 1 \]
  (attenuate effect of disturbances)

- Noise rejection means ratio of norms of errors to input noise is small:
  \[ \frac{\| e(s) \|}{\| n(s) \|} \ll 1 \]
  (attenuate effect of noise)

- These are ensured by making norms of transfer functions small, e.g.:
  \[ \frac{\| e(s) \|}{\| d_V(s) \|} \leq \left\| - (I + CTPK)^{-1} CT \right\| \ll 1 \]

- For example, large gains in controller K can make this small.
Performance Requirements – Time Domain

- **Typical Specifications on Step Response:**
  - Rise Time < X seconds
  - Percent Overshoot < Y %
  - Settling Time < Z seconds (within ε %)
Consider plant used in Bode plots:

- Root Locus diagram shows stability changes with K:
  - Open-loop stable plant
  - Stable closed loop, K=10
  - Unstable closed loop, K=200

Performance Requirements - Stability

\[ r(s) \rightarrow \text{error} \rightarrow K \rightarrow \text{commands} \rightarrow \frac{2e8}{s^3 + 2100 s^2 + 2.2e6 s + 2e8} \rightarrow p(s) \]

Root Locus Diagram (in s-plane)

Increasing gain K => eventually unstable

K=10 poles => stable closed loop
Modern plasma control mixes discrete- and continuous-time systems:

**Approach (1) to Control Design:**
- Treat entire system as continuous time. Develop continuous controller $K(s)$, then convert to discrete controller $K(z)$.
- Issues: Close to original physics models, but sampling rate must be fast enough to justify treating discrete controller as continuous.

**Approach (2) to Control Design:**
- Treat entire system as discrete and develop discrete controller directly. (Methods exist to convert mixed continuous/discrete to all discrete system.)
- Issues: Direct production of discrete controller with given sample rate, but difficult to retain physical intuition.
System Representation – Discrete Time Systems

- Time now represented by integers \( k=1,2,... \) (i.e., time = sample number)

- State-Space models are difference equations:
  \[
  x(k+1) = Ax(k) + Bu(k) \\
  y(k) = Cx(k) + Du(k)
  \]

- Now we have Z-transform instead of Laplace transform
  \[
  F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)z^{-k}
  \]

- Nice properties:
  - Transfer functions now defined on "z"-plane:
    \[
    zX(z) = AX(z) + BU(z) \\
    Y(z) = CX(z) + DU(z) \\
    \Rightarrow Y(z) = C\left((zI - A)^{-1}BU(z)\right) + DU(z)
    \]
Controllers – Example Digital Implementations

• Simple gain multiplier:
  - Command signal \( u(k) = K \times e(k) \) \hspace{1em} (error \( e(k) = r(k) - y(k) \))
  - \( K \) can be scalar (SISO) or matrix (MIMO)

• Digital filter (SISO):
  - Either SISO or MIMO:
    - Output computed from present error and previous state
    - Controller state is updated at each time step
  \[
  u(k) = a_1 u(k-1) + \ldots + a_n u(k-n) + b_0 e(k) + b_1 e(k-1) + \ldots + b_m e(k-m)
  \]
  \[
  \Rightarrow \quad \frac{U(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}}{1 - a_1 z^{-1} - \ldots - a_n z^{-n}}
  \]

• State Space:
  - Either SISO or MIMO:
    - Output computed from present error and previous state
    - Controller state is updated at each time step
  \[
  u(k) = C_c x_c (k-1) + D_c e(k)
  \]
  \[
  x_c (k) = A_c x_c (k-1) + B_c e(k)
  \]
Next – some examples of types of controllers

- **Why different controller types?**
  - Simple versus difficult to use
  - SISO versus MIMO system
  - Highly coupled versus mostly diagonal system
  - How problem is posed (what you "care about")
  - Noise characteristics of system
  - Disturbance sources/effects and characteristics
  - Level of knowledge of system dynamics (model uncertainty)
  - Guaranteed stability including uncertainty versus nominal stability (not accounting for uncertainty)
  - Guaranteed performance including uncertainty versus nominal performance (not accounting for uncertainty)
**Controller Types – PID controllers**

- **PID = Proportional, Derivative, Integral feedback**
  - Ideal: \( u(t) = K_P e(t) + K_D \dot{e}(t) + K_I \int e(t) \, dt \)
  - \( e(t) \) = error signal, \( u(t) \) = command to control actuator

- **Simple and often all that is needed (DO NOT confuse "often" with "always")**

- **Purpose of each term:**
  - \( K_P \): Tracking \((K_P G/(1+K_P G) \sim 1 \text{ over control bandwidth})\)
  - \( K_I \): Regulation \((\text{gain is infinite at } j\omega=0 \Rightarrow \text{steady-state error } = 0)\)
  - \( K_D \): Damping, phase lead

- **Issues:**
  - \( K_P \): can destabilize if too large \( \) (implemented as simple gain multiplier)
  - \( K_I \): integrator windup \( \) (implemented as digital filter)
  - \( K_D \): amplifies noise at high frequencies \( \) (implemented as digital filter)

- **Advantage:**
  - Simple, tunable

- **Disadvantage**
  - Difficult to determine gains in highly coupled systems
Controller Types – LQG controllers

• **LQG= Linear, Quadratic, Gaussian ("optimal control")**
  – Assume the **linear** system has **Gaussian** noise \( v(t), w(t) \):
    \[
    \begin{align*}
    \dot{x}(t) &= Ax(t) + Bu(t) + v(t) \\
    y(t) &= Cx(t) + w(t)
    \end{align*}
    \]
  – Minimize objective functional \( J \) ...
    \[
    J = \int_{0}^{\infty} x(t)^T Q x(t) + u(t)^T R u(t) dt
    \]
  – ... where \( Q>0, R>0 \) (**quadratic** cost)
  – Typically, states \( x \) are variations around a **stable** equilibrium \( x_0 \)
  – Sometimes \( J \) has terms for output \( y \) or error \( e = \) reference - output

• **Main idea: keep signals small "on average" (variation due to noise)**

• **Optimal controller is given by:**
  \[
  \begin{align*}
  \dot{x}(t) &= A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t)) \\
  u(t) &= -L\hat{x}(t)
  \end{align*}
  \]
  – First equation is the **Kalman Filter**, which provides an **optimal** estimate for \( x \)
  – If state measured directly, insert \( x \) in place of \( \hat{x} \) and use 2\(^{nd} \) equation only

• **Advantage:**
  – Straightforward to generate controller optimal against "noise", once \( J \) is defined

• **Disadvantage**
  – Matrices \( Q \) and \( R \) typically determined through trial and error
Controller Types – H-infinity ("robust") controllers

- \( H^\infty \) = method for synthesizing robust controllers ("Hardy space, infinity norm")
- Robust = guaranteed stability/performance with unknown (but bounded) uncertainty in plant model
  - Infinity ("worst case") norm: \( \| \Delta \|_\infty < \text{bound} \)
- Main idea
  - Remove \( \Delta \) from picture ...
  - ... and make transfer function from \( \Delta_{\text{out}} \) to \( \Delta_{\text{in}} \) as small as possible
- Advantage:
  - Guarantees on stability and performance in the deployed feedback system
- Disadvantage:
  - More difficult to understand and to use; some tools produce conservative designs
Summary

• **Control Terminology and Concepts:**
  – Linear/Nonlinear systems, Linear-Time-Invariant system, Discrete time system, System gain/phase, s-plane, z-plane, poles, zeros, pure delay, phase lag, phase lead, SISO, MIMO, feedforward, feedback, open-loop instability, control-driven instability, LHP, RHP, frequency response, roll-off, gain margin, phase margin, stability margin, disturbance, overshoot, rise time, settling time

• **Control Tools and Methods:**
  – Block Diagrams, Transfer Functions, State Space Models, Laplace Transform, Z-Transform, Fourier Transform, Bode plot, derivation of closed-loop transfer function, Root Locus, PID controllers, LQG controllers, H-infinity controllers

• **Multiple Objectives of Control:**
  – Stability,
  – Tracking and Regulation
  – Disturbance Rejection
  – Noise Rejection
  – Robustness
Further Reading

• Free downloadable books:
    – Control System Toolbox, Robust Control Toolbox

• Good entry-level control books:
  – Franklin, Powell, Emami-Naeini, Feedback Control of Dynamic Systems
  – Friedland, Control System Design: An Introduction to State-Space Methods