Fundamentals of Magnetic Island Theory in Tokamaks

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Macroscopic Instabilities

• Two main types of macroscopic instabilities in tokamaks: 
  
  – Catastrophic “ideal” (i.e., non-reconnecting) instabilities, which disrupt plasma in few micro-seconds. Can be avoided by limiting plasma pressure and current.
  – Slowly growing “tearing” instabilities, which reconnect magnetic flux-surfaces to form magnetic islands, thereby degrading their confinement properties. Much harder to avoid.

\[\text{\textsuperscript{a}MHD Instabilities, G. Bateman (MIT, 1978).}\]
Magnetic Islands

- Helical structures, centered on rational magnetic flux-surfaces which satisfy $\vec{k} \cdot \vec{B} = 0$, where $\vec{k}$ is wavenumber of mode, and $\vec{B}$ is equilibrium magnetic field.

- Effectively “short-circuit” confinement by allowing heat/particles to radially transit island region by rapidly flowing along magnetic field-lines, rather than slowly diffusing across flux-surfaces.
Need for Magnetic Island Theory

- Magnetic island formation associated with *nonlinear* phase of tearing mode growth (*i.e.*, when radial island width becomes greater than linear layer width at rational surface).
- In very hot plasmas found in modern-day tokamaks, linear layers so thin that tearing mode already in nonlinear regime when first detected.
- Linear tearing mode theory largely irrelevant. Require nonlinear magnetic island theory to explain experimental observations.
MHD Theory

- Tearing modes are macroscopic instabilities which affect whole plasma. Natural to investigate them using some form of fluid-theory.
- Simplest fluid theory is well-known magnetohydrodynamical approximation, which effectively treats plasma as single-fluid.
- Shall also use slab approximation to simplify analysis.

\(^a\) Plasma Confinement, R.D. Hazeltine, and J.D. Meiss (Dover, 2003).
Slab Approximation

rational surface

"toroidal" $z$

"poloidal" $y$

$x = 0$

$d/dz = 0$

perfectly conducting wall

periodic in $y$–dirn.
Slab Model

- Cartesian coordinates: \((x, y, z)\). Let \(\partial/\partial z \equiv 0\).
- Assume presence of dominant uniform “toroidal” \(\vec{B}_z \vec{Z}\).
- All field-strengths normalized to \(B_z\).
- All lengths normalized to equilibrium magnetic shear-length:
  \[L_s = B_z/B'_y(0)\].
- All times normalized to shear-Alfvén time calculated with \(B_z\).
- Perfect wall boundary conditions at \(x = \pm a\).
- Wavenumber of tearing instability: \(\vec{k} = (0, k, 0)\), so \(\vec{k} \cdot \vec{B} = 0\) at \(x = 0\). Hence, rational surface at \(x = 0\).
Model MHD equations

- Let $\vec{B}_\perp = \nabla \psi \times \vec{z}$ and $\vec{V} = \nabla \phi \times \vec{z}$, where $\vec{V}$ is $\vec{E} \times \vec{B}$ velocity.
- $\vec{B} \cdot \nabla \psi = \vec{V} \cdot \nabla \phi = 0$, so $\psi$ maps magnetic flux-surfaces, and $\phi$ maps stream-lines of $\vec{E} \times \vec{B}$ fluid.
- Incompressible MHD equations:\(^a\)
  \[
  \frac{\partial \psi}{\partial t} = [\phi, \psi] + \eta J,
  \frac{\partial U}{\partial t} = [\phi, U] + [J, \psi] + \mu \nabla^2 U,
  \]
  where $J = \nabla^2 \psi$, $U = \nabla^2 \phi$, and $[A, B] = A_x B_y - A_y B_x$. Here, $\eta$ is resistivity, and $\mu$ is viscosity. In normalized units: $\eta, \mu \ll 1$.
- First equation is $z$-component of Ohm’s law. Second equation is $z$-component of curl of plasma equation of motion.

\(^a\) *Plasma Confinement*, R.D. Hazeltine, and J.D. Meiss (Dover, 2003).
Outer Region

- In “outer region”, which comprises most of plasma, non-linear, non-ideal ($\eta$ and $\mu$), and inertial ($\partial/\partial t$ and $\vec{V} \cdot \nabla$) effects negligible.

- Vorticity equation reduces to

$$[J, \psi] \simeq 0.$$ 

- When linearized, obtain $\psi(x, y) = \psi^{(0)}(x) + \psi^{(1)}(x) \cos(k y)$, where $B_y^{(0)} = -d\psi^{(0)}/dx$, and

$$\left(\frac{d^2}{dx^2} - k^2\right)\psi^{(1)} - \left(\frac{d^2 B_y^{(0)}}{B_y^{(0)}}/dx^2\right)\psi^{(1)} = 0.$$ 

- Equation is singular at rational surface, $x = 0$, where $B_y^{(0)} = 0$. 

Tearing Stability Index

• Find tearing eigenfunction, $\psi^{(1)}(x)$, which is continuous, has tearing parity $[\psi^{(1)}(-x) = \psi^{(1)}(x)]$, and satisfies boundary condition $\psi^{(1)}(a) = 0$ at conducting wall.

• In general, eigenfunction has gradient discontinuity across rational surface (at $x = 0$). Allowed because tearing mode equation singular at rational surface.

• Tearing stability index:

$$\Delta' = \left[ \frac{d \ln \psi^{(1)}}{dx} \right]_{0-}^{0+}.$$  

• According to conventional MHD theory, a tearing mode is unstable if $\Delta' > 0$.

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A rational surface is depicted, showing the behavior of an eigenfunction near critical points. The diagram illustrates the tearing effect at the point $x = 0$, where the function transitions from $x = -a$ to $x = +a$. The vertical line at $x = 0$ indicates a critical point where the eigenfunction experiences a tearing effect. The horizontal axis is labeled with $x = -a$, $x = 0$, and $x = +a$, representing the range of $x$ values for this study.
**Inner Region**

- “Inner region” centered on rational surface, $x = 0$. Of extent, $W \ll 1$, where $W$ is magnetic island width (in $x$).

- In inner region, non-ideal effects, non-linear effects, and plasma inertia can all be important.

- Inner solution must be asymptotically matched to outer solution already obtained.
Constant-$\psi$ Approximation

- $\psi^{(1)}(x)$ generally does not vary significantly in $x$ over inner region:
  \[ |\psi^{(1)}(W) - \psi^{(1)}(0)| \ll |\psi^1(0)|. \]

- Constant-$\psi$ approximation: treat $\psi^{(1)}(x)$ as constant in $x$ over inner region.

- Approximation valid provided
  \[ |\Delta'| W \ll 1, \]
  which is easily satisfied for conventional tearing modes.
Constant-$\psi$ Magnetic Island

- In vicinity of rational surface, $\psi^{(0)} \to -x^2/2$, so

  $$\psi(x, y, t) \simeq -x^2/2 + \Psi(t) \cos \theta,$$

  where $\Psi = \psi^{(1)}(0)$ is “reconnected flux”, and $\theta = ky$.

- Full island width, $W = 4 \sqrt{\Psi}$.

\[\begin{align*}
\text{x-axis:} & \quad y = 0 \quad \text{and} \quad x = 0 \\
\text{X-point:} & \quad ky = 0 \\
\text{O-point:} & \quad ky = \pi \\
\text{separatrix:} & \quad ky = 2\pi \\
\text{W:} & \quad \psi = +\Psi \\
\text{X-point:} & \quad \psi = -\Psi
\end{align*}\]
Flux-Surface Average Operator

- Flux-surface average operator is annihilator of Poisson bracket 
  \[ [A, \psi] \equiv \vec{B} \cdot \nabla A \equiv k x \frac{\partial A}{\partial \theta} \psi \] for any \( A \): i.e., 
  \[ \langle [A, \psi] \rangle \equiv 0. \]

- Outside separatrix:
  \[ \langle f(\psi, \theta) \rangle = \oint \frac{f(\psi, \theta)}{|x|} \frac{d\theta}{2\pi}. \]

- Inside separatrix:
  \[ \langle f(s, \psi, \theta) \rangle = \int_{-\theta_0}^{\theta_0} \frac{f(s, \psi, \theta) + f(-s, \psi, \theta)}{2|x|} \frac{d\theta}{2\pi}, \]

  where \( s = \text{sgn}(x) \), and \( x(s, \psi, \theta_0) = 0. \)
MHD Flow -1

- Move to island frame. Look for steady-state solution: \( \partial / \partial t = 0 \).\(^a\)
- Ohm’s law:

\[
0 \simeq [\phi, \psi] + \eta J.
\]

- Since \( \eta \ll 1 \), first term potentially much larger than second.
- To lowest order:

\[
[\phi, \psi] \simeq 0.
\]

- Follows that

\[
\phi = \phi(\psi): \quad i.e., \text{MHD flow constrained to be around flux-surfaces.}
\]

MHD Flow - II

• Let

\[ M(\psi) = \frac{d\phi}{d\psi}. \]

• Easily shown that

\[ V_y = x M. \]

• By symmetry, \( M(\psi) \) is *odd* function of \( x \). Hence,

\[ M = 0 \]

inside separatrix: *i.e.*, no flow inside separatrix in island frame. Plasma *trapped* within magnetic separatrix.
MHD Flow - III

• Vorticity equation:

\[ 0 \simeq [-M \mathbf{U} + J, \psi] + \mu \nabla^4 \phi. \]

• Flux-surface average, recalling that \( \langle [A, \psi] \rangle = 0 \):

\[ \langle \nabla^4 \phi \rangle \equiv -\frac{d^2}{d\psi^2} \left( \langle x^4 \rangle \frac{dM}{d\psi} \right) \simeq 0. \]

• Solution outside separatrix:

\[ M(\psi) = \text{sgn}(x) M_0 \int_{-\psi}^{\psi} d\psi/\langle x^4 \rangle \bigg/ \int_{-\psi}^{-\infty} d\psi/\langle x^4 \rangle. \]
MHD Flow - IV

• Note that

\[ V_y = x M \rightarrow |x| M_0 \]

as \(|x|/W \rightarrow \infty\).

• V-shaped velocity profile which extends over whole plasma.

• Expect *isolated* magnetic island to have *localized* velocity profile. Suggests that \( M_0 = 0 \) for isolated island.

• Hence, zero MHD flow in island frame: *i.e.*, island propagates at local \( \vec{E} \times \vec{B} \) velocity.
\[ x = -a \quad x = 0 \quad x = +a \]

\[ V_{ExB} - V \]

\[ \text{rational surface} \]

\[ \text{unlocalized profile} \]

\[ \text{localized profile} \]
Rutherford Equation - I

• Asymptotic matching between inner and outer regions yields:

\[ \Delta' \Psi = -4 \int_{+\Psi}^{-\infty} \langle J \cos \theta \rangle \, d\psi. \]

• In island frame, in absence of MHD flow, vorticity equation reduces to

\[ [J, \psi] \simeq 0. \]

• Hence,

\[ J = J(\psi). \]
Rutherford Equation - II

• Ohm’s law:
  \[ \frac{d\Psi}{dt} \cos \theta \simeq [\phi, \psi] + \eta \, J(\psi). \]

• Have shown there is no MHD-flow \([i.e., \phi \sim O(1)]\), but can still be resistive flow \([i.e., \phi \sim O(\eta)]\).

• Eliminate resistive flow by flux-surface averaging:
  \[ \frac{d\Psi}{dt} \langle \cos \theta \rangle \simeq \eta \, J(\psi) \langle 1 \rangle. \]

• Hence,
  \[ \Delta' \psi \simeq -\frac{4}{\eta} \frac{d\Psi}{dt} \int_{+\psi}^{-\infty} \frac{\langle \cos \theta \rangle^2}{\langle 1 \rangle} \, d\psi. \]
Rutherford Equation - III

- Use $W = 4 \sqrt{\Psi}$, and evaluate integral. Obtain *Rutherford island width evolution equation*: \(^a\)

$$\frac{0.823 \ dW}{\eta \ dt} \simeq \Delta'.$$

- According to Rutherford equation, island grows *algebraically on resistive time-scale*.

- Rutherford equation does not predict island saturation.

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Rutherford Equation - IV

• Higher order asymptotic matching between inner and outer regions yields:

\[
\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' - 0.41 \left( -\frac{d^4 B_{y}^{(0)}}{dx^4} \right)_{x=0} \frac{d^2 B_{y}^{(0)}}{dx^2}
\]

• Hence, saturated \((d/dt = 0)\) island width is

\[
W_0 = \frac{\Delta'}{0.41} \left( -\frac{d^2 B_{y}^{(0)}}{dx^2} \right)_{x=0} \frac{d^4 B_{y}^{(0)}}{dx^4}
\]

MHD Theory: Summary

- Tearing mode unstable if $\Delta' > 0$.
- Island propagates at local $\vec{E} \times \vec{B}$ velocity at rational surface.
- Island grows algebraically on resistive time-scale.
- Saturated island width:

$$W_0 = \frac{\Delta'}{0.41} \left( - \frac{d^2 B_y^{(0)}}{dx^2} \frac{d^4 B_y^{(0)}}{dx^4} \right)_{x=0}. $$
Drift-MHD Theory

- In drift-MHD approximation, analysis retains charged particle drift velocities, in addition to $\vec{E} \times \vec{B}$ velocity.
- Essentially two-fluid theory of plasma.
- Characteristic length-scale, $\rho$, is ion Larmor radius calculated with electron temperature.
- Characteristic velocity is diamagnetic velocity, $V_*$, where

$$n_e \vec{V}_* \times \vec{B} = \nabla P.$$ 

- Normalize all lengths to $\rho$, and all velocities to $V_*$. 
Basic Assumptions

- Retain slab model, for sake of simplicity.
- Assume parallel electron heat transport sufficiently strong that $T_e = T_e(\psi)$.
- Assume $T_i/T_e = \tau = \text{constant}$, for sake of simplicity.
Basic Definitions

• Variables:
  – $\psi$ - magnetic flux-function.
  – $J$ - parallel current.
  – $\phi$ - guiding-center (i.e., MHD) stream-function.
  – $U$ - parallel ion vorticity.
  – $n$ - electron number density (minus uniform background).
  – $V_z$ - parallel ion velocity.

• Parameters:
  – $\alpha = (L_n/L_s)^2$, where $L_n$ is equilibrium density gradient scale-length.
  – $\eta$ - resistivity. $D$ - (perpendicular) particle diffusivity. $\mu_{i/e}$ - (perpendicular) ion/electron viscosity.
**Drift-MHD Equations - I**

- Steady-state drift-MHD equations:\(^a\)

\[
\begin{align*}
\psi &= -\chi^2/2 + \Psi \cos \theta, \quad U = \nabla^2 \phi, \\
0 &= [\phi - n, \psi] + \eta J, \\
0 &= [\phi, U] - \frac{\tau}{2} \left\{ \nabla^2[\phi, n] + [U, n] + [\nabla^2 n, \phi] \right\} \\
&\quad + [J, \psi] + \mu_i \nabla^4 (\phi + \tau n) + \mu_e \nabla^4 (\phi - n), \\
0 &= [\phi, n] + [V_z + J, \psi] + D \nabla^2 n, \\
0 &= [\phi, V_z] + \alpha [n, \psi] + \mu_i \nabla^2 V_z. 
\end{align*}
\]

Drift-MHD Equations - II

• Symmetry: $\psi, J, V_z$ even in $x$. $\phi, n, U$ odd in $x$.

• Boundary conditions as $|x|/W \to \infty$:
  - $n \to -(1 + \tau)^{-1} x$.
  - $\phi \to -V x$.
  - $J, U, V_z \to 0$.

• Here, $V$ is island phase-velocity in $\vec{E} \times \vec{B}$ frame.

• $V = 1$ corresponds to island propagating with electron fluid.
  $V = -\tau$ corresponds to island propagating with ion fluid.

• Expect

\[ 1 \gg \alpha \gg \eta, D, \mu_i, \mu_e. \]
Electron Fluid

- Ohm’s law:

\[ 0 = [\phi - n, \psi] + \eta J. \]

- Since \( \eta \ll 1 \), first term potentially much larger than second.

- To lowest order:

\[ [\phi - n, \psi] \simeq 0. \]

- Follows that

\[ n = \phi + H(\psi) : \]

\( i.e., \) electron stream-function \( \phi_e = \phi - n \) is \textit{flux-surface function}. Electron fluid flow constrained to be around flux-surfaces.
Sound Waves

• Parallel flow equation:

\[ 0 = [\phi, V_z] + \alpha [n, \psi] + \mu_i \nabla^2 V_z. \]

• Highlighted term dominant provided

\[ W \gg \alpha^{-1/2} = L_s/L_n. \]

• If this is case then to lowest order

\[ n = n(\psi), \]

which implies \( n = 0 \) inside separatrix.

• So, if island sufficiently wide, sound-waves able to flatten density profile inside island separatrix.
Subsonic vs. Supersonic Islands

- Wide islands satisfying
  \[ W \gg \frac{L_s}{L_n} \]
  termed *subsonic* islands. Expect such islands to exhibit flattened density profile within separatrix. Subsonic islands strongly coupled to both electron and ion fluids.

- Narrow islands satisfying
  \[ W \ll \frac{L_s}{L_n} \]
  termed *supersonic* islands. No flattening of density profile within separatrix. Supersonic islands strongly coupled to electron fluid, but only weakly coupled to ion fluid.
Subsonic Islands\textsuperscript{a}

- To lowest order:
  \[ \phi = \phi(\psi), \quad n = n(\psi). \]

- Follows that both electron stream-function, \( \phi_e = \phi - n \), and ion stream-function, \( \phi_i = \phi + \tau n \), are flux-surface functions. Both electron and ion fluid flow constrained to follow flux-surfaces.

- Let
  \[ M(\psi) = d\phi/d\psi, \quad L(\psi) = dn/d\psi. \]

- Follows that
  \[ V_{E \times B}_y = xM, \quad V_{e y} = x(M - L), \quad V_{i y} = x(M + \tau L). \]

Density Flattening

- By symmetry, both $M(\psi)$ and $L(\psi)$ are odd functions of $x$. Hence,

$$M(\psi) = L(\psi) = 0$$

inside separatrix: i.e., no electron/ion flow within separatrix in island frame.

- Electron/ion fluids constrained to propagate with island inside separatrix.

- Density profile \textit{flattened} within separatrix.
**Analysis - I**

- Density equation reduces to
  \[ 0 \simeq [V_z + J, \psi] + D \nabla^2 n. \]

- Vorticity equation reduces to
  \[ 0 \simeq [ -M U - (\tau/2)(L U + M \nabla^2 n) + J, \psi ] + \mu_i \nabla^4 (\phi + \tau n) + \mu_e \nabla^4 (\phi - n). \]

- Flux-surface average both equations, recalling that \( \left\langle [A, \psi] \right\rangle = 0. \)
Analysis - II

- Obtain

\[
\langle \nabla^2 n \rangle \simeq 0,
\]

and

\[
(\mu_i + \mu_e) \langle \nabla^4 \phi \rangle + (\mu_i \tau - \mu_e) \langle \nabla^4 n \rangle \simeq 0.
\]

- Solution outside separatrix:

\[
M(\psi) = -\frac{(\mu_i \tau - \mu_e)}{\left(\mu_i + \mu_e\right)} L(\psi) + F(\psi),
\]

where

\[
L(\psi) = -\text{sgn}(x) L_0 / \langle x^2 \rangle,
\]

and \(F(\psi)\) is previously obtained MHD profile:

\[
F(\psi) = \text{sgn}(x) F_0 \int_{-\psi}^{\psi} \text{d}\psi / \langle x^4 \rangle \bigg/ \int_{-\psi}^{\psi} \text{d}\psi / \langle x^4 \rangle.
\]
Velocity Profiles

• As $|x|/W \to \infty$ then $xL \to L_0$ and $xF \to |x|F_0$.

• $L(\psi)$ corresponds to *localized* velocity profile. $F(\psi)$ corresponds to *non-localized* profile. Require localized profile, so $F_0 = 0$.

• Velocity profiles outside separatrix (using b.c. on $n$):

\[
V_{yi} \simeq + \frac{\mu_e}{\mu_i + \mu_e} \frac{|x|}{\langle x^2 \rangle},
\]

\[
V_{yE \times B} \simeq - \frac{\left( \mu_i \tau - \mu_e \right)}{(1 + \tau)(\mu_i + \mu_e)} \frac{|x|}{\langle x^2 \rangle},
\]

\[
V_{ye} = - \frac{\mu_i}{\mu_i + \mu_e} \frac{|x|}{\langle x^2 \rangle}.
\]
\[ x = -a \quad x = 0 \quad x = +a \]

The figure illustrates a rational surface with an electric field \( E \times B \) and potentials \( V_y \) and \( V \). The diagram shows the behavior of electrons and ions as they move through the field, with the vertical line at \( x = 0 \) representing the central axis. The distance \( x \rightarrow \) to \( x = 0 \) is labeled with a vertical arrow and the value \( 1 \).
Island Propagation

• As $|\chi|/W \to \infty$ expect $V_y E \times B \to V_{EB} - V$, where $V_{EB}$ is unperturbed (i.e., no island) $E \times B$ velocity at rational surface (in lab. frame), and $V$ is island phase-velocity (in lab. frame).

• Hence

$$V = V_{EB} + \frac{\mu_i \tau - \mu_e}{(1 + \tau)(\mu_i + \mu_e)}.$$

• But unperturbed ion/electron fluid velocities (in lab. frame):

$$V_i = V_{EB} + \tau/(1 + \tau), \quad V_e = V_{EB} - 1/(1 + \tau).$$

• Hence

$$V = \frac{\mu_i}{\mu_i + \mu_e} V_i + \frac{\mu_e}{\mu_i + \mu_e} V_e.$$

So, island phase-velocity is *viscosity weighted average* of unperturbed ion/electron fluid velocities.
Polarization Term - I

- Vorticity equation yields
  \[ J_c \approx \frac{1}{2} \left( x^2 - \frac{\langle x^2 \rangle}{\langle 1 \rangle} \right) \frac{d[M(M + \tau L)]}{d\psi} + I(\psi) \]

  outside separatrix, where \( J_c \) is part of \( J \) with \( \cos \theta \) symmetry.

- As before, flux-surface average of Ohm’s law yields:
  \[ \langle J_c \rangle = I(\psi) \langle 1 \rangle = \eta^{-1} \frac{d\Psi}{dt} \langle \cos \theta \rangle. \]

- Hence
  \[ J_c \approx \frac{1}{2} \left( x^2 - \frac{\langle x^2 \rangle}{\langle 1 \rangle} \right) \frac{d[M(M + \tau L)]}{d\psi} + \eta^{-1} \frac{d\Psi}{dt} \frac{\langle \cos \theta \rangle}{\langle 1 \rangle}. \]
Polarization Term - II

• Asymptotic matching between inner and outer regions yields:

\[ \Delta' \Psi = -4 \int_{-\infty}^{+\Psi} \langle J_c \cos \theta \rangle \, d\psi. \]

• Evaluating flux-surface integrals, making use of previous solutions for \( M \) and \( L \), obtain modified Rutherford equation:

\[ \frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' + 1.38 \beta \frac{(V - V_{EB})(V - V_i)}{(W/4)^3}. \]

• New term is due to polarization current associated with ion fluid flow around curved island flux-surfaces (in island frame). Obviously, new term is zero if island propagates with ion fluid: \( i.e., V = V_i. \)
Drift-MHD Theory: Summary

- Results limited to large islands: \textit{i.e.}, large enough for sound waves to flatten density profile.

- Island propagates at (perpendicular) viscosity weighted average of unperturbed (no island) ion and electron fluid velocities.

- Bootstrap term in Rutherford equation is \textit{destabilizing}.

- Polarization term in Rutherford equation is \textit{stabilizing} provided ion (perpendicular) viscosity greatly exceeds electron (perpendicular) viscosity (which is what we expect), and destabilizing otherwise.