Statistical Properties of the Drift Wave Fluctuations

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Abstract

The nature of turbulence in macroscopically confined plasmas is reviewed and contrasted to turbulence in hydrodynamics. The statistical properties of the fluctuations are analyzed both from the Gibb's distribution for the soliton gas model of the electrostatic field and for the Hamiltonian field theory equilibrium statistics. For that purpose, nonlinear drift wave equations are derived from two-fluid theory. From the reduced nonlinear drift wave equations we construct continuous plasma models with simple Hamiltonians, which allow canonical distributions to be defined explicitly. Partition functions and correlation functions can be calculated analytically in the one-dimensional case as functional integral averages over canonical distributions. The relation of the k-space fluctuation spectrum obtained from canonical distributions with those inferred from the electromagnetic scattering experiments is given. The open problem of saturation levels of fluctuations is discussed in the conclusions.

I. Introduction

Macroscopically confined plasmas are not quiescent in the small and display a complex structure of fluctuations in space and time. These fluctuations are the result of small-scale instabilities which saturate due to the effects of sources, nonlinearities and dissipation. The picture is somewhat similar to turbulence in fluids at high Reynolds numbers. It is important to understand turbulence in order to be able to explain observations in, for example, astrophysics and improve engineering designs for fusion plasmas.

Anomalous heat conduction observed in fusion plasma devices limits the energy confinement time. This effect has to be compensated by increasing the size of the plasma and the device itself. This calls for extensive outlays and severely reduces the economic attractiveness of fusion reactors. Anomalous conduction is caused by the low-frequency fluctuations mentioned above, which are a manifestation of drift wave turbulence. This means that understanding instabilities and turbulence is a crucial question for fusion as an energy source.

There are two basic difficulties in understanding turbulence quantitatively. First, the dynamics is highly nonlinear. Second, the statistics are not at equilibrium as in thermodynamics. More precisely, complex nonlinear and dissipative dynamic systems have, usually, several complex attractors. The statistics on such attractors is not an equilibrium statistics, unless special conditions are met. An instructive example of complex but analytically tractable dynamics is given by a large system of van der Pol-like oscillators interacting through coupling matrices.[1] The location of attractors in a finite region of phase space can be established, the attractors themselves being inaccessible to analysis. When the finite region shrinks to a hypersurface containing the attractor, an equilibrium statistics is then possible. At the same time the Liouville theorem in phase space becomes valid and compensation of driving and damping becomes local.

Usually, turbulence in hydrodynamics is investigated for the case of well-separated sources and sinks. The sources are active at long wavelengths or low k-vectors, and the sinks are effective at large k. The main interest is then focussed on the inertial range, which lies in k-space between the sources and the sinks. This is justified by the fact that, usually, either the boundaries or large-scale instabilities are the cause of sources, and the viscosity is responsible for the sinks. The inertial nonlinearity causes a cascade of energy from the low k to the large k.

In macroscopically confined plasmas the situation is quite different. The sources are due to small-scale instabilities perpendicular to the magnetic field, but large scales along the magnetic field, and the sinks are due mainly to the "shear damping" which occurs at low k-vectors parallel to the magnetic field. The role of nonlinearities lies in the redistribution of the modes. It is then not too bad an approach to consider the driving and the damping as canceling each other locally, and to try to look for continuous, conservative plasma models and their equilibrium statistics. The paper is structured as follows: In Sec. II, nonlinear drift wave models are analyzed in terms of the soliton gas. Section III develops a continuous plasma model, and Section IV deals with the construction of Hamiltonians for those models. Correlation functions and spectra are the subject of Sec. V. The conclusions are presented in Sec. VI.

II. The One-Dimensional Drift Wave Soliton Gas Model

A. Physics of Drift Waves

Drift waves are low frequency ($\omega \ll eB/m_i c$) collective oscillations of a magnetized, inhomogeneous plasma resulting from $\mathbf{E} \times \mathbf{B}$ motions and parallel electron currents $j_\parallel^e$. Here we briefly outline the physics of the drift waves and give in Sec. III a guide to the derivation from the two-component (ion and electron) hydrodynamic equations.

Consider the effect of $\mathbf{E} \times \mathbf{B}$ convection in a magnetized plasma $\mathbf{B} = B\hat{\mathbf{e}}_z$ with gradients of the density $L_n^{-1} = -\partial_x n_0(x)/n_0(x)$ and temperature $L_T^{-1} = -\partial_x T_e(x)/T_e(x)$ along the x-axis. To the lowest order the electrostatic convection $\mathbf{E} = -\nabla\phi$ is incompressible $\nabla \cdot \mathbf{v}_E = 0$; however, due to large ion inertia there is an inertial polarization current given by

$$\mathbf{j}_p = \frac{c^2 m_i n_i}{B^2} \frac{d\mathbf{E}_\perp}{dt} \tag{1}$$

that is balanced in the high density, $\epsilon_\perp = (1 + 4\pi c^2 m_i n_i/B^2) \gg 1$, plasma by a return electron current

$$\nabla_\parallel j_\parallel^e = e\left(\frac{\partial n_e}{\partial t} + \mathbf{v}_E \cdot \nabla n_e + n_e \nabla \cdot \mathbf{v}_E\right) \tag{2}$$

where

$$\mathbf{v}_\mathbf{E} = \frac{c\mathbf{E} \times \mathbf{B}}{B^2} = \frac{c\hat{\mathbf{e}}_z \times \nabla\phi}{B}. \tag{3}$$

In Eqs. (1)–(3) we use that the electric field is electrostatic due to the slow phase velocities of the waves and the low plasma-to-magnetic field pressure ratio $\beta = 8\pi p/B^2 \ll 1$. For thermal electrons with $v_e = (T_e/m_e)^{1/2}$ the drift waves have $k_\parallel v_e \gg \omega$ so that the parallel electron acceleration reduces to the quasi-static force balance $en_e E_\parallel + \nabla_\parallel p_e = -en_e\left(\nabla_\parallel \phi - \frac{T_e}{e}\,\ell n\, n_e\right) = 0$. Thus, the electrons establish the Boltzman distribution

$$n_e(\mathbf{r}, t) \cong n_0(x) \exp[e\phi(\mathbf{r}, t)/T_e(x)] \tag{4}$$

during each phase of the fluctuation. Using Eq. (4) in Eq. (2) to eliminate the fluctuating electron density and balancing the divergence of the parallel and perpendicular currents yields the Petviashvili equation,[2]

$$-\frac{c^2 m_i T_e}{e^2 B^2} \frac{d}{dt} \nabla_\perp^2 \phi + \partial_t \phi + v_d \partial_y \phi - \eta_e v_d \phi \partial_y \phi = 0. \tag{5}$$

The first term in Eq. (5) arises from $\nabla \cdot \mathbf{j}_p$ in Eq. (1) balance with the leading term in Eq. (2) $\nabla_\parallel j_\parallel^e \cong e\partial_t n_e \simeq (n_e e^2/T_e)\partial_t \phi$. This balance defines the ion inertia scale length

$\rho_s = c(m_i T_e)^{1/2}/eB$ which is a scale length associated with coherent structures in drift wave turbulence and with wave dispersion.

In the long wavelength limit $k\rho_s \ll 1$ the small amplitude solutions of Eq. (5) propagate $\phi(y,t) = \phi(y - v_d t)$ where $v_d = cT_e/eBL_n$. The Burger nonlinearity from the temperature gradient (last term) $\eta_e v_d$ in Eq. (5) causes the secular steepening of the drift wave until the wave dispersion from the first balances the nonlinearity $k^2\rho_s^2 k v_d \phi_m \simeq \eta_e k_y v_d \phi_m^2$ at which point coherent soliton-like structures are formed in the fluctuations. Here we investigate the statistical properties of this form of nonlinear turbulence.

The existence of the inverse scattering transform for solving the integrable nonlinear wave equations gives rise to a new description of collective modes of systems. In the new description, which may be called the soliton gas, the basic collective modes are the single particle-like states of the solitary waves rather than the small amplitude waves (plasmons and phonons). Due to the integrability of the KdV and cubic Schrödinger equation we know that the elementary nonlinear structures emerge from scattering events with the same form as before but carrying phase shifts from the change of their relative speed during the period of interaction. The development of the N-soliton solutions of the KdV equation are now well known.[3,4,5]

The multisoliton solutions provides an alternative theoretical framework for the interpretation of the electromagnetic scattering experiment, based on the concept of an (nearly) ideal gas of plasma solitons as the form of the fluctuations spectrum in a background spectrum of small amplitude waves. These ideas have also been studied in the context of condensed matter physics.[6] The condensed matter work shows that solitons contribute to the free energy of nonlinear lattices in equilibrium, and, furthermore, that their effects are experimentally detectable.[7]

For electron plasma waves in an *unmagnetized* plasma, the wavenumber spectrum generated by a gas of Langmuir solitons has been investigated.[8] A review by Makhankov[9] emphasizes the importance in physical systems of the non-integrable solitary wave equations—in contrast to the integrable KdV soliton equation. The point is that an exchange of energy-momentum between constituents on the long time scale is required physically and leads to the establishment of the Gibbs distribution in the long time limit. The drift wave solitary waves are non-integrable but have the robust features emphasized by Makhankov. [The drift wave Eq. (5) reduces to the integrable KdV equation when the first dispersion term is replaced by $\partial_t \partial_y^2 \phi \to -v_d \partial_y^3 \phi$, valid in the limit $\rho_s^2 \nabla_\perp^2 \to 0$]

Meiss and Horton[10,11] have developed the soliton gas description of drift wave turbulence. The electromagnetic scattering experiments[12–17] have led to the identification of the microturbulence in tokamaks with drift wave turbulence. The general features of the dynamical form factor $S(\mathbf{k}, \omega)$ for the electron density fluctuations $\langle |\delta n_e(\mathbf{k}, \omega)|^2 \rangle$ are interpreted in terms of the frequency $k_y v_{de}$, where $v_{de} = cT_e/(eBL_n)$ is the electron diamagnetic drift velocity, and the most unstable wavenumbers of drift wave theory $k_\perp \rho_s \lesssim 1$, where ρ_s is the ion inertial scale length, $\rho_s = c(m_i T_e)^{1/2}/eB$. The peak of the fluctuation spectrum occurs in the long wavelength regime ranging from $k_y \rho_s \sim 0.2$ to 0.5 depending on the experiment. The k_x spectrum is peaked at $k_x = 0$ so it is reasonable to consider models with $k_x \ll k_y$.

Efforts to make a detailed comparison of the scattering data with theory, however, have been frustrated by the fact that for a well-defined $\mathbf{k}$ and scattering volume, the distribution of the scattered power has a peak at a frequency which is two to three times larger than the *linear drift frequency* $\omega^\ell(k)$.[12−18] Weak turbulence theory introduces an assumption of weak correlations e.g., "maximal randomness" of the direct interaction approximation[19] which predicts, for moderate levels, nonlinear frequency shifts $\Delta\omega_{n\ell}$ proportional to integrals over $I(\mathbf{k}) \propto \langle |\delta n_e(\mathbf{k})|^2 \rangle$. This is sharp contrast to the coherent solitary structures with frequency shift proportional to $|\delta n_e|/n_e \sim e\phi_{\max}/T_e$

Meiss and Horton[10,11] derived the $k\omega$-fluctuation spectrum for the drift wave soliton gas assuming that soliton overlap can be neglected. They obtain the dynamical form factor, $S(\mathbf{k}, \omega)$ given in Sec. II.B, due to solitons which arise from an ensemble of initial conditions with a given mean square fluctuation level, $\langle \delta n_e^2 \rangle$, of the electron density. At this time, Horton was motivated to put aside the weak turbulence approach by the patient, but forceful, explanations of Petviashvili during the Kiev theory meeting that the drift waves could develop into solitary wave structures.

Using the same nonlinear drift wave model Tasso develops in Sec. IV the fluctuation spectrum without explicitly breaking up the field into its wave and soliton components. By using the partition function for the Hamiltonian for the KdV drift wave model he explicitly calculates the fluctuation spectrum containing all wave-soliton components. The concepts developed here for the one-dimensional drift wave applies to many other systems with long-lived, localized, coherent structures in any number of dimensions. For the 2D vortex system the concepts developed for soliton gas are often taken to form a point of view for analysis of the numerical and laboratory experiments.[20]

A principal result from the soliton gas description is a formula for the density fluctuation spectrum, $S(\mathbf{k}, \omega)$, which is qualitatively differentfrom formulas based on weakly correlated linear normal modes. The difference follows from the soliton "dispersion" relation, $\omega = ku$, where the soliton velocity u depends linearly upon its amplitude and is independent of k. In a system where a large number of solitons are excited with varying amplitudes, the frequency spread for a given k is $\Delta\omega \sim k \langle (\Delta u)^2 \rangle^{1/2}$, where Δu is the width of the soliton velocity distribution $f_s(u)$. This spectral density contrasts qualitatively with that obtained by renormalized weak turbulence theory.

In the functional integral analysis of Tasso given in Sec. V the principal result is for the two-space point spectral density $S(k) = \int_{-\infty}^{+\infty} S(k, \omega) d\omega / 2\pi$ where $S(k)$ is calculated directly from the equilibrium statistics.

B. Drift Wave Soliton Gas Theory

For radially extended drift modes ($k_x \ll k_y$) Eq. (5) reduces to a one-dimensional form. Comparing the nonlinearity in convection of the vorticity $d_t \nabla^2 \phi$ and the Burger steepening (temperature gradient nonlinearity) indicates that the one-dimensional equation applies in the limit $(k_x \rho_s)(k\rho_s)^2 \ll \eta_e(\rho_s/L_n)$ where $\eta_e = -L_n(\partial_x T_e/T_e)$. The radial dimension of the quasi-one-dimensional solutions is limited by the scale of variation of $v_d(x)$ in Eq. (5). Balancing $v_d \rho_s^2 \partial_x^2 = v_d \rho_s^2/\Delta x^2$ with the variation of $v_d(x)$ about its maximum, $\Delta x^2 \partial_x^2 v_d$, gives

$$(\Delta x)^2 \sim \rho_s L_n. \tag{6}$$

Thus, the one-dimensional drift wave solitons are taken to extend over the radial region Δx, centered at the maximum of v_d, and in addition we assume an axial length $\Delta z = L_c$. Recent particle simulations[21] of drift modes show that the radially extended ($k_x \ll k_y$) mode structures occur in toroidal magnetic field structures and are robust nonlinear structures even in the presence of magnetic shear.

The temperature gradient drift wave is governed by the equation

$$(1 - \rho_s^2 \partial_y^2)\partial_t \varphi + v_d \partial_y \varphi - v_d \varphi \partial_y \varphi = 0, \tag{7}$$

where $\varphi = \eta_e \phi$ and $v_d = \rho_s c_s/L_n$. In neutral fluid studies, where Eq. (7) is called a "regularized" form of the KdV equation, Benjamin *et al.*[22] introduce the name regularized long-wave (RLW) equation. Morrison *et al.*[23] analyze inelastic head-on collisions and essentially elastic overtaking collisions with simulations and a reduced description derived from the least action principle and the Lagrangian density for Eq. (7).

(i) The solitary wave and its energy

Like the KdV equation, Eq. (7) has solitary wave solutions

$$\varphi_s(y, t:\; y_0, u) = -3(u/v_d - 1)\,\mathrm{sech}^2\left[\frac{1}{2\rho_s}\left(1 - \frac{v_d}{u}\right)^{1/2}(y - y_0 - ut)\right], \tag{8}$$

where the velocity u is restricted to the ranges

$$u > v_d \quad \text{or} \quad uv_d < 0. \tag{9}$$

Unlike the KdV equation, these solutions are not solitons in the pure sense, since they are not strictly preserved upon collision due to the generation of a wave component. However for collisions of moderate amplitude solitary waves traveling in the same direction, the inelasticity of collisions is so weak so as to be difficult to detect. In head-on collisions the inelasticity is more pronounced.[23,24]

Therefore, solitary waves of Eq. (7) persist for long times and through many collisions. As has been emphasized by Makhankov[9] and by Currie *et al.*[6] solitary waves which are not strictly solitons still can have an important contribution to the statistical properties of the turbulent fields.

To determine statistical properties we will need the drift wave energy

$$E = \tfrac{1}{2}\int \left[\varphi^2 + (\rho_s\partial_y\varphi)^2\right]\frac{dy}{\rho_s}, \tag{10}$$

which is the physical energy in units of $n_e T_e \rho_s \Delta x L_c/\eta_e^2$. The energy of the solitary wave, Eq. (8), is

$$E_s = \frac{12}{5}\left(\frac{u}{v_d}\right)^2\left(1-\frac{v_d}{u}\right)^{3/2}\left(6-\frac{v_d}{u}\right), \tag{11}$$

For $u/v_d \gg 1$, E_s increases quadratically with u. While as $u \to 0^-$ Eq. (11) reduces to $E_s \simeq 12/5(v_d/|u|)^{1/2}$. The minimum E_s for $u<0$ occurs at $u = v_d(2-\sqrt{10})/12 \cong -0.1v_d$, where $E_s = 14.0$.

(ii) Solitary Wave Spectrum

A turbulent state described by Eq. (7) will consist of a broad wavenumber spectrum of small-amplitude modes together with an ensemble of solitary waves. For each linear mode, the frequency spectrum will be peaked about $\omega_k^\ell = kv_d/[1+(k\rho_s)^2]$ with some width determined, for example, by renormalized turbulence theory. Each solitary wave, however, contributes frequencies which depend upon its velocity (and hence its amplitude) through $\omega = ku$. By virtue of Eq. (10) these frequencies will range over $\omega > \omega_{*e} = kv_d$ and $\omega\omega_{*e} < 0$, which is complementary to the range of the linear dispersion relation $0 < \omega < \omega_{*e}$. The situation in k, ω-fluctuation space is shown in Figure 1.

As a first approximation we ignore the small-amplitude wave component supposing that its spectrum can be merely added to that for the solitary waves. The interaction between solitary waves and linear modes may act to renormalize the solitary wave parameters giving "dressed solitons."[6]

The potential is written as a superposition of solitary waves

$$\varphi(y,t) = \sum_{n=1}^{N_s} \varphi_s(y,t;\; y_n, u_n). \tag{12}$$

The assumption in writing Eq. (12) ignores the strong interactions between these essentially nonlinear objects which occur whenever they overlap. To the extent that the solitary waves act as KdV solitons, ($\bar{k}^2\rho_s^2 \ll 1$), the only effect of this interaction is a phase shift of the soliton positions.

The spectral density is the Fourier transform of the two-point correlation function:

$$S(\xi,\tau) = \langle \varphi(x+\xi, t+\tau)\varphi(x,t)\rangle = \frac{1}{(2\pi)^2}\int dk \int d\omega\, S(k,\omega)\exp(ik\xi - i\omega\tau), \tag{13}$$

where the average is over the ensemble specified below. Utilizing the complete field from Eq. (12) with the solution Eq. (8), gives for $u > v_d$ and $uv_d < 0$

$$S(k,\omega) = \frac{1}{L}\sum_{n=1}^{N_s}\left\langle \left[12\pi k\rho_s\left(\frac{u_n}{v_d}\right)\operatorname{csch}\left(\frac{\pi k\rho_s}{(1-v_d/u_n)^{1/2}}\right)\right]^2 \delta(\omega - ku_n)\right\rangle, \tag{14}$$

and $S(k,\omega) = 0$ for $0 \le u \le v_d$. In deriving Eq. (14) it is assumed that the solitary wave positions, y_n, are randomly distributed along the length L. Generally $L = 2\pi r$, the circumference of the confinement device at the radius r which gives the maximum v_d with the axial magnetic field $B\hat{\mathbf{z}}$. For a large number of solitary waves, $N_s \gg 1$, the sum in Eq. (14) may be converted to an integral over the distribution function, $f_s(u)$, of solitary waves in u space:

$$S(k,\omega) = \frac{1}{L} f_s\left(\frac{\omega}{k}\right)\left\{12\pi\rho_s\frac{\omega}{v_d}\mathrm{csch}\left[\pi k\rho_s\left(\frac{\omega}{\omega - kv_d}\right)^{1/2}\right]\right\}^2, \tag{15}$$

where $\int_{-\infty}^{\infty} f_s(u)du = N_s$. Before determining $f_s(u)$, we can deduce the qualitative shape of $S(k,\omega)$ directly from Eq. (15)) when $k\rho_s \ll 1$.

$$S(k,\omega) \sim \begin{cases} 0, & 0 < \omega < kv_d \\ \exp\left[-2\pi k\rho_s\left(\dfrac{kv_d}{\omega - kv_d}\right)^{1/2}\right], & kv_d < \omega \ll kv_d\left[1 + (\pi k\rho_s)^2\right] \\ \left(\dfrac{\omega}{kv_d}\right)^2\left(1 - \dfrac{kv_d}{\omega}\right) f_s\left(\dfrac{\omega}{k}\right), & \omega \gg kv_d[1 + (\pi k\rho_s)^2] \quad \text{or} \quad \omega < 0. \end{cases} \tag{16}$$

Note the $S = 0$ just in the range where small-amplitude excitations contribute. If there is some maximum-amplitude solitary wave, $-|\varphi_{\max}|$, then Eqs. (8) and (9) imply $S(k,\omega) = 0$ for $\omega > (1+\frac{1}{2}|\varphi_{\max}|)\omega_{*e}$. Extending the field (12) to include the wave component $\sum_k \varphi_k e^{-i\omega_k t}$ within weak turbulence approximation fills the spectral gap $0 < \omega < kv_d$.

A sketch of the contours of $S(k,\omega)$ is given in Fig. 1. In addition, the dispersion relation for linear modes is shown by the dashed line. The analysis and figure clearly show the qualitative difference between the solitary wave spectrum and the weak turbulence spectrum $\langle|\varphi_{k\omega}|^2\rangle$ concentrated along the linear modes $\omega = \omega(k)$. Here, for clarity, we omit the calculation of the wave spectrum in the range $0 < \omega < kv_d$ which follows the usual renormalized weak turbulence theory calculations.

(iii) Canonical Distribution Function of Solitary Waves

A quantitative formula for $S(k,\omega)$ requires knowledge of the distribution function $f_s(u)$. Now we suppose that the solitary waves can be characterized by a Gibbs ensemble. To the extent the solitary waves are solitons they each have an associated conserved quantity. For soliton-bearing equations such as KdV, the inverse spectral transform acts as a canonical transform to action-angle coordinates in which each soliton is represented by a single degree-of-freedom (J, θ), at least for the weakly coupled case. A general theory of action-angle variables for all equations solvable by the inverse scattering transform has been given by Flaschka and Newell.[25] In terms of these coordinates there are N_s conserved quantities, the N_s actions, for a system with N_s solitons. These conserved quantities span the soliton component of the infinite dimensional phase space.

For the true soliton system these degrees-of-freedom are independent, and thus the Gibbs ensemble factors as

$$P_n(J_1, J_2, \ldots, J_{N_s}, \theta_1, \ldots, \theta_{N_s}) = \prod_{i=1}^{N_s} P(J_i, \theta_i),$$

$$P(J, \theta) = (1/Z)\exp[-\beta_s E_s(J)], \tag{17}$$

where P is the single soliton probability distribution and Z is the partition function (normalization constant). The effective inverse temperature β_s fixes the mean energy in the solitary wave component.

The solitary wave degrees-of-freedom are of course not independent for the Petviashvili equation. Soliton-soliton collisions may in fact provide at least part of the randomization necessary for validity of a statistical description. Numerical computations show that moderate amplitude solitary waves can survive many collisions with only small modifications of its parameters. Thus our description is meaningful on the timescale with which the solitary waves maintain their integrity, and is not a long-time equilibrium calculation. In the long-time limit the system may reduce to equipartition with energy $1/\beta$ per degree-of-freedom.

Since the inverse spectral transformation probably does not exist for Petviashvili equation, the canonical transformation to the action-angle coordinates is not possible. Meiss and Horton calculate the probability distribution P using the following procedure. Integration of Eq. (17) over θ and a transformation of coordinates from J to E_s gives

$$P(E_s) = \frac{2\pi}{Z} \left| \left(\frac{\partial E_s}{\partial J} \right)^{-1} \right| \exp(-\beta_s E_s). \tag{18}$$

Since J is a canonical variable, the derivative $\partial E_s/\partial J$ is the frequency $\dot{\theta}$. For a soliton, which acts as a free particle

$$\dot{\theta} = 2\pi(u/L),$$

and therefore

$$P(u) = \frac{L}{Z} \left| \frac{\partial E_s}{\partial u} \frac{1}{u} \right| \exp[-\beta_s E_s(u)]. \tag{19}$$

Note that the soliton energy is now expressed as a function of its velocity, which is an easily calculable function. The one-soliton distribution function used in Eq. (15) is defined by

$$f_s(u) = N_s P(u). \tag{20}$$

For low temperatures, $\beta_s \gg 1$, the distribution function simplifies to the KdV form since the negative velocities components have exponentially small weight:

$$P_{\beta_s \gg 1}(u) \propto \begin{cases} \left(\frac{u}{v_d} - 1 \right)^{1/2} \exp\left[-12\beta_s \left(\frac{u}{v_d} - 1 \right)^{3/2} \right], & u > v_d \\ 0, & u < v_d. \end{cases} \tag{21}$$

Negative velocity solitary waves become significantly excited for $\beta_s \gg 1$ with the distribution peaked in the region $-0.1v_d < u < 0$ and

$$P_{\beta_s \gg 1}(u) \propto \begin{cases} \left(\frac{v_d}{-u} \right)^{5/2} \exp\left[-\frac{12}{5}\beta_s \left(\frac{v_d}{-u} \right)^{1/2} \right], & u < 0 \\ 0, & u > 0. \end{cases} \tag{22}$$

The velocity distribution $P(u)$ vanishes at $u \cong -0.1v_d$ due to the zero of the Jacobian factor $\partial E_s/\partial u$ in Eq. (19).

(iv) Solitary Wave Number Density and Temperature

To utilize the distribution function, Eq. (20), it is necessary to know the effective temperature, $T_s = \beta_s^{-1}$, which fixes the mean energy, $\langle E_s \rangle$. The relationship between these quantities is obtained through

$$\langle E_s \rangle = \int_{-\infty}^{\infty} du\, E_s(u) P(u), \tag{23}$$

where $P(u)$ is given in Eq. (19). This integral can be done approximately utilizing Eq. (21) and Eq. (22), yielding

$$\langle E_s \rangle \sim \begin{cases} T_s, & T_s \ll 1 \\ \\ 3T_s, & T_s \gg 1, \end{cases} \tag{24}$$

where in the upper (lower) relation only the $u > v_d (uv_d < 0)$ solitary waves contribute.

Since the solitary waves represent independent degrees-of-freedom, energy is equipartitioned

$$N_s \langle E_s \rangle \simeq \left(\frac{L}{\rho_s} \right) \left\langle \varphi^2 \right\rangle = \left(\frac{L}{\rho_s} \right) \varphi_0^2, \tag{25}$$

where we assume that the fluctuation energy represented by φ_0^2 is entirely due to the solitary waves which have a number density $n_s = N_s/L$.

The total available thermal energy for the drift wave field may be estimated using thermodynamic arguments. There are two energy sources, the diamagnetic kinetic energy and the free energy of expansion, arising from the background temperature and density gradients. When the radial scale of the drift waves is large, $\Delta x \gg \rho_s$, expansion energy dominates and a thermodynamic bound is

$$\varphi_0^2 \lesssim \eta_e^2 (\Delta x / L_n)^2 \simeq \eta_e^2 (\rho_s / L_n). \tag{26}$$

Equation (6) has been used for Δx for the radial extent of the drift wave solitons.

Once the energy available to the field is known, we only need to calculate N_s to obtain the temperature through Eqs. (24) and (25). This calculation requires knowledge of the number of solitons emerging from a particular initial state, $\varphi(x)$. Since the initial value problem for the Petviashvili equation remains unsolved, we turn again to the KdV limit, recalling that the results will be correct for small T_s.

The inverse scattering transform allows the determination of the number of solitons emitted by any particular initial state. The number N_s of solitons from a field $\varphi(x)$ in the KdV quation is given by the number of bound states in the associated Schrödinger equation with the potential taken as $\varphi(x)$. For moderate amplitude initial states,

$$\varphi_{\max} \gg (\rho_s/L_n)^2,$$

the number of solitons produced is large and a WKB approximation of the inverse problem can be used to obtain

$$N_s[\varphi] = \frac{1}{\sqrt{6}\pi\rho_s} \int_{\varphi<0} dx[-\varphi(x)]^{1/2}. \tag{27}$$

This result is only strictly valid when $\varphi < 0$ for all x. The $\varphi > 0$ regions of the potential generate non-soliton excitations in the one-dimensional case. In the analogous 2D vortex problem this restriction is removed.

To compute the mean number of solitons, we average Eq. (27) using a Gibbs ensemble with the KdV energy

$$E_{\mathrm{KdV}} = \tfrac{1}{2} \int \varphi^2 \frac{dy}{\rho_s}, \tag{28}$$

which is obtained from Eq. (11) when $k\rho_s \ll 1$. The mean square potential is fixed to agree with the available energy of Eq. (26), $\langle\varphi^2\rangle = \varphi_0^2$. The mean number of solitons is determined by a functional integral which upon discretization becomes

$$\langle N_s\rangle = \frac{1}{Z} \prod_{i=1}^{n} d\varphi_i N_s[\varphi] \exp\left[-\tfrac{1}{2}\left(\frac{\varphi_i}{\varphi_0}\right)^2\right], \tag{29}$$

where $Z = (2\pi\varphi_0^2)^{n/2}$ is the normalization and $\varphi_i = \varphi(x_i)$. We then obtain

$$n_s = \langle N_s\rangle/L = \alpha(\varphi_0^{1/2}/\rho_s); \qquad \alpha = \Gamma\left(\tfrac{3}{4}\right)/(12\sqrt{2}\pi^3)^{1/2} = 0.053. \tag{30}$$

Combining Eqs. (25) and (30) gives the mean energy

$$\langle E_s \rangle = (1/\alpha)\varphi_0^{3/2}, \tag{31}$$

which, in conjunction with Eq. (24), gives T_s.

Using the estimate of φ_0 in Eq. (26) gives

$$\begin{aligned} n_s &= \frac{\alpha \eta_e^{1/2}}{\rho_s} \left(\frac{\rho_s}{L_n} \right)^{1/4} \\ \langle E_s \rangle &= \frac{\eta_e^{3/2}}{\alpha} \left(\frac{\rho_s}{L_n} \right)^{3/4} . \end{aligned} \tag{32}$$

Typical values for tokamak experiments are $\rho_s/L_n \simeq 0.01$ and $\eta_e \simeq 1$, for which $n_s = 1.68 \times 10^{-2}\rho_s^{-1}$ and $\langle E_s \rangle = 0.597$. We take $T_s = \langle E_s \rangle = 0.5$ in the following. Significant excitation of negative frequency modes propagating in the ion diamagnetic direction occurs when there is sufficient fluctuation energy to overcome the required creation energy, $E_{\min} = 14.01$.

The frequency shift of the positive spectral peak, $\omega_{\max}$, is given by $\omega_{\max} \cong kv_d(1.2+0.1T_s)$, which gives $\omega_{\max}/kv_d \simeq 1.25$ for $T_s = 0.5$ and 1.5 for $T_s = 3.0$. The frequency shift relative to the linear mode frequency ω_k^ℓ depends upon k, and increases rapidly as $k\rho_s \to 1$. If we use the finite Larmor radius formula for ω_k^ℓ, a frequency shift of $\omega_{\max}/\omega_k^\ell \simeq 3$ is obtained when $k\rho_s \simeq 0.5$ and $T_s = 0.5$, which is in quantitative agreement with the experiments.[12−17]

To summarize this section, the root-mean-square fluctuation levels typical of the saturated state of drift wave turbulence, is such that the inverse scattering theory for the soliton (KdV) equation indicates that a large number, $N_s \gg 1$, of solitons can evolve from the drift wave fields. Each drift wave soliton introduces a spatially localized infinite order set of correlations, due to its intrinsic coherence. These correlations are lost in the truncations of statistical turbulence theory, and give rise to new features in the fluctuation spectrum even in the limit of an ideal gas approximation to the many-soliton system. The soliton gas theory is now developed following Tasso[26,28] for the one-dimensional drift wave models using the functional integral analysis. We now turn to a review of the Tasso theory of the fluctuation spectrum.

III. Continuous Plasma Models from Two-Component Magnetohydrodynamics

In the search for continuous plasma models, the first idea which comes to mind is to look for the well-known macroscopic descriptions such as ideal magnetohydrodynamics (MHD), the two fluid theory or the Vlasov-Maxwell system. These models are fine but the problem with them is that they cannot be described by a "faithful" Hamiltonian formalism in terms of Euler or Clebsch variables. In short, Euler variables give rise to degenerate Poisson brackets and Clebsch variables are not a single-valued representation of Euler variables. See Tasso[28] and Morrison *et al.*[29] for a discussion of this question.

On the other hand, many scalar equations such as Korteweg-de Vries equation do have essentially faithful Hamiltonians.[30] Fortunately, important fluctuations in plasmas seem to be well approximated by drift wave equations described by a single scalar, the electrostatic potential.

This leads us to concentrate on the latter kind of equations, which can be derived from the two-fluid and Maxwell system. The equations of motion of the two-fluid system read

$$n_i m_i \left(\frac{\partial \mathbf{v}_i}{\partial t} + \mathbf{v}_i \cdot \nabla \mathbf{v}_i \right) = e n_i (\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) - \nabla p_i, \tag{33}$$

$$n_e m_e \left(\frac{\partial \mathbf{v}_e}{\partial t} + \mathbf{v}_e \cdot \nabla \mathbf{v}_e \right) = -e n_e (\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \nabla p_e. \tag{34}$$

Let us apply system (33) and (34) augmented with continuity, Maxwell equations and appropriate equations of state to the case of a slab of low-pressure plasma immersed in a strong, homogeneous magnetic field $\mathbf{B}$ pointing in the z-direction. Any time-dependent perturbation about the slab geometry is restricted to being electrostatic, $\mathbf{E} = -\nabla\phi$, so that

Maxwell equations reduce essentially to quasineutrality if the perturbation wavelength is larger than the Debye length. In addition, we assume low-frequency perturbation, for which the inertia of the ions can be neglected in first approximation, and the electrons can be taken as isothermal along $\mathbf{B}$. The relevant equations for a low-β (β is the ratio of kinetic pressure to magnetic pressure) plasma are then given by

$$n_i \mathbf{v}_{i\perp} = -\frac{n_i \nabla\phi \times \mathbf{B}}{B^2} - \frac{\nabla p_i \times \mathbf{B}}{eB^2}, \tag{35}$$

$$n_e \mathbf{v}_{e\perp} = -\frac{n_e \nabla\phi \times \mathbf{B}}{B^2} + \frac{\nabla p_e \times \mathbf{B}}{eB^2}, \tag{36}$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_{i\perp}) = 0, \tag{37}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0, \tag{38}$$

$$n_e = n_0(x) \exp\left[\frac{e\phi}{k_B T_e(x)}\right], \tag{39}$$

$$n_i = n_e, \tag{40}$$

$$\mathbf{B} = B\mathbf{e}_z, \tag{41}$$

where n_e and n_i are the electron and ion densities, $\mathbf{v}_e$ and $\mathbf{v}_i$ are the electron and ion macroscopic velocities, T_e is the electron temperature, ϕ is the electrostatic potential, x is the coordinate perpendicular to the slab and y is the coordinate perpendicular to both x and z, n_0 is the unperturbed density, e is the charge of the proton, and k_B is the Boltzmann constant.

Equations (35) and (36) solve the equations of motions for neglected inertia. The parallel motion of the ions is small in view of their large mass. The continuity equations for ions and electrons are expressed by equations (37) and (38), while the electrons behave along z according to a Boltzmann distribution given by equation (39). Quasineutrality, easily restored by the electrons along the field lines, is ensured by equation (40).

Elimination of $\mathbf{v}_i$ and n_i from equation (37) using equations (35) and (40) leads to the following equation for ϕ:

$$\frac{e}{kT}\frac{\partial\phi}{\partial t} - \frac{1}{B}\left(\frac{n'}{n} - \frac{T'}{T}\frac{e\phi}{kT}\right)\frac{\partial\phi}{\partial y} = 0. \tag{42}$$

The subscripts as well as the explicit indication of x-dependence have been dropped in Eq. (42). The prime denotes the derivative with respect to x.

Equation (42) is essentially the inviscid Burgers equation in which x appears as a parameter. It is the simplest model of a nonlinear drift wave equation, and was discovered in 1967 by one of the authors.[31] The nonlinearity is due to the temperature gradient of the electrons, which is present in any confined hot plasma, and is called "scalar nonlinearity" in the literature. In regions of flat temperature profiles, Eq. (42) becomes linear. In this case, however, higher-order terms due to ion inertia produce a so-called "vector nonlinearity," which is, in essence, two-dimensional and first appeared in Hasegawa and Mima.[32]

The solutions of the inviscid Burgers equation are known to develop infinitely steep gradients at finite times, which can be prevented by adding some of the neglected physical terms such as ion inertia or gyroviscosity, thus limiting attention to nondissipative effects.

A first attempt[33] to take such terms into account was to consider the case of cold ions and concentrate on the first inertial term in Eq. (33). On the assumption of solutions with weak x-dependence, the correction due to ion inertia is obtained by iteration of Eq. (33), inserting in the inertial term the approximate solution given by Eq. (35) for zero ion pressure. This leads to

$$\mathbf{v}_{i\perp} = \mathbf{v}_{i\perp 0} + \mathbf{v}_{i\perp 1} = -\frac{\nabla\phi\times\mathbf{B}}{B^2} + \mathbf{e}_y\frac{m_i}{eB^2}\frac{\partial^2\phi}{\partial y\partial t}. \tag{43}$$

Let us insert $\mathbf{v}$ from Eq. (43) in Eq. (37), using equations (39) and (40) to obtain

$$\frac{eB}{kT}\frac{\partial\phi}{\partial t} - \left(\frac{n'}{n} - \frac{T'}{T}\frac{e\phi}{kT}\right)\frac{\partial\phi}{\partial y} - \frac{m_i}{eB}\left(\frac{\partial^3\phi}{\partial t\partial y^2} - \frac{e}{kT}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y\partial t}\right) = 0. \tag{44}$$

Equation (44) is the same as Eq. (6) of Oraevsky *et al.*[33] if solutions of the form $\phi(y-ut)$ are sought. The discussion of such solutions led to the existence of drift solitons[33] and other nonlinear waves.

To go beyond Eq. (44), keeping the assumption of cold ions, it will be necessary to go to higher-order terms in the expansion in the inertial terms. In Hasegawa and Mima,[32] the so-called vector nonlinearity has been included as another correction to Eq. (43), which yields

$$\mathbf{v}_{i\perp} = \mathbf{v}_{i\perp 0} + \mathbf{v}_{i\perp 1} + \mathbf{v}_{i\perp 2} = -\frac{\nabla\phi \times \mathbf{B}}{B^2} - \frac{m_i}{eB^2}\nabla\frac{\partial\phi}{\partial t} + \\ + \frac{m_i}{eB^2}\frac{\nabla\phi\times\mathbf{B}}{B^2}\cdot\nabla\left(\frac{\nabla\phi\times\mathbf{B}}{B^2}\right). \tag{45}$$

Similarly to the derivation of Eq. (44), we insert $\mathbf{v}$ from equation (45) in Eq. (37), using Eqs. (39) and (40), and obtain a two-dimensional equation of the type

$$\frac{\partial}{\partial t}(\phi - \nabla^2\phi) + (\nabla\phi \times \mathbf{e}_z \cdot \nabla)\nabla^2\phi = 0. \tag{46}$$

The coefficients in equation (46) have been omitted, to give the equation the same form as in Hasegawa and Mima.[32] The scalar nonlinearity of equations (42) and (44) is absent in equation (46) because of problems of ordering, i.e. this nonlinearity is of zero order and would dominate the vector nonlinearity.

Equations (44) and (46) are too simple to describe real situations. Since the ions are not cold, diamagnetic and gyroviscous terms should be taken into account. Also the parallel velocity of the ions could cause acoustic waves along the magnetic field and, if different from the parallel velocity of the electrons, contributes to the creation of electric currents and related magnetic fields. On the other hand, the electrons do not need to behave "adiabatically" as expected in equation (39). Friction between electrons and ions decouples density and potential fluctuations. Finally, the slab geometry and the homogeneous magnetic field are too simple an assumption to represent real toroidal situations. An account of sophisticated drift waves models is given in Horton.[34]

All these effects tend to complicate the approximate equations in such a way that they become as difficult to handle as the the original two-fluid-Maxwell system. The Vlasov-Maxwell or the Fokker-Planck-Maxwell system would, of course, be much less tractable either analytically or numerically.

Our aim is to extract statistical information about the system, but this does not seem possible to achieve on such difficult equations, especially if dissipative terms are included.[1] On the other hand, nondissipative equations do not always have faithful Hamiltonians in useful variables. This leads us to look for models containing an essential part of the physics and possessing faithful Hamiltonians simple enough to be able to carry, on canonical distributions constructed with them, calculations of statistical averages such as correlation functions. Note that such correlations are functional integrals, which can be very difficult to evaluate.

IV. Hamiltonians

To construct canonical Gibbs distributions, first we need to have Hamiltonians. Usually Hamiltonians are derived from Lagrangians using Legendre transformations. Lagrangian formulation has the advantage of ensuring invariance properties such as Galilean invariance of the original exact equations. The penalty, however, lies in the complexity, if not chaoticness, of Lagrangian variables. This makes the corresponding Hamiltonian highly nonlinear and useless for evaluating functional integrals. It may be of a great analytical advantage to remain in Eulerian variables[27,30] and elaborate drift wave approximations as we did in the previous section. The penalty is now that some of the model equations obtained may, for instance, not be Galilean invariant. As long as we remain in the reference system in which the approximations made are physically valid, we are safe.

Equation (42), as mentioned above, is the inviscid Burgers equation. In reduced form it reads

$$\frac{\partial u}{\partial t} - (a + bu)\frac{\partial u}{\partial y} = 0. \tag{47}$$

Its Hamiltonian in terms of Eulerian variables is given by the functional

$$H = \int \left(a\frac{u^2}{2} + b\frac{u^3}{6} \right) dy, \tag{48}$$

so that equation (47) can be written as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial y}\frac{\delta H}{\delta u} = [u, H], \tag{49}$$

where the brackets denote the generalized Poisson brackets of two functionals and are defined

$$[F, G] = \int \frac{\delta F}{\delta u}\frac{\partial}{\partial y}\frac{\delta G}{\delta u} dy \tag{50}$$

as given by Gardner.[35]

Brackets (50), though noncanonical, have the property that their symplectic operator, i.e. $\partial/\partial y$, is independent of the dynamic variable u. This property is important for defining simple volume elements in phase space and for satisfying the Liouville theorem despite the noncanonical formulation.[36] The symplectic operator of brackets (50) is essentially nondegenerate, i.e. the brackets possess few or no Casimir invariants.[28]

As discussed above, the solutions of equation (42) may contain infinite gradients, which led us to introduce equation (44). Unfortunately, equation (44) has mixed time and space derivatives together with nonlinearities, which makes the introduction of a Hamiltonian formulation very difficult. In order to proceed, we have to make assumptions in addition to the expansion in the strength of the inertial terms, assumptions which cannot be justified by expansions in small parameters.

We now state that equation (44) can be modelled by Korteweg-de Vries equation for the following reasons: First, both equations have solitary wave solutions. Second, the time derivative in the two terms within the parentheses of the last part of equation (44) can be approximated by the space derivative for long wavelengths. Third, the second of these two terms is smaller than the first for small amplitudes. Our model equation is then of the form[37]

$$u_t - C_1 u u_y + C_2 u_{yyy} = 0. \tag{51}$$

For the coefficients and further calculations see Tasso.[26]

Equation (51) has the following Hamiltonian[35]:

$$H = \int \left(\frac{C_1}{6} u^3 + \frac{C_2}{2} u_y^2 \right) dy, \tag{52}$$

together with bracket (50), so that

$$u_t = \frac{\partial}{\partial y} \frac{\delta H}{\delta u}. \tag{53}$$

Hamiltonian (52) is not bounded because of the cubic term and the fact that u can have both signs. Note that, in some cases, unbounded Hamiltonians can be used in equilibrium statistics, but at the expense of a rather difficult analysis.[38]

There is another[39] symplectic representation of equation (51) with Hamiltonian

$$H_E = \frac{1}{2} \int u^2 dy \tag{54}$$

and bracket

$$[F, G] = - \int \frac{\delta F}{\delta u} \left(C_2 \frac{\partial^3}{\partial y^3} - \frac{2C_1}{3} u \frac{\partial}{\partial y} - \frac{C_1}{3} u_y \right) \frac{\delta G}{\delta u} dy, \tag{55}$$

so that equation (51) can be written as

$$u_t = [u, H_E] = - \left(C_2 \frac{\partial^3}{\partial y^3} - \frac{2C_1}{3} u \frac{\partial}{\partial y} - \frac{C_1}{3} u_y \right) \frac{\delta H_E}{\delta u}. \tag{56}$$

Representation (54)-(56) has a bounded Hamiltonian (54), but a rather complicated symplectic operator $-\left(C_2 \frac{\partial^3}{\partial y^3} - \frac{2C_1}{3} u \frac{\partial}{\partial y} - \frac{C_1}{3} u_y \right)$, which depends upon u, so that nice properties, such as the Liouville theorem and the simple volume element in phase space, are lost. Fortunately, a Miura[40] transformation

$$u = \frac{C_1}{6} v^2 + C_2^{\frac{1}{2}} v_y \tag{57}$$

exists which induces a new phase space v with $\frac{\partial}{\partial y}$ as symplectic operator. The field v obeys the modified Korteweg-de Vries equation, which can be written as

$$v_t = \frac{C_1^2}{6} v^2 v_y - C_2 v_{yyy} = \frac{\partial}{\partial y} \frac{\delta H_E}{\delta v}, \tag{58}$$

with

$$H_E(v) = \frac{1}{2} \int \left(\frac{C_1^2}{36} v^4 + C_2 v_y^2 \right) dy, \tag{59}$$

which is formally identical to the one-dimensional Ginzburg-Landau potential.[41] Note, however, that any solution of equation (58) is a solution of (51), but the opposite is not true, which means that solutions of the Korteweg-de Vries equation may be lost through transformation (57). This will have to be remembered when averages over phase space v are taken later.

The last equation for which we would like to have a Hamiltonian is equation (46). In view of the problems already mentioned, concerning the Hamiltonian formulation[42] of equation (44), one is discouraged from looking for a simple canonical Hamiltonian. There is, however, a noncanonical formulation for equation (46). The brackets in this formulation are noncanonical and depend upon the dynamic variable, so that they are not very useful for the calculation of averages.

Instead of equation (46), it is more practical to use an ad hoc model containing terms reminiscent of the scalar nonlinearity of equation (42) as well as two-dimensional dispersive terms reminiscent of the one-dimensional dispersive term of equation (44) or (51). This ad hoc equation has, at the same time, a simple Hamiltonian formulation and appeared for the first time[43] in (1994). It reads

$$u_t = \frac{\partial}{\partial y} \frac{\delta H}{\delta u}, \tag{60}$$

with a Hamiltonian of the type

$$H = \int \left[\gamma u^4 + \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy \tag{61}$$

and Poisson brackets identical to those given by equation (50). Hamiltonian (61) is this time formally identical to the two-dimensional Ginzburg-Landau potential.[41]

V. Correlation Functions and Spectra in One Dimension

As stated above, statistical averages for continuous systems necessitate the calculation of functional integrals. Functional integrals are the limit of multiple integrals when the number of integrations goes to infinity in a proper way. It is known[44] that such integrations can usually be done explicitly for one-dimensional problems and only occasionally for two-dimensional systems. Though observations suggest two-dimensional behavior of drift waves, for technical reasons we would like to start with one-dimensional systems.

Following Tasso,[45,26] we calculate first the partition function Z for a system described by equations (58) and (59),

$$Z = \int D(v) \exp(-\beta H_E(v)), \tag{62}$$

where β^{-1} is the "temperature" or energy of the fluctuations. The Hamiltonian $H_E(v)$ from equation (59) is rewritten as

$$H_E = \int \xi^{-1}(bv^4 + cv_y^2)dy, \tag{63}$$

with

$$\frac{b}{\xi} = \frac{C_1^2}{72}, \qquad \frac{c}{\xi} = \frac{C_2}{2}, \tag{64}$$

and the length ξ will later be related to the correlation length. Expression (62) is written first in discretized form as

$$Z = \lim_{N\to\infty} D^{-N} \int \prod_{i=-N}^{N} dv_i \exp\left\{-\frac{\beta\Delta y}{\xi}\left[bv_{i+1}^4 + c\left(\frac{v_{i+1}-v_i}{\Delta y}\right)^2\right]\right\}, \tag{65}$$

where N is defined by $N = \frac{L}{\Delta y}$, L being the periodicity length of the function v. When N goes to infinity, Δy goes to zero with L fixed. The integration in (65) can be reduced to a product of single integrals by using the eigenvalues of the transfer integral operator[46]:

$$D^{-1}\int dv_{i-1}\exp[-\beta f(v_i,v_{i-1})]\psi_n(v_{i-1})=\exp\left(-\beta\epsilon_n\frac{\Delta y}{\xi}\right)\psi_n(v_i). \tag{66}$$

Operator (66) reduces to the one-dimensional Schrödinger operator in the limit of $N\to\infty$ and $\Delta y\to 0$.[46] It reads

$$\left(-\frac{1}{4}\frac{d^2}{dv^2}+v^4\right)\psi_n(v)=\epsilon_n\beta^{\frac{4}{3}}\beta_0^{-\frac{1}{3}}\psi_n(v)\equiv E_n\psi_n(v), \tag{67}$$

where $\beta_0^{-1}=b$ and ξ has been chosen as $\frac{c}{b}$. Obviously, it holds that $E_n=\epsilon_n\beta^{\frac{4}{3}}\beta_0^{-\frac{1}{3}}$. A good approximation to Z is

$$Z\approx\exp\left(-\frac{L}{\xi}\beta\epsilon_0\right), \tag{68}$$

where ϵ_0 is the lowest eigenvalue of the anharmonic oscillator.

We can now proceed to the evaluation of the space (equal time) correlation function. The definition is given by

$$C(y)=\langle\delta u(y)\delta u(0)\rangle=\int D(v)\delta u(y)\delta u(0)\frac{\exp(-\beta H_E)}{Z}, \tag{69}$$

where

$$\delta u=u-\langle u\rangle,\langle u\rangle=\int D(v)\delta u\frac{\exp(-\beta H_E)}{Z}, \tag{70}$$

and u is expressed in terms of v through equation (57). The explicit calculation of correlation function (69) closely follows the evaluation of partition function (62) using the transfer integral operator technique (66). It closely follows the method given in Scalapino *et al.*[46] One obtains

$$C(y)=\frac{C_1^2}{36}\sum_{n=1}^{\infty}\left[\left\langle\psi_n|v^2|\psi_0\right\rangle-\left(\frac{\beta_0}{\beta}\right)^{\frac{2}{3}}(E_n-E_0)^2\langle\psi_n|v|\psi_0\rangle\right]\times$$
$$\exp\left[-\frac{y}{\xi}\left(\frac{\beta_0}{\beta}\right)^{\frac{1}{3}}(E_n-E_0)\right], \tag{71}$$

where the Dirac brackets denote scalar products in the Hilbert space of eigenvectors.

We are now in a position to calculate the k-spectrum of the fluctuations by taking the Fourier transform of correlation function (71):

$$S(k) = \int_{-\infty}^{\infty} dy \exp(iky) C(y) = 2 \sum_{n=1}^{\infty} \frac{q_n}{k^2 + p_n}, \tag{72}$$

where

$$p_n = \xi^{-1} \left(\frac{\beta_0}{\beta}\right)^{\frac{1}{3}} (E_n - E_0), \tag{73}$$

$$q_n = \frac{C_1^2}{36} \left[\left\langle \psi_n | v^2 | \psi_0 \right\rangle - \langle \psi_n | v | \psi_0 \rangle (p_n \xi)^2 \right]. \tag{74}$$

To subject spectrum (72) to observation, it is first necessary to relate the "constants" C_1 and C_2 of equation (51) to the experiment. For C_1 we use the coefficient in front of the steepening term of equation (42), originally discovered by Tasso (1967). For C_2 we can use the coefficient of the linear "dispersive" term of equation (44) or generalize to a gyroviscous dispersion effect due to the finite gyroradii of the ions.[26] The choices of β and L are related to the observed level of fluctuations and to the major radius of the toroidal tokamak experiment,[26] respectively. This procedure yields essentially a Lorentz spectrum in k since the terms for $n > 1$ are negligible in equation (72). For more details see Ref.[26].

In the meantime, experiments[47] confirm the plateau behavior of the spectrum for small k but give a k^{-3} behavior for large k, in disagreement with the k^{-2} behavior of spectrum (72). In the next section it will be seen that the disagreement is due to the one-dimensional calculation, which contradicts the two-dimensionality of the observed turbulence.

The Lorentz spectrum and the related exponential decay of correlation function (71) seem to have a kind of universal character. Hamiltonians of the form

$$H = \int [g(u) + \alpha u_y^2] dy, \tag{75}$$

with $g(u)$ higher than quadratic and H bounded or $H > 0$ give rise to an exponential shape for the correlation function and to a Lorentz shape for the spectrum.[6,28] For example, equations of the type

$$u_t - u^{2n} u_y + u_{yyy} = 0, \tag{76}$$

with n an integer, would belong to the "universality" class. This fact gives us some freedom for modeling physical systems in two dimensions, as demonstrated in the next section.

VI. Spectra in Two Dimensions

As already mentioned above, it is important to have a model containing the main physical effects and possessing a simple Hamiltonian formulation. It turns out that the simplest two-dimensional extension is given by Hamiltonian (61).[43] Unfortunately, functional integrals with Hamiltonian (61) in the canonical distribution like

$$C(x,y) = \langle \delta u(x,y)\delta u(0,0)\rangle = \int D(u)\delta u(x,y)\delta u(0,0)\frac{\exp(-\beta H)}{Z} \tag{77}$$

are not tractable analytically for two-dimensional models. Note that fluctuation δu is defined similarly to the one-dimensional case as in equation (70).

At this point, we have to guess the right behavior for the correlation function. Inspired from two-dimensional calculations in spin systems[48] and from the one-dimensional result (71), we assume for moderate to large values of r

$$C(r) \approx \exp(-\mu r), \tag{78}$$

where $r = \sqrt{x^2 + y^2}$. A weak divergence of $C(r)$ at small values of r will be cancelled by a residual viscosity. To obtain the spectrum we take the two-dimensional Fourier transform of correlation (78), which according to theorem (56) of Bochner[49] is

$$S(k) = \int_0^\infty \exp(-\mu r) r J_0(kr) dr, \tag{79}$$

where J_0 is the zeroth-order Bessel function and $k = \sqrt{k_x^2 + k_y^2}$. Integral (79) is known,[49] which gives

$$S(k) = \frac{\mu}{[\sqrt{\mu^2 + k^2}]^3}. \tag{80}$$

Spectrum (80) has the observed k^{-3} behavior for large k with a plateau for small k, in agreement with the experiment.[47] This result is quite encouraging for the pursuit of a macroscopic modeling of drift wave turbulence. Though equilibrium statistics cannot deliver the height of the spectrum, which is related to the saturation level of the turbulence and is responsible for the observed anomaly in diffusion, it gives a consistent picture of the spectrum, which may increase our understanding of the scalings to large plasmas.

VII. Conclusions

Because of the special nature of drift wave turbulence, as discussed in the introduction, it is possible to apply the equilibrium statistical approach successfully. The introduction of approximate nonlinear drift wave equations together with adequate model equations having simple Hamiltonians is a basic starting element of this approach. Basic difficulties relating to the explicit analytic calculation of functional integrals as averages over canonical distributions can only be overcome for one-dimensional continuous systems. The Lorentz spectrum obtained in this case disagrees with the k^{-3} fluctuation spectrum observed in large tokamaks.[47] Though the two-dimensional case is not completely tractable analytically, the reasonable assumption of exponential behavior of the space correlation function leads to a spectrum in excellent agreement with observation.

Evidence for coherent structures in tokamak turbulence has been difficult to find until recently. New multi-probe arrays (16 Langmuir probes and 16 optic channels for D_α-recombination light) by the ASDEX-team[50] have provided direct evidence for intermittent, poloidally localized coherent features. The bi-orthogonal decomposition analysis of these structures by Benkadda *et al.*[51] indicates that they have correlation features similar to those found in drift mode simulations of the edge plasma. Further work on the Endler data sets is required to show that the observed structures are into the nonlinear self-focused regime of solitary waves.

New microwave reflectometry methods have shown the rather large radial correlation lengths consistent with Eq. (6) in the core ($r/a \simeq 0.3$) of the large ($I_p = 1.2\,\mathrm{MA}$, $B = 4.2$ to $4.8\,\mathrm{T}$) TFTR tokamak.[52] The measured radial correlation length is $\ell_{cr} = 2 - 4\mathrm{cm}$ which is approximately equal to $(\rho_i a)^{1/2} = (1.0\,\mathrm{mm} \times 0.8\,\mathrm{m})^{1/2} \simeq 3\mathrm{cm}$. Thus, the description of radially correlated fluctuations with $k_\theta \rho_s > 0.1$, $\omega = k_\theta u \geq k v_{de}$ appears to be consistent with the reflectometry measurements.

This statistical approach ignores the difficult problem of the saturation level of fluctuations in assuming a local balance between driving and damping.[1] This makes the problem analytically solvable, but gives a partial answer, which is the k-spectrum of fluctuations. The determination of the saturation level remains an important open problem whose solution is crucial for a quantitative estimate of the observed anomaly in diffusion. Its solution must involve the sources and sinks of turbulence. The statistical part of the problem will have to be done out of equilibrium. This is a major question in turbulence which seems to resist all attempts at solution, including the "Maximum Entropy Principle" advocated by Jaynes.[53] The application of this principle implies the introduction of side-conditions compatible with the dynamics.[1] The side-conditions are, in general, impossible to find and formulate for nonlinear dissipative systems. The saturation level of drift wave fluctuations and the related anomalous diffusion will have to be extracted from the subtleties accompanying the application of such a principle or any other nonequilibrium approach.

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FIGURES

FIG. 1. Contours of $S(k, \omega)$ for the 1D solitary drift wave showing the complementary regions of wave and soliton propagation.