Rotation and Locking of Magnetic Islands

F. L. Waelbroeck and R. Fitzpatrick

Institute for Fusion Studies, University of Texas, Austin, Texas 78712

(January 28, 1997)

Abstract

The polarization drifts arising from the acceleration of plasma flowing alongside an island are shown to be destabilizing. A critical island width is found above which the polarization drifts (inertial forces) defeat the stabilizing effect of the viscous forces and prevent islands from being unlocked by increasing the plasma rotation.

52.30.-q, 52.30.Jb, 52.35.Py

Stationary Magnetic Perturbations (SMP) are frequently observed to be followed by major disruptions. [1–3] They occur as a result of the locking of the plasma rotation, either by a resonant error field or through the interaction between a magnetic island and the resistive vessel wall. [4,5] This paper examines the possibility of unlocking islands by injection of momentum. We find that inertial forces are destabilizing and lead to a threshold island-width above which unlocking through momentum injection is impossible. Our results are in agreement with experimental observations. [3]

The key property governing island rotation is that the plasma cannot flow across flux surfaces once the displacement amplitude exceeds the visco-resistive layer width. [5] It follows that the electrostatic potential, which acts as a stream function for the flow, must be constant on flux surfaces in the rest frame of the perturbation. The plasma velocity is thus proportional to the transverse magnetic field on each flux surface. The velocity inside the island, in particular, has odd parity about the O-point. Since the velocity is continuous in the presence of viscosity, the even part of the velocity distribution must vanish immediately outside the separatrix. This boundary condition was violated in previous studies of island rotation. [6–9]

The distortion of the flow caused by a magnetic perturbation gives rise to an inertial force proportional to the cross product of the velocity with the vorticity. [8] This force is balanced by a current directed along the axis of the island, $J_z = -\rho v_y v_y'/B_y$ where v_y and B_y are the transverse velocity and magnetic field respectively, and ρ is the mass density. The current perturbation reaches its maximum negative amplitude outside the separatrix on either side of the O-point, where the stream lines are compressed. It thus constitutes a drive for magnetic reconnection. The origin of this drive is clearly the same as for the Kelvin-Helmholtz instability, namely the reduction in kinetic energy resulting from the interchange of fast and slow moving fluid elements.

The inertial destabilization must compete with the stabilizing effect of the viscous force. [5] This force opposes the change in plasma rotation caused by the island. It is proportional to the discontinuity in the gradient of the plasma velocity across the island region. The magnitude of the discontinuity (and thus of the viscous force) is determined by solving the momentum diffusion equation away from the island, $\nabla \cdot (\nu \nabla \delta \mathbf{v}) = \mathbf{0}$, with the boundary condition that the velocity be continuous across the island. Here δv is the perturbation in the equilibrium plasma flow.

The stabilizing effect of the viscous force may be understood as follows: In the case of islands driven by the current gradient (positive tearing stability index Δ'), viscosity causes the island to rotate with the plasma: this reduces Δ' by virtue of the wall stabilization effect. In the case of islands induced by an error field, the viscous force produces a phase lag between the island and the error field, thereby reducing its effective amplitude. The object of this paper is to determine the conditions under which viscous stabilization dominates inertial destabilization.

We consider a configuration invariant under translation along the z axis, $\partial/\partial z = 0$, and periodic in the y direction with wavenumber $k = 2\pi/L_y$, where L_y is the wavelength. The magnetic field is expressed as $\mathbf{B} = \mathbf{B_z}\widehat{\mathbf{z}} + \widehat{\mathbf{z}} \times \nabla\psi$, where the flux ψ is related to the vector potential by $\psi = -A_z$. The current is $\mathbf{J} = \mathbf{J}\widehat{\mathbf{z}}$ where $J = \mu_0^{-1}\nabla^{\mathbf{2}}\psi$. The velocity is given in terms of a stream function ϕ by $\mathbf{v} = \widehat{\mathbf{z}} \times \nabla\phi$. In the presence of a strong guide field $B_z \gg B_y$, the stream function is approximately equal to $1/B_z$ times the electrostatic potential.

The structure of magnetic islands is governed by the equation for the axial component of the vorticity, $U = \nabla^{\mathbf{2}}\phi$, and by Ohm's law:

$$\rho\left(\frac{\partial U}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{U}\right) = \bar{k}B\cdot\bar{k}\nabla J + \nu\bar{k}\nabla^2 U, \tag{1}$$

$$\frac{\partial\psi}{\partial t} + \bar{k}v\cdot\bar{k}\nabla\psi = \eta J - E_0. \tag{2}$$

Here ρ is the constant mass density and E_0 is the inductive electric field in the reference state. In the absence of perturbations $E_0 = \eta J_0$, where J_0 is the current in the reference state. Since we seek steady state solutions of these equations, we set the two terms involving time derivatives equal to zero. The equations may then be viewed as magnetic differential equations for J and ϕ respectively:

$$\bar{k}B\cdot\bar{k}\nabla J = \rho\bar{k}v\cdot\bar{k}\nabla U - \nu\bar{k}\nabla^2 U; \tag{3}$$

$$\bar{k}B \cdot \bar{k}\nabla\phi = -\eta\,\delta J, \tag{4}$$

where $\delta J = J - J_0$. In order for a solution to exist it is necessary that

$$\langle \delta J \rangle = 0 \tag{5}$$

$$\nu\langle \bar{k}\nabla^2 U\rangle = \rho\langle \bar{k}v \cdot \bar{k}\nabla U\rangle, \tag{6}$$

where the angular braces denote the average over the flux surface

$$\langle f \rangle = \frac{k}{2\pi}\oint \frac{dy}{B_y} f. \tag{7}$$

For flux surfaces inside the island, the integration $\oint$ should be interpreted as the extending over the complete cross section of the flux surface rather than over a wave period. When the solubility conditions Eqs. (5)-(6) are satisfied the current J and stream function ϕ may be determined, aside from two "flux functions" $I(\psi)$ and $\Phi(\psi)$, by integration of Eqs. (3)-(4) along the field lines,

$$J(\psi, y) = I(\psi) + \int_0^y \frac{d\hat{y}}{B_y}(\rho\bar{k}v \cdot \bar{k}\nabla U - \nu\bar{k}\nabla^2 U) \tag{8}$$

$$\phi(\psi, y) = \Phi(\psi) - \eta\int_0^y \frac{d\hat{y}}{B_y}\delta J \tag{9}$$

The flux functions $I(\psi)$ and $\Phi(\psi)$ are determined by the solubility conditions, Eqs. (5)-(6).

Velocity distribution

Recall that in the nonlinear regime, by definition, the island width W is much larger than the linear visco-resistive layer width, $\delta = (\eta\nu/k^2 B_y'^2)^{1/6}$. Assuming that the magnitude of the perturbed current can be estimated using the last term of Eq. (8) and using $\bar{k}\nabla^2 U \sim v_y'/W^2$, the diffusive term in Eq. (9) is of the order $\eta\delta J/(kB_y) \sim v_y'\delta^6/W^4$. In the nonlinear regime, therefore,

$$\phi(\psi, y) = \Phi(\psi)(1 + O(\delta/W)^6) \tag{10}$$

That is, the plasma is compelled to flow within the flux surfaces in the bulk of the island (except for a narrow sub-layer around the separatrix). This allows the inertial force to be written as an exact differential along the field lines, $\rho\bar{k}v \cdot \bar{k}\nabla U = \rho\bar{k}B \cdot \bar{k}\nabla(U\dot{\Phi})$, where $\dot{\Phi} = d\Phi/d\psi$. An important consequence is that the average of the convective acceleration around a flux surface vanishes. The velocity profile is thus determined by $\langle\bar{k}\nabla^2 U\rangle = 0$. We conclude that *inertial forces do not affect the velocity profile.*

For small amplitude perturbations the ordering $k \ll \partial/\partial x$ may be invoked to simplify the form of the viscous torque: $\nu\bar{k}\nabla^2 U = \nu\partial^2 U/\partial x^2$. We introduce the Alfvènic Mach number distribution $M(\psi) = (\mu_0\rho)^{1/2} d\Phi/d\psi = v_y(x)/v_A(x)$, where $v_A(x) = B_{y0}(x)/(\mu_0\rho)^{1/2}$. The torque may be written in terms of the flux variables as

$$\nu\bar{k}\nabla^2 U = \nu v_A \partial^2(B_y^3 \dot{M})/\partial\psi^2. \tag{11}$$

Here we have neglected δJ compared to J_0. Substituting this in the solubility condition and integrating yields

$$M(\psi, \sigma) = D + C\sigma H(\psi - \psi_1) \int_{\psi_1}^{\psi} d\widehat{\psi} \langle B_y^4\rangle^{-1}, \tag{12}$$

where C and D are integration constants, ψ_1 is the value of ψ on the separatrix, H is the Heaviside function and $\sigma = \mathrm{sign}(x)$.

The constants C and D are determined by matching to the even and odd parts, respectively, of the velocity profile in the bulk plasma. We require continuity of the velocity, but allow for a jump in the even part of the vorticity. This jump implies that a singular viscous force is acting on the separatrix. The viscous force singularity is resolved by resistivity, which allows the plasma to flow across magnetic surfaces in a boundary layer around the separatrix. The effect of the singularity on the equilibrium is negligible in the nonlinear regime ($W \gg \delta$). Note that a jump in the odd part of the vorticity would be rapidly relaxed through spin-up of the plasma inside the island. The odd part of the velocity does not contribute to the force balance and we henceforth set it equal to zero.

The constant C is determined by the jump in the derivative of the velocity across the island region:

$$C = M_\infty \left(\int_{\psi_1}^{\infty} d\widehat{\psi}/\langle B_y^4\rangle \right)^{-1}, \tag{13}$$

where $M_\infty = (\mu_0\rho)^{1/2}[dv/dx]_-^+/(2B'_{y0})$ is half the jump in the Mach number determined by the solution of the transport problem in the bulk plasma. The velocity profile determined by Eqs. (12)–(13) has the characteristic "V" shape observed in numerical simulations. [10]

Island equations of state

We restrict consideration to the case where the current perturbation is small, so that the constant-ψ approximation applies: $\psi = B'_{y0}\, x^2/2 + \psi_1 \cos ky$. The magnetic field is thus given in terms of the flux by $B_y = [2B'_{y0}(\psi - \psi_1 \cos ky)]^{1/2}$, and the island width by $W = 4(\psi_1/B'_{y0})^{1/2}$. The amplitude of the island and its phase shift with respect to the external currents are given by matching the total current induced in the layer to the jump in the perturbed magnetic field,

$$\Delta'\psi_1 = 4\mu_0 \int_{-\psi_1}^{\infty} d\psi \langle \delta J \exp iky \rangle. \tag{14}$$

The parameter Δ' is a generalization of the well-known stability parameter for tearing modes. It measures the ratio of the total current induced in the tearing layer to the perturbed flux: $\Delta' = [d\psi_1/dx]_-^+/\psi_1$, where $[\,\cdot\,]_-^+$ represents the jump across the layer. We will consider specific forms of Δ' below.

The current profile is determined by substituting the velocity distribution, Eq. (12), into the first integral of the shear-Alfven law along the field line, Eq. (8), and using the solubility condition, Eq. (5), to determine the integration constant $I(\psi)$. Using $U = (B_y^2 \dot{M} + \mu_0 J M)/(\mu_0\rho)^{1/2}$, we find $\delta J = \delta J_r + \delta J_i$, where the in-phase component of the current perturbation,

$$\delta J_r = \mu_0^{-1} \left(B_y^2 - \langle B_y^2 \rangle / \langle 1 \rangle \right) M \dot{M},$$

is caused by the inertial forces and the component in phase quadrature,

$$\delta J_i = \nu \left(\frac{\mu_0}{\rho} \right)^{1/2} \int_0^y d\widehat{y}\, \partial^2 (B_y^3 \dot{M}) / \partial \psi^2,$$

is caused by the viscous torque. Note that we have made the approximation $1 - M^2 \approx 1$. With this approximation the constant, even part of the Alfvènic Mach number distribution contributes only to the odd part of the current.

The island amplitude and unlocking thresholds are determined by substituting the current into the matching relation, Eq. (14). The matching condition for the component of the current in phase quadrature with the flux perturbation yields the azimuthal force balance equation, [11,5]

$$k\Im(\Delta')W^4 = W_\nu^2 M_\infty, \tag{15}$$

as may be verified by direct evaluation of the imaginary part of Eq. (14) by successive integrations by parts. Here $W_\nu^2 = 256\nu/\rho\omega_A$ and $\omega_A = B'_{y0}/(\mu_0\rho)^{1/2}$.

The matching condition for the component of the current in phase with the flux perturbation yields

$$\Re(\Delta') = -.792\,\frac{M_\infty^2}{W}. \tag{16}$$

Equation (16) is the principal result of this paper: it shows that the inertial forces contribute a *destabilizing* term inversely proportional to the island width. This term must compete with the stabilizing effect of viscosity. The mechanism for viscous stabilization depends on whether the island is intrinsically unstable or whether it is produced by an error-field. We consider each case in turn.

We first consider the case of a saturated island produced by a tearing instability in a periodic cylindrical plasma. The Δ' stability index depends on the degree of penetration of the flux perturbation through the conducting wall surrounding the plasma. [11,5] The flux penetration is related to the current in the wall, I_w, by $\psi_w = i\mu_0 I_w/2m\omega\tau_w$ provided that $\delta_w/r_w \ll m\omega\tau_w \ll r_w/\delta_w$, where δ_w is the thickness of the wall, r_w is the radius of the wall, m is the mode number, ω is the mode rotation frequency and τ_w is the skin time of the wall. The stability index is then given by

$$\Delta'(W) = \Delta'_s(W) - \frac{\Delta'_0 - \Delta'_s(0)}{1 + 2im\omega\tau_w\Delta_w'^{-1}}$$

where Δ'_s is the stability index for a perfectly conducting wall, while Δ'_0 is the stability index in the absence of wall.

Substituting the above expression for Δ' into the matching equations we obtain the equations of state for the island,

$$\frac{1+\beta\widehat{\omega}^2}{1+\widehat{\omega}^2} - w = -\kappa\frac{\mathcal{V}'^2}{w}; \tag{17}$$

$$(1-\beta)\frac{\widehat{\omega}w^4}{1+\widehat{\omega}^2} = -\mathcal{V}', \tag{18}$$

where $\mathcal{V}' = W_\nu^2 M_\infty / k\Delta'_0 W_0^4$ is the normalized velocity gradient, $\beta = \Delta'_s(0)/\Delta'_0$, and $\widehat{\omega} = 2m\omega\tau_w/r_w\Delta'_w$. We have modeled the quasi-linear saturation by $\Delta'_s(W) = \Delta'_s(0)(1 - W/\beta W_0)$ and normalized the island width to the saturation width in the absence of wall W_0: $w = W/W_0$. The normalized saturated width for a perfectly conducting wall is thus $w = \beta < 1$. The coefficient

$$\kappa = .792 k^2 \Delta'_0 W_0^7 / W_\nu^4 > 0 \tag{19}$$

parametrizes the relative strength of the inertial destabilization compared to the viscous stabilization term. Figure 1 shows curves of constant κ defined by Eqs. (17)–(18) in the $\mathcal{V}'$-$\widehat{\omega}$ plane. The rotation frequency and amplitude of an island can be determined from the intersection of these curves with the straight line $\mathcal{V}' = \lambda(\widehat{\omega}_0 - \widehat{\omega})$, where

$$\lambda = \frac{W_\nu^2}{k\Delta'_0 W_0^4} \frac{r_w \Delta'_w}{2m\tau_w\omega_A} \frac{[v'_y]_-^+}{2k(v_{y0} - v_y)}.$$

This line represents the linear relation between the torque acting on the island and the change in the rotation frequency from its value in the absence of an island. For $\kappa = 0$ the torque curve peaks at $\widehat{\omega} = 1/3$, leading to a forbidden band of rotation frequencies indicated in Fig. 1 for $\lambda = .2$ by the pair of vertical dotted lines. For $\kappa > \kappa_{\rm crit}(\beta)$, by contrast, the torque curves separate in two. The low-frequency, high-drag branch extends along the entire normalized torque ($\mathcal{V}'$) axis. For such κ it is impossible to unlock the island by increasing the plasma rotation velocity.

We next consider the case of islands resulting from the penetration of error fields into the plasma. In this case the viscous torque reduces the island amplitude by introducing a phase shift φ between the island and the error field. The stability index is now

$$\Delta'(\psi_1) = \Delta'_0 \left[1 + \mathcal{A}^2(\psi_{\rm vac}/\psi_1)\exp(i\varphi)\right],$$

where $\psi_{\rm vac}$ is the vacuum flux perturbation due to the error field and $\mathcal{A}$ measures the amplification of the island with respect to its vacuum width. It is again convenient to normalize the island width to its value in the absence of flow, $w = W/W_0$ where $W_0 = \mathcal{A}W_{\rm vac}$, In terms of these quantities the island equations of state, (15)–(16), are

$$w^2 - \cos\varphi = \kappa w \mathcal{V}'^2; \tag{20}$$

$$w^2 \sin\varphi = \mathcal{V}' \tag{21}$$

where κ is defined as above except for the replacement of Δ'_0 by its absolute value ($\Delta'_0 < 0$ is assumed here). The variation of the island amplitude with the velocity gradient is then obtained by eliminating the phase from these equations. The result is shown in Fig. 2. We find that rotation decreases the island amplitude for $\kappa < 1/2$. In a narrow intermediate regime, $.5 < \kappa < .602$, the island is amplified by moderate rotation but unlocks as the rotation is increased. For $\kappa > .602$, however, the island remains locked for all values of the velocity, in agreement with the numerical results of Parker [10].

In summary, we have shown that destabilizing inertial effects dominate the viscous effects for $\kappa > \kappa_{\rm crit}$. In terms of island width, this condition may be translated to $W_0 > W_{\rm crit}$, where $W_{\rm crit} = (\kappa_{\rm crit} W_\nu^4/.792 k^2 \Delta'_0)^{1/7}$. We use the vacuum value of Δ'_0 in order to evaluate this width for the Joint European Torus (JET). For an island with poloidal and toroidal mode numbers $m = 2$ and $n = 1$, we find $W_{\rm crit} \simeq 4$ cm, corresponding to a magnetic perturbation near the vessel wall of 5.6 G. Taking into account the neoclassical renormalization of the inertial term by a factor of $(B_z/B_\theta)^2$ reduces the critical magnetic perturbation to 1.5 G. This should be compared with the experimental value of 3 G. [3]

Previous authors have considered step-function velocity profiles in order to describe cases where the rotation velocity of the ions outside the island differs from that of the island itself.

[7–9] Unfortunately, these authors incorrectly omitted the polarization current contributed by the velocity discontinuity. Including this contribution reverses the effect of the polarization current, so that it becomes destabilizing in the absence of diamagnetic drifts. We conclude that the role of the polarization drift in providing a threshold against the growth of neoclassical bootstrap-current driven islands [12] needs to be reexamined.

Instructive discussions with A. Smolyakov and H. Wilson are gratefully acknowledged. This work was supported by the United States Department of Energy under contract DE-FG03-96ER-54346.

REFERENCES

[1] J. A. Snipes et al., Nucl. Fusion **28**, 1085 (1988).

[2] J. T. Scoville et al., Nucl. Fusion **21**, 875 (1991).

[3] G. M. Fishpool and P. S. Haynes, Nucl. Fusion **34**, 109 (1994).

[4] T. C. Hender et al., Nucl. Fusion **32**, 2091 (1992).

[5] R. Fitzpatrick, Nucl. Fusion **33**, 1049 (1993).

[6] R. Fitzpatrick and T. C. Hender, Phys. Fluids B **3**, 644 (1991).

[7] M. Zabiego and X. Garbet, Phys. Plasmas **1**, 1890 (1994).

[8] A. I. Smolyakov et al., Phys. Plasma **2**, 1581 (1995).

[9] H. R. Wilson et al., Phys. Plasmas **3**, 248 (1996).

[10] R. D. Parker, Proceedings of the 19th EPS-ICPP conf., Innsbruck, Vol. 1, 427 (1992).

[11] M. F. F. Nave and J. Wesson, Nucl. Fusion **30**, 2575 (1990).

[12] Z. Chang et al., Phys. Rev. Lett. **74**, 4663 (1995).

FIGURES

FIG. 1. Torque curves as a function of the normalized island rotation frequency $\widehat{\omega}$ for various values of κ. The lowest curve corresponds to $\kappa = 0$. The two dashed lines correspond to $\lambda = .2$. The vertical dotted lines show the forbidden frequency band for $\kappa = 0$.

FIG. 2. Island amplitude as a function of the imposed rotation shear for various values of κ. The dashed portions of the lines indicate unstable branches.