Plasma Analog of Particle–Pair Production

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Abstract. It is shown that the plasma axial shear flow instability satisfies the Klein-Gordon equation. The plasma instability is then shown to be analogous to spontaneous particle-pair production when a potential energy is present that is greater than twice the particle rest mass energy. Stability criteria can be inferred based on field theoretical conservation laws.

Key Words: particle-pair production, analog, axial shear flow, temperature gradient instability, Lagrangian, conservation

1. Quantum Relativistic Analogy for the Axial Shear Instability

The purpose of this note is to illustrate a remarkable similarity between the equation for axial shear instability in an inhomogeneous plasma and the Klein-Gordon equation in relativistic quantum mechanics.

$$\left[(\hbar\partial_t + ie\varphi)^2 - \hbar^2 c^2 \partial_z^2 + m^2 c^4\right]\psi = 0, \tag{1}$$

In the presence of a large enough potential variation, the Klein-Gordon equation creates particle-antiparticle pairs. We will show that this creation process in the Klein-Gordon equation, is the instability mechanism for the axial shear flow plasma problem. Further, we will discuss how the conservation laws associated with the quantum mechanical problem gives us insight into the plasma stability problem.

First we summarize the plasma problem. Instability from axial shear flow was first recognized by Kadomtsev.[1] He investigated a flute mode for a plasma on an open homogeneous magnetic field line when there is an equilibrium $\boldsymbol{E}\times\boldsymbol{B}$ flow that varies along the magnetic field line. When the ends are terminated by an insulating end-wall, the response that arises due to ion inertia and quasineutrality leads to the condition,

$$\int_0^L \frac{d^2\delta\varphi}{dt^2}\,dz = -\delta\varphi \int_0^L [\omega - \omega_E(z)]^2\,dz = 0,$$

where $\delta\varphi$ is the perturbed potential, and $\omega_E(z) = \boldsymbol{k}_\perp \cdot \boldsymbol{v}_E = \boldsymbol{k}_\perp \cdot \frac{c}{B^2}(\boldsymbol{E}(z) \times \boldsymbol{B})$. This dispersion relation is always **unstable** when $\omega_E(z)$ varies along the axis. Such variation is consistently described by the two fluid MHD equation[2] when there is present both a longitudinal electron pressure gradient $\nabla_\parallel p_e$ and a transverse electron temperature gradient $\nabla_\perp T_e$. The axial shear instability has also been investigated in later works[3,4] in application to tandem mirror stabilization due to finite Larmor radius effects. In a more recent paper[5] the axial shear instability was investigated for the semi-infinite space problem. Both the insulating end-wall and conducting end-wall with the Kunkel-Guillory[6] boundary condition was studied for a plasma with zero resistance but with finite Larmor radius (FLR).

In Tsidulko *et al.*,[5] the following linear equation was derived and analyzed,

$$\left[\left(\frac{\partial}{\partial t} + i\omega_E + i\omega_*\right)\left(\frac{\partial}{\partial t} + i\omega_E\right) - V_A^2 \frac{\partial^2}{\partial z^2}\right]\psi = 0, \tag{2}$$

where the field line displacement is related to ψ by $\boldsymbol{\xi} = i(\boldsymbol{k}_\perp \times \boldsymbol{B}/B)\psi$ and

$$\omega_*(z) = \frac{\boldsymbol{k} \cdot (\boldsymbol{B} \times \boldsymbol{\nabla} p_i)}{Bn\, m_i \omega_{Bi}}$$

is the ion diamagnetic drift frequency. The boundary condition for an insulating end-wall is $\partial\psi/\partial z = 0$, while the Kunkel-Guillory boundary condition for a conducting wall relates the perturbed longitudinal current,

$$j_\parallel \propto \frac{\partial \psi}{\partial z}$$

to the perturbation of the potential drop across the Debye sheath,

$$\delta\varphi_L \equiv \delta\varphi + \boldsymbol{\xi} \cdot \boldsymbol{\nabla}\varphi \,\propto\, i\omega\psi,$$

Specifically, one finds,

$$\frac{\partial \psi}{\partial z} = i\omega \frac{v_\parallel}{(V_A k_\perp \rho_i)^2}\psi, \tag{3}$$

where $v_\parallel$ is longitudinal plasma flow velocity into the conducting wall. Note that the boundary condition takes on its simplest form when expressed in terms of the Lagrangian displacement $\delta\varphi_L$. In fact in terms of this variable the boundary condition is mathematically identical to the Kunkel-Guillory condition which was derived without consideration of an equilibrium transverse flow.

It was found that the axial shear instability appears when the change of the function ω_E along z is greater than ω_* and when the end-wall coefficient $\alpha \propto v_\parallel$ is either sufficiently large or sufficiently small.

Another instability resulting from Eq. (2) arises even when there is no variation in the flow speed (i.e. ω_E is spatially homogeneous) but there are conducting wall boundary conditions (given by Eq. (3)). This instability was studied in Berk *et al.*[7,8,9] Such a flow is readily established in plasmas on open field lines when there is a temperature gradient perpendicular to the magnetic field.

When the Debye sheath conductivity coefficient goes to zero (an insulator) or infinity (a common choice for the conducting boundary condition in ideal MHD theory) the growth rate goes to zero. The latter two boundary conditions are what has been most frequently used in the literature, and this accounts for the relative lateness of the realization that there is such an instability mechanism in the scrape-off layer of a plasma.

To explicitly exhibit the analogy between Eq. (1) and Eq. (2) we make the following associations:

$$V_A \leftrightarrow c; \quad \omega_E + \frac{\omega_*}{2} \leftrightarrow \frac{e\varphi(z)}{\hbar}; \quad \frac{|\omega_*|}{2} \leftrightarrow \frac{mc^2}{\hbar}. \tag{4}$$

Then the only the difference between Eqs. (1) and (2) is that in the plasma equation [Eq. (2)] V_A and ω_* can vary along z-axis, whereas in the Klein-Gordon equation, c, the speed of light and the particle mass m are constants. Henceforth for simplicity, we shall neglect variation of V_A, but ω_* will be assumed to be a function of space.

2. Positive and Negative Energy Waves in an Axially Homogeneous System

Let us consider Alfvén waves, using the usual definitions of wave momentum and wave energy found in the theory of electrodynamics of continuous media.

We use for the dielectric function the definition,

$$\epsilon_{\alpha\beta}(\omega, \boldsymbol{k})\delta E_\beta \equiv \delta E_\alpha + \frac{4\pi i}{\omega}\delta j_\alpha, \tag{5}$$

which leads to the following form for the Maxwell wave equation

$$\left(k_\alpha k_\beta - k^2\delta_{\alpha\beta} + \frac{\omega^2}{c^2}\epsilon_{\alpha\beta}\right)\delta E_\beta = 0. \tag{6}$$

The wave energy expressed in terms of Hermitian part of the dielectric function tensor $\epsilon_{\alpha\beta}(\omega, \boldsymbol{k}) = \left(\epsilon_{\alpha\beta} + \epsilon^*_{\beta\alpha}\right)/2$, is given by the following[10]

$$\mathcal{W} = \frac{\partial}{\partial\omega}\left[\frac{c^2}{\omega}\left(k_\alpha k_\beta - k^2\delta_{\alpha\beta}\right) + \omega\epsilon_{\alpha\beta}\right]\frac{\delta E^*_\alpha \delta E_\beta}{16\pi}. \tag{7}$$

The simplest way to calculate the Alfvén wave energy is the following. We first calculate the energy density $\mathcal{W}'$ in a moving frame where the equilibrium electric field $\boldsymbol{E}$ is zero and we then transform to the laboratory frame, where the frequency is $\omega = \omega' + \boldsymbol{k}\cdot\boldsymbol{v}_E$, using the general relation

$$\mathcal{W} = \mathcal{W}' + \boldsymbol{v}_E \cdot \boldsymbol{\mathcal{P}}' = \frac{\omega}{\omega'}\mathcal{W}',$$

where $\boldsymbol{\mathcal{P}}' = \frac{\boldsymbol{k}}{\omega'}\mathcal{W}'$ is wave momentum density (prime corresponds to the frame with $\boldsymbol{E} = 0$).

The dielectric function tensor in the case $\boldsymbol{E} = 0$ for a low pressure plasma, where $v_s k_\parallel \ll \omega' \ll \omega_{Bi}$ and $\rho_i \ll k_\perp^{-1} \ll L_\perp$ (where v_s is sound speed, ω_{Bi} the cyclotron frequency, $\rho_i = v_{Ti}/\omega_{Bi}$ is ion Larmor radius and $L_\perp$ is scale length of transverse variation of plasma parameters) has the form,

$$\epsilon_{\alpha\beta} = \epsilon_\perp(\delta_{\alpha\beta} - b_\alpha b_\beta) + \epsilon_\parallel b_\alpha b_\beta \tag{8}$$

with b_α the component of unit vector along the equilibrium field. For the plasma we find that $\epsilon_\perp = \omega_{pi}^2 \left(1 - \omega_*/\omega'\right)/\omega_{Bi}^2$ where $\omega_{pi}^2 = 4\pi n e^2/m_i$ and $\epsilon_\parallel \to \infty$, when m_e and the collision frequency are taken as arbitrarily small.

From Eqs. (6) and (8) we obtain the dispersion relations for the compressional Alfvén waves respectively,

$$\omega'(\omega' - \omega_*) = k^2 V_A^2, \qquad \omega'(\omega' - \omega_*) = k_\parallel^2 V_A^2. \tag{9}$$

The energy density is found to be,

$$\mathcal{W}' = \frac{|\delta E_\perp|^2}{8\pi}\frac{\omega_{pi}^2}{\omega_{Bi}^2}\left(1 - \frac{\omega_*}{2\omega'}\right) = m_i n \frac{|\xi|^2}{2}\,\omega'\left(\omega' - \frac{\omega_*}{2}\right), \tag{10}$$

where $\boldsymbol{\xi} = ic[\delta\boldsymbol{E}\times\boldsymbol{B}]/(\omega' B^2)$ is displacement. In the laboratory frame the shear Alfvén wave dispersion relation is,

$$(\omega - \omega_E)(\omega - \omega_E - \omega_*) = k_\parallel^2 V_A^2 \tag{11}$$

and the wave energy density

$$\mathcal{W} = m_i n \frac{|\xi|^2}{2}\,\omega\left(\omega - \omega_E - \frac{\omega_*}{2}\right). \tag{12}$$

Now let us use this dispersion relation to express the frequency $\omega(v_{gr})$ as a function of the group velocity

$$v_{gr} \equiv \frac{\partial\omega}{\partial k_\parallel} = \frac{k_\parallel V_A^2}{\omega - \omega_E - \omega_*/2}. \tag{13}$$

Then eliminating $k_\parallel$ we find the following relation,

$$\omega = \omega_E + \omega_*/2 \pm \left|\frac{\omega_*/2}{\sqrt{1 - v_{gr}^2/V_A^2}}\right|, \tag{14}$$

where the upper sign corresponds to the case with $\omega > \omega_E + \omega_*/2$ and the lower sign corresponds to the opposite case. Then we can rewrite Eq. (12), the expression for the energy density, as follows, $\mathcal{W} = \mathcal{N}\mathcal{E}$, where

$$\mathcal{N} = m_i n \frac{|\xi|^2}{2}\,|\omega - \omega_E - \frac{\omega_*}{2}|\,\frac{1}{\hbar}$$

and

$$\mathcal{E} = \pm\hbar\,\omega = \hbar\left|\frac{\omega_*/2}{\sqrt{1 - v_{gr}^2/V_A^2}}\right| \pm \hbar\left(\omega_E + \frac{\omega_*}{2}\right).$$

If we now use the transformations to the relativistic quantum system [Eq. (4)] we immediately see that this relation corresponds to the standard expression for energy of an elementary charged particle in an electrostatic potential φ where we have,

$$\mathcal{E} = \left| \frac{mc^2}{\sqrt{1 - v_{gr}^2/c^2}} \right| \pm e\varphi. \tag{15}$$

We see that we have an analogy where $\omega > \omega_E + \omega_*/2$ corresponds to a particle of charge of say $+e$ and then $\omega < \omega_E + \omega_*/2$ corresponds to antiparticle with charge $-e$. The positive value $\mathcal{N}$ is to be interpreted as the particle (or antiparticle) density. If $e|\varphi| > mc^2/(1 - v_g^2/c^2)^{1/2}$ the antiparticles have negative energy if $e\varphi > 0$ and the particles have negative energy if $e\varphi < 0$. Note, that the range $e\varphi - mc^2 < \hbar\omega < e\varphi + mc^2$ corresponds to imaginary value of $k_\parallel$ ($v_g^2 < 0$) in our problem and to virtual particles (antiparticles) in the quantum analogy.

3. Lagrangian Approach for Complex ω and Spatial Inhomogeneity

In order to study in more detail the properties of our equation, we examine the Lagrangian for equations (1) and (2). To simplify the expressions we denote

$$s = \int \frac{dz}{|V_A|} \leftrightarrow \frac{z}{c}, \quad u(s) = \omega_E + \frac{\omega_*}{2} \leftrightarrow \frac{e\varphi}{\hbar}, \quad m(s) = \frac{|\omega_*|}{2} \leftrightarrow \frac{mc^2}{\hbar},$$

and we assume that V_A varies sufficiently slowly $\left(\frac{\partial_z V_A}{V_A}\right) \ll \min\left(\frac{\omega}{V_A}, \frac{1}{L_\parallel}\right)$. Then, both our equations can be written in the form,

$$\left[(\partial_t + iu)^2 - \partial_s^2 + m^2\right]\psi = 0. \tag{16}$$

This equation can be derived from the well-known Lagrangian density[11] for zero spin charged particles. In the case where there is only an electrostatic potential, the appropriate Lagrangian is

$$\mathcal{L}(\psi, \psi^*, ...) = |\partial_t\psi + iu\psi|^2 - |\partial_s\psi|^2 - m^2|\psi|^2 \tag{17}$$

and its Euler-Lagrange equation is Eq. (16).

Clearly, this Lagrangian is invariant respectively to time translation, $t \to t + \delta t$ and gauge transformation, $\psi \to \psi\exp(i\Lambda)$, where Λ is real. Hence, according the Noether's theorem if the action $S = \int \mathcal{L} dt\, ds$ is invariant with respect to transformations given by $x^\mu \to x^\mu + X^\mu_\nu \lambda^\nu$ with $x^\mu \equiv [t(\mu = 0), s(\mu = 1)]$, double index implies summation and $\psi_k \to \psi_k + \Psi_{k,\nu}\lambda^\nu$ (ψ_k are ψ and ψ^* in our case) the solutions exhibit current conservation $\partial_\mu J^\mu_\nu = 0$, where currents are $J^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi_k)}\Psi_{k,\nu} - \Theta^\mu_\sigma X^\sigma_\nu$ and $\Theta^\mu_\sigma = \delta^\mu_\sigma \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi_k)}\partial_\sigma\psi_k$ is energy-momentum tensor.

We find two conservation relations. The first is energy conservation (the parameters for Noether's theorem are $X^t_t = 1$, with the other $X^\nu_\mu = 0$ and $\Psi_{k,\nu} = 0$)

$$\partial_t\widetilde{\mathcal{W}} + \partial_s\widetilde{\mathcal{S}} = 0, \tag{18}$$

where

$$\widetilde{\mathcal{W}} = |\partial_t\psi|^2 + (m^2 - u^2)|\psi|^2 + |\partial_s\psi|^2 = (\omega_r^2 + \Gamma^2 + m^2 - u^2)|\psi|^2 + |\partial_s\psi|^2 \tag{19}$$

is the energy density and

$$\widetilde{\mathcal{S}} = -(\partial_t\psi\partial_s\psi^* + \text{c.c.}) = \omega_r(i\psi\partial_s\psi^* + \text{c.c.}) - \Gamma\,\partial_s|\psi|^2 \tag{20}$$

is energy flux density. The forms in Eqs. (19) and (20) use $\psi \propto \exp(-i\omega t), \quad \omega = \omega_r + i\Gamma$.

Let us redefine the energy density and the energy flux density as follows,

$$\mathcal{W} = \widetilde{\mathcal{W}} - \partial_s\,\Re(\psi^*\partial_s\psi), \quad \mathcal{S} = \widetilde{\mathcal{S}} + \partial_t\,\Re(\psi^*\partial_s\psi).$$

Then the substitution into Eq. (19) yields,

$$\mathcal{W} = 2\omega_r(\omega_r - u)|\psi|^2 \,\propto\, |\xi|^2\omega_r\left(\omega_r - \omega_E - \frac{\omega_*}{2}\right) \tag{21}$$

in agreement with our previous expressions calculations of the wave energy given in Eq. (12) for real ω. The expression for the energy flux density $\mathcal{S}$ for $\psi \propto \exp(-i\omega t + ik_\parallel z)$ is $\mathcal{S} = \omega_r(i\psi\partial_s\psi^* + \text{c.c.})$, in agreement with usual relation $\mathcal{S} = v_{gr}\mathcal{W}$, where v_{gr} is defined in Eq. (13).

The gauge transformation leads to "electric charge" conservation (the parameters in Noether's theorem are $X_\mu = 0$, $\Psi_1 = ie\psi$ and $\Psi_2 = -ie\psi^*$)

$$\partial_t\mathcal{Q} + \partial_s\mathcal{J} = 0, \tag{22}$$

where

$$\mathcal{Q} = e(i\psi^*\partial_t\psi - u|\psi|^2 + \text{c.c.}) = 2e(\omega_r - u)|\psi|^2 \tag{23}$$

is the "electric charge" density and

$$\mathcal{J} = e(i\psi\partial_s\psi^* + \text{ c.c.}) \tag{24}$$

is "electric current" density. (If $\psi \propto \exp(-i\omega t)$ the "charge" conservation condition gives us additional information only for the case $\omega_r = 0$.)

4. Reflection Problem

We can now make the following conclusions regarding the nature of wave reflection, for a wave with real frequency ω for the situation illustrated in Fig. 1. In this figure $u(s)$ is drawn as a monotone function of s with $u_\infty > u_{-\infty}$, but in principle the shapes of $u(s)$ and $m(s)$ are arbitrary, except that they asymptotic to constant values as $|s| \to \infty$.

The solution for $s \to -\infty$ consists of the incident and reflected waves

$$\psi_{-\infty} = \frac{A_0}{|k_0|^{1/2}}\exp\left(-i\omega t + ik_0 s\right) + \frac{A_1}{|k_1|^{1/2}}\exp\left(-i\omega t - ik_1 s\right) \tag{25}$$

with an "electric current"

$$\mathcal{J}_{-\infty} = e(i\psi\partial_s\psi^* + \text{ c.c.}) = 2e\left[\frac{k_0}{|k_0|}|A_0|^2 + \frac{k_1}{|k_1|}|A_1|^2\right]. \tag{26}$$

The solution for $s \to +\infty$ consists of the transmitted outgoing wave only

$$\psi_\infty = \frac{A_2}{|k_2|^{1/2}} \exp\left(-i\omega t + ik_2 s\right) \tag{27}$$

with "electric current" $\mathcal{J}_\infty = 2ek_2/|k_2||A_2|^2$. The direction of wave propagation is defined by sign of the group velocity which equals $v_{gr} \equiv \partial\omega/\partial k_\| = \mathcal{J}/\mathcal{Q} = \mathcal{S}/\mathcal{W}$, i.e. it is the direction where "particles" ("antiparticles") "carry their own charge and energy."

There are five regions (a)–(e) to discuss. We introduce the following notation for the longitudinal wavenumbers of either particles or antiparticles in the asymptotic regions (where u and m are constant). The wavenumber, k_0, is associated with the positive group velocity wave as $s \to -\infty$, the wavenumber, k_1, is associated with negative group velocity wave as $s \to -\infty$ and the wavenumber k_2 with positive group velocity wave as $s \to \infty$ or the decaying wave in the case when k_2 is imaginary.

The region (a) depicted in Fig. 1 is where $\omega > u_\infty + m_\infty$ we consider an incoming particle from $s = -\infty$. Then,

$$k_0 = [(\omega - u_{-\infty})^2 - m_{-\infty}^2]^{1/2} > 0, \quad k_1 = -[(\omega - u_{-\infty})^2 - m_{-\infty}^2]^{1/2} < 0,$$

and $k_2 = [(\omega - u_\infty)^2 - m_\infty^2]^{1/2} > 0$. From $\mathcal{J}_{-\infty} = \mathcal{J}_\infty$, we conclude $|A_0|^2 - |A_1|^2 = |A_2|^2$ and therefore the reflection coefficient $|r| \equiv \frac{|A_1|}{|A_0|} < 1$. Thus, if there are $\mathcal{I}$ incoming particles, $|r|^2\mathcal{I}$ particles are reflected and $(1-|r|^2)\mathcal{I}$ particles are transmitted.

In region (b), $u_\infty - m_\infty < \omega < u_\infty + m_\infty$, we are still only dealing with particles and $k_0 > 0$, $k_1 < 0$. Now, however k_2 is pure imaginary. In this case $\mathcal{J}_\infty = 0$, so that $A_2 = 0$, and we conclude that $|A_0|^2 - |A_1|^2 = 0$. Hence, $|r| = 1$, and all the incoming particles are reflected.

In region (c), (arising only if $u_\infty - u_{-\infty} > m_\infty + m_{-\infty}$) $u_{-\infty} + m_{-\infty} < \omega < u_\infty - m_\infty$, k_0 and k_1 are the same as before, but in order for k_1 to have $\partial\omega/\partial k > 0$ as $s \to \infty$, we need to choose, $k_2 = -[(\omega - u_\infty)^2 - m_\infty^2]^{1/2} < 0$. In this case we have an incoming particle, a reflected particle and a transmitted antiparticle. Further, from $\mathcal{J}_{-\infty} = \mathcal{J}_\infty$ and using Eq. (26), we have $|A_0|^2 - |A_1|^2 = -|A_2|^2$. Hence, $|A_1|^2 > |A_0|^2$ so that $|r| > 1$. We see that for a given input flux of incoming particles, even more particles are reflected. Notice also that particles and antiparticles are created in pairs, with the total flux $\propto |A_1|^2 - |A_0|^2$ of particles going to the left, is equal to the flux of antiparticles $\propto |A_2|^2$ going to the right. The specific effect described here is a special case of general description described in Kull *et al.*[12] which shows how a reflection coefficient greater than unity arises in stable systems when there are positive and negative energy waves. Of course, there is possible the reversed process of particle annihilation in the region (c), when waves coming from $+\infty$ and $-\infty$ simultaneously are considered.

In region (d), $u_{-\infty} - m_{-\infty} < \omega < u_{-\infty} + m_{-\infty}$, we cannot have any incident wave from the left.

Finally, in region (e), $\omega < u_{-\infty} - m_{-\infty}$, $k_0 = -[(\omega - u_{-\infty})^2 - m_{-\infty}^2]^{1/2} < 0$, corresponds to an incoming antiparticle, and

$$k_1 = [(\omega - u_{-\infty})^2 - m_{-\infty}^2]^{1/2} > 0, \qquad k_2 = -[(\omega - u_\infty)^2 - m_\infty^2]^{1/2} < 0,$$

corresponds to a reflected and transmitted antiparticles respectively. The condition $\mathcal{J}_{-\infty} = \mathcal{J}_\infty$ implies $|A_0|^2 - |A_1|^2 = |A_2|^2$. Hence, $|r| \equiv |A_1|/|A_0| < 1$.

The picture described above applies in an analogous way to antiparticles ($\omega < u_{-\infty} - m_{-\infty}$) and particles ($\omega > u_{-\infty} + m_{-\infty}$) when they impinge on the potential structure from the right, with the order (a) to (e) reversed.

As an illustration, we present the expression for the reflection coefficient of a step profile. We consider the potential

$$u(s) = u_{-\infty} + (u_{\infty} - u_{-\infty})H(s),$$

where $H(x) = 0$ for $x < 0$, $H(x) = 1$ for $x > 0$ and if we assume $m = \text{const}$, we have $r = (k_0 + k_2)/(k_0 - k_2)$, (the choice of the signs for the roots k_0 and k_2 in the different regions was discussed above). Note, that the step profile can have an infinitely large reflection coefficient in the region (c).

Further, the absolute value of the reflection coefficient can be easily calculated analytically in the case when the WKB approximation is valid everywhere excluding the vicinities of points s_1 and s_2 where the lines $u(s) \pm m(s) - \omega$ vanish. If the linear approximation for these lines can be used in the vicinities, then in region (c) WKB technique yields

$$|r|^2 = 1 + \exp\left[-2\int_{s_1}^{s_2} \left|m^2 - (\omega - u)^2\right|^{1/2} ds\right].$$

In this case the largest value of the reflection coefficient can be $|r|^2_{\max} = 2$.

The existence of regions where reflection coefficient is larger than unity clarifies the appearance of the instabilities we have discussed. In particular, if a profile for some ω has two reflection regions separated by a space interval where the WKB approximation is valid and the product of the reflection coefficients satisfies the condition $|r_1 r_2| > 1$, then there has to exist an unstable solution.

5. Instability in the Case with Conservative Boundary Conditions

In this section we study stability properties that can be inferred from the conservation of energy and "charge." Such conservation applies to either a full space case, a half space case or to a space of finite length. For the cases when $s \to \infty$ or $s \to -\infty$ the boundary condition is ψ vanishes as $|s| \to \infty$. For the cases when one or two end-walls are present, we take for the boundary condition on the walls either $\partial\psi/\partial s = 0$ (insulating boundary condition) or $\psi = 0$ (infinitely large Debye sheath conductivity coefficient).

We are searching for an unstable solution. Obviously, an unstable solution has all conserved values equal zero:

$$\int \mathcal{Q}\, ds = 0, \quad \int \widetilde{\mathcal{W}}\, ds = 0, \quad \int \mathcal{W}\, ds = 0.$$

We define the averaging operator

$$\overline{f} \equiv \int f|\psi|^2 ds \Big/ \int |\psi|^2 ds$$

with the integration over the entire range of the variable s. Then from our conservation relations we have,

$$\omega_r = \overline{u} \quad \text{and} \quad 0 < \Gamma^2 = \overline{(u-\omega_r)^2 - m^2 - |k|^2} < \overline{(u-\omega_r-m)(u-\omega_r+m)}\,, \tag{28}$$

where $k \equiv -i\partial_s\psi/\psi$. Using the inequality

$$\left[\overline{\left(f-\overline{f}\right)^2}\right]^{1/2} \leq \frac{1}{2}(f_{\max} - f_{\min}),$$

we find that instability is possible only if the potential drop

$$\Delta u = u_{\max} - u_{\min} > 2m_{\min}, \tag{29}$$

in agreement with the well-known statement that only when the a potential drop is larger than two particle masses can particle–antiparticle pairs be produced. In plasma terms it means that the considered instability is possible only when

$$(\boldsymbol{k}_\perp \cdot \boldsymbol{v}_E)_{\max} - (\boldsymbol{k}_\perp \cdot \boldsymbol{v}_E)_{\min} > |\omega_*|_{\min}. \tag{30}$$

In addition, we conclude that eigenfrequency of the unstable solution has to satisfy the following conditions:

1) The curve $u - \omega_r$ (see Fig. 2) has to vanish somewhere in order to satisfy the zero "charge" condition $\omega_r = \overline{u}$.

2) In order to have $\overline{(\omega-u-m)(\omega-u+m)} > 0$, either $\omega - u - m$ or $\omega - u + m$ must vanish somewhere.

Formally, the conservation relations in the form given by Eq. (28), give expressions for the real and imaginary parts of the eigenfrequency. The approximate averaging in these expressions can be made analytically in the asymptotic case when $\Delta u\,\Delta s \ll 1$ and $m\,\Delta s \ll 1$, where, Δs is either the scale length of the region in which the functions $u(s)$ and $m(s)$ significantly differ from their asymptotic values in the infinite length case, $s \to \pm\infty$ or $\Delta s = s_r - s_l$ is the distance between end-walls placed at positions s_l and s_r.

In these cases the problem can be either stable or have only one unstable solution. Now, we briefly discuss this calculations for the different boundary conditions. Under the assumed condition the solution ψ is almost constant so that in the region where u and m vary, the contribution from the term $\propto |k|^2$ in the second expression in Eq. (28) is negligible. Hence, for the case when Δs is the distance between two insulating end-walls the growth rate is given by the following expression,

$$\Gamma^2 \simeq \int\limits_{s_l}^{s_r} \left[(u-\omega_r)^2 - m^2\right]\, ds\big/\Delta s, \text{ with } \omega_r \simeq \int\limits_{s_l}^{s_r} u\, ds\big/\Delta s.$$

The case where Δs is taken as a short distance between two end-walls and when the Debye sheath conductivity is arbitrarily large at least at one of the walls, is stable. This follows because the function ψ is constrained to vary linearly within the distance Δs and therefore starting from $\psi = 0$ at an end-wall with the Debye conductivity, we are unable to satisfy the conservative boundary condition at the other end wall.

For the general full space case, with the conditions $\Delta u \, \Delta s \ll 1$ and $m \, \Delta s \ll 1$, the eigenfrequency can be obtained by using a technique derived in Tsidulko *et al.*.[5] Here we only present the result for the case when $u_{-\infty} = u_\infty$ and $m_{-\infty} = m_\infty$, where there is an unstable solution when the following condition is satisfied,

$$Y \equiv \int_{-\infty}^{\infty} \left[(u - u_\infty)^2 - (m - m_\infty)^2 \right] ds > 2m_\infty. \tag{31}$$

Then the solution has the growth rate given by

$$\Gamma \simeq \frac{1}{2} \sqrt{(Y^2 - 4m_\infty)}. \tag{32}$$

In the opposite case when $\Delta u \Delta s \gg 1$ (Δs is now the minimal scale of variation of the function $u(s)$ and $m(s)$) instability always arises when the condition $|r_1 r_2| > 1$ is satisfied. Note, that when there are end-walls (which have reflection coefficient equal to unity, because we consider only conservative boundary conditions in this section), one of the reflection coefficients in this inequality can be the reflection coefficient of an end-wall.

Suppose we have instability for the full space problem with even functions $u(s)$ and $m(s)$ as it shown in Fig. 2. Clearly, we can then place an insulating end-wall boundary condition for an even solution in the center of the potential well where $\partial_s \psi = 0$ while an ideal conducting condition can be placed at the zero of ψ if an unstable solution with a null exists. In this sense, the full space potential well problem is equivalent to the instability that has been found in previous work.[5]

We also see that in principle the instability is not connected with the existence of an end-wall. It takes place when a proper potential $u(s) - m(s) \propto \omega_E(z) \propto \nabla_\perp T_e$ exists in the plasma and the FLR term ("particle mass") $m \propto \nabla_\perp (nT_i)$ is sufficiently small.

A similar instability takes place in the potential of a nucleus[13] with a large charge number Z although this case is somewhat outside our analogy. With increasing Z the lowest eigenvalue reduces and when $Z > 137$ it becomes lower than $-m$. As a result, two electron-positron pairs are born, the positrons escape to infinity and the total charge in vicinity of the nucleus becomes less than 137.

6. Summary

We have shown that there is remarkable analogy between the Alfvén wave of the two-fluid magnetohydrodynamic equation and the one-dimension Klein-Gordon equation for zero spin charged particles in an electrostatic potential. The role of the particle rest mass is played by the term in the Alfvén wave equation, which arises from finite Larmor radius effects. The role of the electrostatic potential in the Klein-Gordon equation is played by the parameter $u(z) = \boldsymbol{k}_\perp \cdot \boldsymbol{v}_E + \omega_*/2$.

In contrast with the Klein-Gordon equation, the analog of mass for Alfvén waves can be a function of the space coordinate. However, one can still construct conservation relations in both the Klein-Gordon problem and the plasma problem. The conservation relations follow from invariance of the appropriate Lagrangian with respect to time translation and the gauge transformation. For the Klein-Gordon equation these conservation relations lead to energy and charge conservation respectively. In the analogous plasma problem, time translation symmetry can be related to wave energy conservation. The analogous charge conservation condition in the plasma does not lead to an obvious physical quantity, but we have found it useful in the determination of stability properties and estimates of growth rates. Further, we have shown that the traditional expressions in continuous medium theory for the wave momentum and energy correspond to the usual definitions of particle momentum and energy in relativistic quantum theory.

The analogy is useful for analyzing and interpreting the stability problem. In particular, "the axial shear instability" for the relativistic particle analogy, is the well known effect of spontaneous emission of particle pairs, which can occur if the variation of the potential energy $eu(z)$ is greater than $2mc^2$. Qualitatively one can say that "pair production" occurs in the vicinity of points where $k_\parallel$ vanishes. Here when the wave reflection coefficient is larger than unity there can be instability. The "particle–antiparticle" pairs created by such reflection move in an opposite direction away from the place of birth and the total excited energy is zero.

Note, that in the previous analysis of the "axial shear instability"[5] the plasma-wall interaction was emphasized. However, as our consideration shows, the "axial shear instability" is not really connected with the presence of an end-wall. In contrast the "temperature-gradient instability"[7] is due to a plasma wall interaction. The boundary condition given by Eq. (3) leads to the absorption of energy at the wall with an energy flux given by,

$$\mathcal{S} = \frac{v_\parallel}{|V_A|}\frac{2\omega_r^2}{(k_\perp \rho_i)^2}|\psi|^2. \tag{33}$$

As $\mathcal{S}$ has the sign of $v_\parallel$ this energy flux is always flowing into the wall. As a result, the reflection coefficient for a WKB wave impinging on the Debye sheath becomes larger than unity for a negative energy wave, therefore the two-wall problem becomes unstable. It is an instability arising from the self-excitation of a negative energy wave caused by sheath energy absorption by a Debye sheath at the conducting end-wall. In the quantum analogy we have the production of negative energy anti-particles inside the calculated domain that is caused by a peculiar boundary condition (whose justification is only sensible in the plasma problem) that will allow positive energy particles to be produced and absorbed in a region that is outside the domain of the quantum calculation.

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FIGURE CAPTIONS

FIG. 1. The curves $u(s) \pm m(s)$ with the five discussed regions (a)–(e) are shown. In this example the ω-value is placed in region (c).

FIG. 2. Example of pair-production instability with the functions $u(s)$ and $u(s) \pm m(s)$, where $u(s) = 0.23\,f(s)$ and $m(s) = 0.03\,f(s)$, with $f(s) = 1 + 4\,[1 - \exp(-|s-20|/3)]^2$. This result shows an unstable even eigenfunction, with the eigenvalue $\omega_1 \simeq 0.598 + i0.069$. For this case there is also an odd unstable eigenfunction with the eigenfrequency $\omega_2 \simeq 0.968 + i0.011$. These results are obtained in a parameter range where the simplifying assumptions made in the text do not apply.