On the Saturation of Multihelicity Modes

Abstract

Nonlinear interactions between unstable modes localized around rational magnetic surfaces with different helicities are studied by using a simple set of mode amplitude equations. Stability analyses of stationary solutions of the model equations show that, when low mode number rational surfaces corresponding to unstable modes with large radial widths are densely distributed, not all of these modes are allowed to equally contribute to the transport. In that case, some of the linear unstable modes are suppressed by nonlinear multi-helicity interactions in such a way that, in the radial profile of fluctuation amplitude, only a single peak appears within the radial width of the mode structure even if there exist other low mode number surfaces in the vicinity. These predictions are consistent with results of the resistive g turbulence simulations.

Keywords: multihelicity modes, anomalous transport, rational magnetic surfaces, resistive g turbulence

Microinstabilities, which are responsible for anomalous transport in magnetically confined plasmas, are usually localized in the vicinity of their mode rational (or resonant) surfaces. For nonlinear simulation of plasma turbulence in the sheared magnetic field, two-dimensional single-helicity assumption is often employed because of less computer memory and time, in which only modes with the same helicity (or the same mode rational surface) are included. However, in order to consider the real three-dimensional plasma turbulence and the resultant anomalous transport, it is important to clarify the effects of multi-helicity modes, *i.e.*, modes localized around different rational surfaces. If the mode amplitude saturation depends mainly on nonlinear interactions between the same helicity modes but weakly on those between different helicity modes, the turbulence level and resultant anomalous transport are considered to be proportional to the radial distribution density of the unstable mode rational surfaces as is argued by Beklemishev and Horton.[1,2] For the opposite case where the nonlinear coupling between the modes with different helicities is strong, a symple conjecture from the mixing length argument is that the magnitude of the anomalous transport is independent of the density of the rational surfaces since the fluctuation energy for each mode homogeneously decreases because of energy equipartition among the multi-helicity modes. Here, a simple set of model equations are used to study nonlinear interactions between the multi-helicity modes and to qualitatively explain the nature observed in multi-helicity turbulence simulations.[3,4]

Nonlinear amplitude equations of the multi-helicity modes near marginally stable states are derived for resistive g modes by Sugama *et al.*[5] as

$$dA_n/dt = \gamma A_n - \sum_{n_1,n_2} V_{n-n_1,n-n_2} A_{n-n_1+n_2} A_{n_1} A_{n_2} \tag{1}$$

where $A_n(t)$ $(n = 0, \pm 1, \pm 2, \cdots)$ denote the amplitudes of the multi-helicity modes with the lowest poloidal mode number $m = 1$ and the toroidal mode number n for the time t. The potential and pressure fluctuations are approximately given in terms of the linear eigenfunctions ϕ_1 and p_1 for the dominant $m = 1$ modes as $\widetilde{\phi} = \sum_{n=-\infty}^{\infty} A_n \phi_1(x + n\Delta) \sin[2\pi(y/L_y + nz/L_z)]$ and $\widetilde{p} = \sum_{n=-\infty}^{\infty} A_n p_1(x + n\Delta) \cos[2\pi(y/L_y + nz/L_z)]$, respectively, where x, y, and z are the coordinates in the radial, poloidal, and toroidal directions. Here $m = 1$ and $n = 1$ corresponds to the largest poloidal and toroidal wavelengths not in the whole toroidal region but in the local slab system, which are denoted by L_y and L_z. The radial interval between the adjacent $m = 1$ mode rational surfaces is given by $\Delta \equiv L_s L_y / L_z$ (L_s: the magnetic shear length) while all the $m = 1$ modes have the same linear growth rate $\gamma (> 0)$. The coefficients V_{mn} of the nonlinear term is calculated from the linear eigenfunctions. The coefficient V_{00} represents the intensity of the self-interaction or the nonlinear coupling of the modes with the same helicity and V_{mn} for $(m,n) \neq (0,0)$ represents that of the interaction between the modes with different helicities. For other types of instabilities such as drift wave instabilities, the marginally stable state can have temporal dependence represented by a real eigenfrequency ω. Then, by interpreting $A_n(t)$ as envelope of amplitude oscillations, equations similar to Eq. (1) can be obtained. It is easily found that the form of Eq. (1) is invariant under the following transformation

$$T : A_n \rightarrow (TA)_n = A_{n+1}. \tag{2}$$

This symmetry property results from the fact that most fluid model equations in the sheared slab geometry are invariant under the transformation[5]

$$T : f(x,y,z) \rightarrow (Tf)(x,y,z) = f(x-\Delta, y-\Delta z/L_s, z) \tag{3}$$

where f denotes any fluctuating field variable.

If we use

$$\tau = \gamma t, \qquad a_n = (V_{00}/\gamma)^{1/2} A_n \tag{4}$$

and retain only the self-interaction and interaction between adjacent modes, Eq. (1) is rewitten as

$$da_n/d\tau = a_n(1 - a_n^2 - \lambda a_{n-1}^2 - \mu a_{n+1}^2) \tag{5}$$

where λ and μ represent the ralative magnitude of the interaction between adjacent modes to the self-interaction defined by

$$\lambda = (V_{01} + V_{11})/V_{00}, \qquad \mu = (V_{0-1} + V_{-1-1})/V_{00}. \tag{6}$$

The parameters λ and μ are considered as increasing functions of the ratio of the radial width W of the $m=1$ mode structure $\phi_1(x)$ to the interval Δ between the adjacent $m=1$ mode rational surfaces. In the case of resistive g modes, a certain symmetry property gives $V_{mn} = V_{-m-n}$ and therefore $\lambda = \mu$. Equations for $X_n \equiv a_n^2 (\geq 0)$ are easily obtained as

$$\frac{1}{2}\frac{dX_n}{d\tau} = X_n(1 - X_n - \lambda X_{n-1} - \mu X_{n+1}). \tag{7}$$

Now, let us consider the stationary solutions and their stabilities of Eq. (7). A trivial solution is given by $X_n = 0$ for all n. Clearly, this solution is linearly unstable and has the positive growth rate of 2. Another simple stationary solution of Eq. (7) is one which satisfies T-invariance, *i.e.*, $X_{n+1} = X_n$ and is given by

$$X_n = 1/(1 + \lambda + \mu) \text{ for all } n. \tag{8}$$

Here and hereafter we assume that both λ and μ are positive. Using x_n as deviation from the stationary solution and linearizing Eq. (7), we obtain

$$\frac{1}{2}\frac{dx_n}{d\tau} = -(1 + \lambda + \mu)^{-1}(x_n + \lambda x_{n-1} + \mu x_{n+1}). \tag{9}$$

Assuming the the perturbation of the form $x_n \propto \exp(in\xi)$, the linearized equations reduce to

$$\frac{1}{2}\frac{dx_n}{d\tau} = -(1 + \lambda + \mu)^{-1}(1 + \lambda \mathrm{e}^{-i\xi} + \mu \mathrm{e}^{i\xi})x_n \tag{10}$$

Then, the perturbation x_n depends on the time t and the radial position n as $\exp[\gamma\, t - i(\omega t - \xi n)]$ where the linear growth rate γ and the frequency ω are given by

$$\gamma - i\omega = -2\frac{1 + \lambda \mathrm{e}^{-i\xi} + \mu \mathrm{e}^{i\xi}}{1 + \lambda + \mu} = -2\frac{1 + (\lambda + \mu)\cos\xi + i(-\lambda + \mu)\sin\xi}{1 + \lambda + \mu}. \tag{11}$$

Thus, the stability condition for the T-invariant stationary solution given by Eq. (8) is written as as $|\lambda + \mu| \leq 1$. When $\lambda + \mu > 1$, the perturbation with the *radial wavenumber* ξ satisfying $\cos\xi < -1/(\lambda + \mu)$ grows exponentially in time and propagates radially with the *phase velocity* ω/ξ (In the case of the resistive g modes, the frequency ω and accordingly the phase velocity vanish since $\lambda = \mu$). This stability condition implies that, when the interaction between adjacent modes is large, it is difficult to realize a radially homogeneious distribution of $m = 1$ modes with the same amplitude.

Next, we consider a T^2-invariant stationary solution of Eq. (7), which satisfies $X_{n+2} = X_n$ and is given by

$$X_n = \begin{cases} 0 & \text{for even } n \\ 1 & \text{for odd } n. \end{cases} \tag{12}$$

Linearizing Eq. (7) around this stationary solution yields

$$\frac{1}{2}\frac{dx_n}{d\tau} = \begin{cases} (1-\lambda-\mu)x_n & \text{for even } n \\ -x_n - \lambda x_{n-1} - \mu x_{n+1} & \text{for odd } n. \end{cases} \tag{13}$$

Then it is shown that this T^2-invariant stationary solution is unstable for $\lambda + \mu < 1$ (when the T-invariant stationary solution is stable) and stable for $\lambda + \mu > 1$ (when the T-invariant stationary solution is unstable).

Thus stabililty is exchanged between two stationary solutions (8) and (12) at $\lambda + \mu = 1$. Radial profiles of the nonlinear transport flux $\langle \tilde{p}\tilde{v}_x \rangle$ ($\tilde{v}_x \equiv -(c/B)\tilde{\phi}/\partial y$, $\langle \cdot \rangle \equiv \int_0^{L_y} \int_0^{L_z} dydz/L_y L_z$) corresponding to these solutions are schematically shown in Figs.1 (a) and (b). Similar behaviours of the $m = 1$ modes are observed in simulations of the resistive g turbulence as shown in Figs.2 and 3. Figures 2 and 3 show radial profiles of the nonlinear transport flux $\langle \tilde{p}\tilde{v}_x \rangle$ and contours of the electrostatic potential $\tilde{\phi}$ on the (x, y)-plane obtained by multi-helicity simulations of the resistive g turbulence for different values of the interval Δ (see Ref. 3 for details of the simulations). It is clearly seen that some of adjacent $m = 1$ modes almost disappears when their interval is decreased. Although that simulation is far from the marginally stable states, the $m = 1$ modes have dominant contributions to the total energy in the saturated state and model equations like Eq. (1) is considered to be qualitatively a good approximation.

It is shown in the same way as above that, when the relative intensity of adjacent mode is so large that $\lambda > 1$ and $\mu > 1$, there appears another stable stationary

(T^3-invariant) solution consisting of sets of three modes in which two have vanishing amplitudes and one has an amplitude of unity. In this case, random combinations of this solution and (12) also yield stable stationary solutions. An example of such stable solutions is schematically shown in Fig.4. Strong interaction between the modes with different helicities suppresses two $m = 1$ modes in a row. However it cannot damp three modes in a row since the middle one of them is always linearly unstable.

For simplicity, let us assume that $\lambda = \mu$ as in the case of resistive g modes. For $\lambda < 1/2$, all stationary solutions including some $m = 1$ modes with vanishing amplitudes are unstable since pertubations given to such $m = 1$ modes grow exponentially in time. On the other hand, for $\lambda > 1$, interaction between different helicity modes does not permit stablity of two modes in a row with finite amplitudes and stable solutions are only (12) and those as given in Fig.4. We find that there exist a variety of other types of stationary solutions which is stable only for some finite parameter region with both lower and upper boundaries. For example, it is shown that some stationary solutions as given in Fig. 5 is stable for $(-1 + \sqrt{5})/2 < \lambda < 1$. There we can see the two modes in a row with finite amplitudes.

In summary, we have found several stationary solutions of (7) and their stability parameter region. There are still stable stationary solutions other than described above, although they have only finite stability region inside $1/2 < \lambda < 1$. Here let us consider what results obtained here suggest to multi-helicity turbulence and to resultant anomalous transport. We may expect that there are several patterns of radial mode distribution profiles in saturated states of multi-helicity turbulence, which depends on relative intensity of interaction between the same helicity modes and that between different helicity modes. We should note that only self-interaction and adjacent mode interactions are taken into account in (7) although higher order multi-helicity interactions should be included when the ratio of the mode width to the mode interval is so large. However, it is considered as a general tendency that, when the interaction with different-helicty modes is large, homogeneous distribution of the modes with the same amplitude is unlikely to occur and solutions as in (12) and Fig.4 are preferable. As mentioned previously, this tendendcy has been confirmed by the multi-helicity simulations of the registive g turbulence.[3,4] Analytical results here give support to the conclusion in Ref. 3 (also in Ref. 4) that, within the width of the $m = 1$ modes, there exists only a single peak in the convective flux even when there are other $m = 1$ mode rational surfaces within the mode width. Thus, when the multi-helicity interactions increase, simple pictures described earlier are not valid, *i.e.*, the total transport is not simply proportional to the radial distribution density of the unstable modes, which is not because all unstable mode amplitudes decrease homogeneously but because some unstable modes are eaten by others.

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Figure Captions

FIGURE 1 Radial profiles of the nonlinear transport flux $\langle\widetilde{p}\widetilde{v}_x\rangle$ caused by the $m = 1$ modes for $\lambda+\mu < 1$ (a) and for $\lambda+\mu > 1$ (b). Radial positions of the $m = 1$ modes are denoted by crosses. Stationary solutions given by (8) and (12) are shown in the left- and right-hand sides, respectively. Solid (dashed) curves correspond to stable (unstable) solutions.

FIGURE 2 Radial profiles of the nonlinear transport flux $\langle\widetilde{p}\widetilde{v}_x\rangle$ (top) and contours of the electrostatic potential $\widetilde{\phi}$ on the (x, y)-plane (bottom) obtained by multi-helicity simulations of the resistive g turbulence. Solid (dotted) curves correspond to positive (negative) values of the potential. Positions of the $m = 1$ mode rational surfaces are represented by dashed vertical lines. The interval between adjacent $m = 1$ mode rational surfaces is $\Delta = 20$.

FIGURE 3 Radial profiles of the nonlinear transport flux $\langle\widetilde{p}\widetilde{v}_x\rangle$ (top) and contours of the electrostatic potential $\widetilde{\phi}$ on the (x, y)-plane (bottom) obtained by multi-helicity simulations of the resistive g turbulence. Solid (dotted) curves correspond to positive (negative) values of the potential. Positions of the $m = 1$ mode rational surfaces are represented by dashed vertical lines. The interval between adjacent $m = 1$ mode rational surfaces is $\Delta = 7.5$.

FIGURE 4 Radial profile of the nonlinear transport flux $\langle\widetilde{p}\widetilde{v}_x\rangle$ caused by the $m = 1$ modes which correspond to a stable stationary solution for $\lambda = \mu > 1$.

FIGURE 5 Radial profile of the nonlinear transport flux $\langle\widetilde{p}\widetilde{v}_x\rangle$ caused by the $m = 1$ modes which correspond to a stable stationary solution for $(-1 + \sqrt{5})/2 < \lambda = \mu < 1$.