

Plasma transport near the separatrix of a magnetic island

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Abstract

The simplest non-trivial model of transport across a magnetic island chain in the presence of collisionless streaming along the magnetic field is solved by a Wiener-Hopf procedure. The solution found is valid provided the boundary layer about the island separatrix is narrow compared to the island width. The result demonstrates that when this assumption is satisfied the flattened profile region is reduced by the boundary layer width. The calculation is similar to the recent work by Fitzpatrick [R. Fitzpatrick, *Phys. Plasmas* **2**, 825 (1995)] but is carried out in the collisionless, rather than the collisional, limit of parallel transport, and determines the plasma parameters on the separatrix self-consistently.

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I. Introduction

The dynamics of magnetic islands in tokamaks is currently a topic of intense research. It was shown by Carrera, Hazeltine, and Kotschenreuther,¹ and by Qu and Callen,² that perturbing the bootstrap current caused by an island tends to make the island grow further (if the magnetic shear is positive), thus providing a powerful drive for instability. More recent calculations^{3,4} have included the effect of the ion polarization drift, which was shown to be able to stabilize sufficiently narrow islands. However, islands whose initial width exceeds some threshold grow because of the bootstrap drive. Such a threshold appears to have been observed experimentally in the Tokamak Fusion Test Reactor (TFTR).⁵ The need to re-examine the existence of a threshold due to the ion polarization drift has also been pointed out.⁶

An alternative explanation of the threshold has recently been proposed by Fitzpatrick,⁷ and Gorelenkov and co-workers,⁸ who point out that the density and temperature profiles are not flattened across a sufficiently narrow island because of cross-field transport. The bootstrap current is therefore not significantly perturbed and the instability drive never appears. Fitzpatrick⁷ also discussed the detection of magnetic islands by electron cyclotron emission and argued that a narrow island should be virtually undetectable because the electron temperature profile is not flattened over the island. In contrast, the earlier calculations¹⁻⁴ considered islands wide enough to cause complete flattening of the profiles by ignoring the details of the boundary layer about the island separatrix.

The transport properties associated with the plasma in the neighborhood of a magnetic island are important, especially as the nonlinear growth of magnetic islands poses a serious threat to the development of reactor relevant devices. Experiments are planned in a number of existing tokamaks to apply localized current drive and heating to control the growth of

islands.

The purpose of the present paper is to clarify the nature of transport across a magnetic island by solving a simplified kinetic equation in full island geometry. For simplicity we treat only the limit of large island width, in which the transport boundary layer surrounding the island separatrix is narrow in comparison with the island—a limit considered in all the aforementioned theoretical papers. Even this problem is mathematically non-trivial because the magnetic field lines change topology across the separatrix and because boundary data cannot be specified on the separatrix itself.

Our calculation demonstrates the essential features of transport across a magnetic island, which enforce a certain structure on spatial gradients in the island vicinity. While the paper by Fitzpatrick⁷ considers cross-field transport in the limit of collisional parallel transport, we are interested in the opposite, collisionless limit more relevant to high temperature plasmas. This limit was treated in the paper by Gorelenkov *et al.*,⁸ who, however, did not solve the transport equation in the boundary layer.

In Sec. II, the model kinetic equation is presented and the formalism necessary for dealing with the island geometry is developed. In the following four sections, the boundary conditions are discussed and the equation is solved by a Wiener-Hopf technique,^{9,10} with the details of the factorization given in Sec. V and the results presented in Sec. VI. These results are summarized in the last section.

II. Island coordinates

We suppose that diffusion across flux surfaces competes on an equal footing with free streaming along the magnetic field. Guiding center drifts due to finite Larmor radius are neglected. Hence we study the kinetic equation

$$v_{\parallel} \nabla_{\parallel} f = \nabla \cdot (D \nabla f), \tag{1}$$

where the \parallel subscript refers to the direction of the magnetic field \mathbf{B} , f is the distribution function and D is a diffusion coefficient. The diffusion is assumed to be caused by small-scale plasma turbulence, and can be represented as in (1) if the characteristic time step in the random walk taken by a particle in the turbulent field is shorter than the transit time around the island.

Diffusion is ordinarily a much slower process than parallel streaming; the two processes compete here only because the radial scale-length, w , of f —the scale-length for variation normal to the magnetic flux surfaces—is assumed to be exceptionally short in the region of interest. Thus we have the basic ordering

$$\frac{v_t}{L_{\parallel}} \sim \frac{D}{w^2}, \quad (2)$$

where v_t is the ion thermal speed and L_{\parallel} is the parallel scale length. This ordering is conventional for tokamak boundary layers, such as that arising near a bounding wall; here we apply it to the neighborhood of a magnetic island separatrix. The scale-length L_{\parallel} is the characteristic length of a magnetic field line for one turn around the island, estimated in Eq. (18) below. We suppose that this length is larger than the island width W , which, in turn, exceeds the radial scale length w :

$$L_{\parallel} \gg W \gg w.$$

Toroidal curvature is not important for small W , so we can model the magnetic field using cylindrical coordinates (r, β, z) :

$$\mathbf{B} = B_0 [\hat{\mathbf{z}} + \hat{\mathbf{z}} \times \nabla\psi(r, \beta)]. \quad (3)$$

The z -axis gives the direction of the equilibrium field \mathbf{B}_0 , r corresponds to the minor radius of the torus and β is the helical angle on which the perturbation depends. The field perturbation is introduced through its perturbed (helical) flux, measured by ψ .

Since the ψ in (3) has the dimensions of length, and since it satisfies

$$\mathbf{B} \cdot \nabla \psi = 0,$$

we can use ψ as a radial coordinate in the perturbed field. Since we assume the distribution to have helical symmetry, its natural coordinates are ψ and s ,

$$f(r, \beta, z, \mathbf{v}) \rightarrow f(\psi, s, \mathbf{v}),$$

where s measures distance along \mathbf{B} , and where

$$\frac{\partial f}{\partial \psi} \sim \left(\frac{L_{\parallel}}{w} \right) \frac{\partial f}{\partial s}.$$

It follows in particular that, to lowest order in w/W ,

$$\nabla \cdot (D \nabla f) \cong D |\nabla \psi|^2 \frac{\partial^2 f}{\partial \psi^2}$$

and our kinetic equation becomes

$$\sigma u \frac{\partial f}{\partial s} = D |\nabla \psi|^2 \frac{\partial^2 f}{\partial \psi^2} \tag{4}$$

where $\sigma = \pm 1$ is the sign of the parallel velocity and $u = |v_{\parallel}|$ its magnitude. We choose the perturbation ψ to correspond to an $m = 2$ magnetic island (for concreteness):

$$\psi(r, \beta) = \frac{1}{L_s} (\Delta r^2 - W^2 \cos 2\beta) \tag{5}$$

Here L_s is the shear length of the equilibrium field, W is the (half-) island width and $\Delta r \equiv r - a$ is the radial distance from the surface at $r = a$ where islands are centered. Note that ψ has the value W^2/L_s on the island separatrix. The field-line trajectories are given by $\Delta r(\psi, \beta)$, where ψ labels the contour and β varies along it. It is convenient to introduce $k(\psi)$ where

$$k^2 = \frac{L_s \psi + W^2}{2W^2}.$$

Then we have

$$\Delta r(\psi, \beta) = \sqrt{2}kW\sqrt{1 - k^{-2}\sin^2\beta} \quad (6)$$

We see that the region inside the island (where β has a limited range) corresponds to $k < 1$, with $k = 0$ at the island magnetic axis and $k = 1$ at the separatrix.

In the next section we solve (4) by Fourier transformation in its radial variable. Here we note that in its present form (4) is not amenable to Fourier transformation because its coefficients are strong functions of radius. Although we assume, for simplicity,

$$D = \text{constant},$$

the quantity $\nabla\psi$ necessarily varies from its nominal value, ψ/W , to zero at the island x -points. Hence we introduce new, dimensionless coordinates $(\psi, s) \rightarrow (x, \theta)$ according to

$$\frac{\partial\theta}{\partial s} \equiv N(\psi)\frac{D|\nabla\psi|^2}{u}, \quad (7)$$

$$\frac{\partial x}{\partial\psi} \equiv N^{1/2}(\psi), \quad (8)$$

where the normalizing factor N is a slow function of ψ that will be chosen presently. We shall see that the second term in

$$\frac{\partial f}{\partial\psi} = \frac{\partial x}{\partial\psi}\frac{\partial f}{\partial x} + \frac{\partial\theta}{\partial\psi}\frac{\partial f}{\partial\theta} \quad (9)$$

makes a relatively small contribution (outside a very narrow neighborhood of the separatrix).

Hence the kinetic equation (4) is transformed into

$$\frac{\partial f}{\partial\theta} = \sigma\frac{\partial^2 f}{\partial x^2}. \quad (10)$$

for any choice of N .

We fix N by requiring θ to have a natural periodicity outside the island:

$$\frac{D}{u}N\int|\nabla\psi|^2 ds = \pi \quad (11)$$

where the integral is performed at fixed ψ over a distance of one island length. Using

$$\nabla = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\beta}} \frac{1}{a} \frac{\partial}{\partial \beta},$$

we find that

$$|\nabla\psi|^2 = \frac{4}{L_s^2} \left(\Delta r^2 + \frac{W^4}{a^2} \sin^2 2\beta \right)$$

Because $a \sim L_{\parallel} \gg W$ the second term is small and neglected for simplicity. Then, since

$$ds = B \frac{d\beta}{\mathbf{B} \cdot \nabla \beta} \cong \frac{aL_s}{2\Delta r} d\beta$$

our periodicity condition becomes

$$\frac{2aDN}{uL_s} \int_{-\pi/2}^{\pi/2} d\beta \Delta r = \pi$$

or, in view of (6),

$$N = \frac{\pi u L_s}{4\sqrt{2} a D W k E(k^{-2})} \quad (12)$$

where E is the complete elliptic integral of the second kind. Note that this quantity is finite for $k \rightarrow 1$.

Now (7) provides the angle variable (outside the island)

$$\theta(\psi, \beta) = \frac{\pi E(\beta, k^{-2})}{2E(k^{-2})} \quad (13)$$

in terms of the incomplete elliptic integral $E(\beta, k^{-2})$.

Inside the island, (12) and (13) are meaningless, but we have a conventional prescription for θ . First define $\phi(\beta, \psi)$ according to

$$\sin \phi \equiv k^{-1} \sin \beta. \quad (14)$$

and then use the identity

$$kE(\beta, k^{-2}) = E(\phi, k^2) + (k^2 - 1)F(\phi, k^2)$$

whose right-hand side is meaningful inside the separatrix. Here F is the incomplete elliptic integral of the first kind. Thus, inside the island,

$$\theta(\psi, \beta) = \frac{\pi}{2} \frac{E(\phi, k^2) + (k^2 - 1)F(\phi, k^2)}{E(k^2) + (k^2 - 1)K(k^2)} \quad (15)$$

where K is the complete elliptic integral of the first kind.

While the range of β is limited inside the island, (14) allows ϕ to vary from $-\pi$ to π over one loop of an interior surface. It can be seen to follow from (15) that θ also has the range $-\pi \rightarrow \pi$. Note that both angles remain well-defined at the island separatrix, except at the endpoints where both suffer logarithmic singularity. Indeed, the definition of θ given by (13) and (15) makes that function fully continuous across the separatrix. It is not a conventional island angle, because of the $|\nabla\psi|^2$ -factor in (7). Without that factor the angle variable would be singular on the island separatrix, where field lines become indefinitely long.

The radial variable is found from (8), which, because of the square root, allows a free choice of signs. We choose $x = 0$ on the separatrix, and $x > 0$ inside it. The two separated exterior regions, corresponding to $r > a$ and $r < a$, will both have $x < 0$. Hence $x(\psi)$ is defined by

$$x(\psi) = 2 \left(\frac{\pi}{\sqrt{2}} \right)^{1/2} \left(\frac{uW^3}{aDL_s} \right)^{1/2} \int_{k(\psi)}^1 \frac{k^{1/2} dk}{[E(k^{-2})]^{1/2}} \quad (16)$$

outside the island, and

$$x(\psi) = 2 \left(\frac{\pi}{\sqrt{2}} \right)^{1/2} \left(\frac{uW^3}{aDL_s} \right)^{1/2} \int_{k(\psi)}^1 \frac{k dk}{[E(k^2) + (k^2 - 1)K(k^2)]^{1/2}} \quad (17)$$

inside.

These definitions do not distinguish the two regions outside the separatrix. In fact we do so not through x but instead using θ : we associate the range $-\pi < \theta < 0$ with the region “below” the island chain ($r < a$), and the range $0 < \theta < \pi$ with the upper region ($r > a$). This arrangement is faithful to the true island topology provided we imagine the line along $\theta = 0$ for $x < 0$ to be an impenetrable barrier.

It is convenient at this point to re-examine our orderings. First note from (8) that the layer width in ψ is measured by

$$\Delta\psi \sim \frac{1}{\sqrt{N}} \sim \sqrt{\frac{aDW}{v_t L_s}}$$

Alternatively we can estimate $\Delta\psi$ from (4) with the result

$$\Delta\psi \sim \frac{W}{L_s} \sqrt{\frac{L_{\parallel} D}{v_t}}.$$

The two expressions agree since

$$L_{\parallel} \sim \frac{aL_s}{W} \gg L_s. \quad (18)$$

They are also consistent with the layer width measured in ordinary radius r ,

$$w \sim \frac{\Delta\psi}{\partial\psi/\partial r},$$

as can be seen from (2):

$$w \sim \sqrt{\frac{DL_{\parallel}}{v_t}}.$$

Figure 1 shows the resulting configuration. The island interior is labeled as region II; the regions outside the separatrix, below and above the island chain, are labelled I and III respectively.

A key assumption in the consistency of these estimates is that the separatrix layer be thin compared to the island, $w \ll W$. This requires an island of some size; in view of (18), we must have

$$W \gg \left(\frac{aL_s D}{v_t}\right)^{1/3}. \quad (19)$$

When (19) is satisfied we can assume $\Delta r \sim W$ and $k \sim 1$ throughout the region of interest, including the asymptotic domains where $x \gg 1$.

It remains to justify the neglect of the second term on the right in (9), which requires

$$\frac{\partial \ln f}{\partial \psi} \gg \frac{\partial \theta}{\partial \psi}.$$

Since

$$\frac{\partial \ln f}{\partial \psi} = \frac{\partial r}{\partial \psi} \frac{\partial \ln f}{\partial r} = O\left(\frac{L_s}{Ww}\right),$$

$$\frac{\partial \theta}{\partial \psi} = \frac{\partial k^2}{\partial \psi} \frac{\partial \theta}{\partial k^2} = O\left(\frac{L_s}{W^2} \ln|1-k|\right)$$

this condition is equivalent to

$$\ln|1-k| \ll W/w,$$

which is satisfied everywhere but very close to the separatrix: it fails only in an exponentially narrow [$\sim \exp(-W/w)$] vicinity of the separatrix. The point is that the boundary layer associated with the transport is considerably wider than this sublayer, which has to do with the logarithmic singularity of the parallel scale length L_{\parallel} . The kinetic equation (9) is therefore sufficiently accurate for our purposes.

III. Boundary conditions

Of course the diffusion equation (10) is simple and conventional. What is distinctive about island transport is the nature of the boundary conditions, which change across the separatrix. Inside the separatrix, where $x > 0$, the distribution is periodic in θ with period 2π ; outside the separatrix there are two separated regions, corresponding to $r > a$ and $r < a$, in which the period is π . Referring to Fig. 1 we can state the periodicity conditions on $f(x, \theta)$ as

$$f(x, -\pi) = f(x, 0-), \text{ in region I}; \tag{20}$$

$$f(x, -\pi) = f(x, \pi), \text{ in region II}; \tag{21}$$

$$f(x, 0+) = f(x, \pi), \text{ in region III}. \tag{22}$$

Outside the island chain (but in its vicinity) we suppose that the distribution uniformly increases in radius. It is the interruption of this constant gradient, by the island separatrices

and interior, that we wish to study. For convenience we consider the f in our kinetic equation to be the difference between the actual distribution and its value, f_0 , at the island o-point; this is permissible since (4) is unchanged by the addition of a (spatial) constant to f . Then, outside the island chain, $f(r, \beta)$ is odd with respect to the variable $r - r_s$ and $f(x, \theta)$ changes sign across the barrier separating regions I and III :

$$f(x, -\theta) = -f(x, \theta), \quad \text{for } x < 0. \quad (23)$$

The change in boundary conditions across the separatrix forces the distribution to vary with θ in a layer of width w surrounding the θ -axis. Outside that layer we expect f to become constant on flux surfaces (independent of θ); the diffusion equation then requires f to be linear in x for large x . Hence the asymptotic boundary conditions are

$$f(x, \theta) \rightarrow 0, \quad x \rightarrow \infty; \quad (24)$$

$$f(x, \theta) \rightarrow c_1 x + c_0, \quad x \rightarrow -\infty, \theta < 0; \quad (25)$$

$$f(x, \theta) \rightarrow -(c_1 x + c_0), \quad x \rightarrow -\infty, \theta > 0. \quad (26)$$

Note that, when viewed on the macroscopic scale, the distribution appears discontinuous across the separatrix layer. It is this jump in f ,

$$\Delta f \equiv f_{III} - f_I = -2c_0 \quad (27)$$

that drives the diffusion process being considered.

A salient conclusion of our analysis will be that the coefficients c_0 and c_1 cannot be set independently by conditions far from the layer. Instead the diffusion equation with its boundary data enforce a linear relation between them.

IV. Fourier analysis

The mixed boundary data forces us to consider half-range Fourier transforms, defined by integrals over positive or negative x . These functions have simple analyticity properties, allowing the full solution to be extracted from the boundary data by function theoretic argument. Our procedure, based on the Wiener-Hopf technique, has been used frequently in plasma kinetic theory; a previous study¹¹ of magnetic trapping in tokamaks is especially close to the present analysis.

Thus we express the Fourier transform of $f(x, \theta)$ as

$$F(p, \theta) = F_u(p, \theta) + F_l(p, \theta)$$

with

$$F_u = \int_0^\infty dx e^{ipx} f(x, \theta) \quad (28)$$

$$F_l = \int_{-\infty}^0 dx e^{ipx} f(x, \theta) \quad (29)$$

Here the subscripts refer to analyticity properties: F_u (F_l) is analytic in the upper-(lower-)half p -plane.¹⁰ Our differential equation (10) becomes

$$\frac{\partial F}{\partial \theta} = -\sigma p^2 F.$$

It is convenient to express the solution in terms of $F_0 = F(p, 0-, \sigma)$. Then we have, for $\theta < 0$,

$$F(p, \theta, \sigma) = F_0(p, \sigma) e^{-\sigma p^2 \theta}. \quad (30)$$

(The dependence on σ will be left implicit when it is not essential to the argument.) Since the form of F for positive θ can be found from (30) and symmetry, our only remaining task is to determine $F_0(p)$, using the boundary data.

To simplify notation we introduce the abbreviations

$$U(p) = F_u(p, -\pi),$$

$$L(p) = F_l(p, -\pi),$$

$$M(p) = F_u(p, 0).$$

It can be seen that

$$F_0 = e^{-\pi\sigma p^2}(U + L) \tag{31}$$

so it suffices to determine U and L .

The large- x behavior of f , given by (24)–(26), fixes the small- p behavior of its transform; in particular we find that

$$L(p) \rightarrow \frac{c_1}{p^2} - i\frac{c_0}{p}, \tag{32}$$

for $p \rightarrow 0$. On the other hand $U(0)$ is finite.

We also observe that regularity of $f(x, \theta)$ at $x = 0$ requires both U and L to decay at least as fast as $1/p$ for large $|p|$:

$$L \sim U \sim \frac{1}{p} \tag{33}$$

for $|p| \rightarrow \infty$.

Considering next the angular data, we note that (20)–(22) imply

$$F(p, -\pi) = L + U,$$

$$F(p, 0-) = L + M,$$

$$F(p, 0+) = -L + M,$$

$$F(p, \pi) = -L + U$$

We combine these results with (30) to infer

$$\begin{aligned} L + M &= (L + U)e^{-\pi\sigma p^2}, \\ -L + U &= (-L + M)e^{-\pi\sigma p^2}. \end{aligned}$$

or, after eliminating M ,

$$\frac{U(p)}{L(p)} = \sigma V(p) \tag{34}$$

where

$$V(p) \equiv \tanh \frac{\pi p^2}{2}.$$

The procedure starting from (34) is conventional.¹⁰ One supposes that V possesses a Wiener-Hopf factorization of the form

$$V(p) = \frac{V_l(p)}{V_u(p)} \tag{35}$$

where V_l (V_u) is analytic in the lower (upper) half p -plane, and where both factors are linear in p for large p :

$$V_l \sim V_u \sim p, \quad p \rightarrow \infty. \tag{36}$$

Now (34) implies

$$U(p)V_u(p) = \sigma L(p)V_l(p). \tag{37}$$

The Wiener-Hopf argument then constructs a function $A(p)$ that is defined by the left-hand side of (37) in the upper half p -plane, and by its right-hand side in the lower half. Then $A(p)$ is analytic in the cut-plane; since (37) states that A is also continuous across the cut, we can infer (under mild mathematical restrictions⁹) that $A(p)$ is an entire function. But note that (33) and (36) imply that $A(p)$ approaches a constant, C_σ , for $|p| \rightarrow \infty$. Since the only entire function bounded by a constant at infinity is constant everywhere, we conclude that

$$U(p) = \frac{C_\sigma}{V_u(p)}, \tag{38}$$

$$L(p) = \sigma \frac{C_\sigma}{V_l(p)}. \quad (39)$$

Thus both functions U and L are determined from the single equation (34).

It is now clear that, as we have remarked, the coefficients c_0 and c_1 of (25) cannot be independent: only a single free constant enters our expressions for U and L . From (32) we see that the relation between c_0 and c_1 is fixed by the form of V_l for small p . Hence we need to make the factorization explicit.

V. Factorization

We find the Wiener-Hopf factors V_u and V_l by a conventional procedure,⁹ observing first that the function

$$V(z) = \frac{\sinh(\pi z^2/2)}{\cosh(\pi z^2/2)}$$

has poles at

$$z = \pm\sqrt{2n+1}e^{i\pi/4},$$

and at

$$z = \pm\sqrt{2n+1}e^{i3\pi/4},$$

where $n = 0, 1, 2, \dots$. It has zeroes at

$$z = \pm\sqrt{2n}e^{i\pi/4},$$

and at

$$z = \pm\sqrt{2n}e^{i3\pi/4}.$$

Note that the zero at the origin ($n = 0$) is second order. It follows that the function

$$q(z) \equiv \log \left[\left(\frac{z^2+1}{z^2} \right) V(z) \right], \quad (40)$$

is analytic and nonvanishing in a neighborhood of the real- z axis. Within this neighborhood we construct the path C_a , parallel to the real axis but displaced a short distance above it, in order to define

$$q_l(p) \equiv \frac{1}{2\pi i} \int_{C_a} dz \frac{q(z)}{z-p}.$$

It is clear that this function is analytic in the lower-half p -plane. Similarly we define

$$q_u(p) \equiv \frac{1}{2\pi i} \int_{C_b} dz \frac{q(z)}{z-p}$$

where the path C_b is displaced a short distance below the real axis. Since Cauchy's theorem implies

$$q_u(p) - q_l(p) = q(p)$$

we have found the relation

$$\left(\frac{p^2 + 1}{p^2} \right) V(p) = e^{q_u(p) - q_l(p)}$$

and it is straightforward to identify

$$V_l(p) = e^{-q_l(p)} \frac{p^2}{p-i}, \quad (41)$$

$$V_u(p) = e^{-q_u(p)} (p+i). \quad (42)$$

Notice that these functions have the asymptotic behavior anticipated in (36).

Recall that the asymptotic slope of the distribution near the layer is fixed by the behavior of $V_l(p)$ for small p . Therefore we consider the Taylor expansion of V_l :

$$V_l(p) = ie^{-q_l(0)} p^2 \left[1 - ip - q'_l(0)p + O(p^2) \right]. \quad (43)$$

Here

$$\begin{aligned} q'_l(0) &\equiv \left. \frac{dq_l(k)}{dk} \right|_{k=0} \\ &= \frac{1}{2\pi i} \int_{C_a} \frac{dz}{z^2} q(z) \\ &= \frac{1}{2\pi i} \int_{C_a} \frac{dz}{z} q'(z). \end{aligned} \quad (44)$$

From (40) we find

$$q'(z) = \frac{2z}{z^2 + 1} - \frac{2}{z} + \frac{2\pi z}{\sinh \pi z^2}$$

so (44) yields

$$q'_l(0) = -i - i \int_{C_a} \frac{dz}{\sinh \pi z^2}.$$

It is shown elsewhere¹¹ that

$$\int_{C_a} \frac{dz}{\sinh \pi z^2} = \sqrt{2} (\sqrt{2} - 1) \zeta(1/2)$$

where ζ denotes the Riemann zeta-function: $\zeta(1/2) \cong -1.46$. Substituting these results into (43) we have

$$V_l(p) = i e^{-a_l(0)} p^2 \left[1 + i \sqrt{2} (\sqrt{2} - 1) \zeta(1/2) p + O(p^2) \right]. \quad (45)$$

It follows in particular that $U \propto V/V_l$ is finite at $p = 0$, as required.

We can also verify that the distribution becomes independent of θ for large x . The point is that

$$\frac{\partial f}{\partial \theta} = \frac{-\sigma}{2\pi} \int dp e^{-ipx} p^2 F(p),$$

where $p^2 F$, unlike F itself, is regular for all real p . Hence phase-mixing will make its inverse transform vanish for large x .

VI. Distribution function

Having determined

$$F_0 = C_\sigma e^{-\pi\sigma p^2} \left(\frac{1}{V_u} + \sigma \frac{1}{V_l} \right) \quad (46)$$

we could now in principle find the complete boundary layer distribution by inverse Fourier transformation. However the resulting integration problem would bring scant reward, since the details of layer structure have little observable effect. Hence we are content to point out salient features of $f(x, \theta)$.

Note first that (46) can be expressed as

$$F_0 = \frac{\sigma C_\sigma}{V_l(p)} [1 + \sigma V(p)] e^{-\sigma p^2 \pi}$$

Thus, after straightforward manipulation using the explicit form of $V(p)$, we can write the Fourier transform of the distribution as

$$F(p, \theta, \sigma) = \frac{\sigma C_\sigma}{V_l(p)} \frac{e^{-\sigma p^2 (\theta + \pi/2)}}{\cosh(\pi p^2 / 2)} \quad (47)$$

We next evaluate C_σ . Equation (32) fixes the small- p behavior of V_l according to

$$V_l(p) = \frac{\sigma C_\sigma}{c_1} p^2 \left(1 + \frac{i c_0}{c_1} p \right). \quad (48)$$

Comparing this result to (45) we infer

$$\alpha \equiv -\frac{c_0}{c_1} = -\sqrt{2} (\sqrt{2} - 1) \zeta(1/2) \quad (49)$$

$$\cong 0.855, \quad (50)$$

and

$$\sigma C_\sigma = -i \frac{c_0}{\alpha} e^{-q_l(0)}.$$

To compute $q_l(0)$ we let the contour C_a approach the real axis and use the Plemelj formula to conclude

$$q_l(0) = -\frac{1}{2} \log \frac{\pi}{2}.$$

Thus

$$\sigma C_\sigma = -i \frac{\sqrt{\pi/2}}{\alpha} c_0 = -i 1.46 c_0.$$

We expect the asymptotic distribution, c_0 , to be even in parallel velocity and therefore infer

$$C_+ = -C_- \quad (51)$$

With (51) in mind we return to (47) and note the symmetry $F(p, \theta, \sigma) = F(p, -\theta - \pi, -\sigma)$, which implies

$$f(x, \theta, \sigma) = f(x, -\theta - \pi, -\sigma). \quad (52)$$

This relation is consistent with the asymptotic boundary data.

More interesting is the relation (49) between the distribution outside the layer and its slope. In a linear tearing mode, conditions inside the tearing layer determine the change in asymptotic slope of the field perturbation, $d\psi/dx$, across it; the value of ψ itself is continuous across the layer. The present, nonlinear description of the distribution function is similar, differing only in that the distribution itself, and not just its slope, will appear discontinuous across the layer when viewed on the macroscopic scale. Thus macroscopic views of the upper and lower separatrices, insensitive to the boundary layer structure, would show a jump in the distribution:

$$\Delta f = -2c_0$$

To characterize this jump we consider region I for definiteness; from (49),

$$\Delta f = 2c_1\alpha,$$

with

$$c_1 = \frac{\partial f}{\partial x} = \frac{1}{\sqrt{N}} \frac{\partial f}{\partial \psi}. \quad (53)$$

Thus

$$\Delta f = 2\alpha \left(\frac{1}{\sqrt{N}} \frac{\partial f}{\partial \psi} \right)_I \quad (54)$$

Recalling that $1/\sqrt{N} = \Delta\psi$ we obtain the estimate

$$\Delta f \sim \Delta\psi \frac{\partial f}{\partial \psi} \sim w \frac{\partial f}{\partial r}.$$

The estimate is not surprising, but note that it involves only the layer width w , rather than the much larger island width.

The same macroscopic view will ascribe the value

$$f_s = f_0 + c_0$$

to the distribution function on the inner (region I) island separatrix. Here f_0 denotes the value of the distribution on the island o-point—the locally constant distribution that was introduced to make f change sign across the island chain. It is consistent with our $W \gg w$ ordering to consider f_0 as an experimentally measurable quantity. The slope of the distribution as it approaches the separatrix in region I, according to (53), is

$$\frac{\partial f}{\partial \psi} = -\frac{\sqrt{N}}{\alpha}(f_s - f_0)$$

with a corresponding expression in region III. Here, to lowest order in w/W , N can be replaced by its value at $k = 1$,

$$N(1) = \frac{\pi u L_s}{4\sqrt{2}aDW},$$

whence

$$\frac{\partial f}{\partial \psi} = -0.875\sqrt{\frac{uL_s}{aDW}}(f_s - f_0) \quad (55)$$

We observe in particular that the gradient is steepest for rapidly streaming (large u) particles.

VII. Discussion

This work demonstrates rigorously an unsurprising circumstance: at low collisionality, the change in the distribution function across an island chain occurs almost entirely in the thin boundary layer surrounding each separatrix. Therefore, as long as the boundary layer is small compared to the island width, the radial extent of the flattened profiles is proportional to the island width, and the peak temperature and density that can be sustained in the core for given edge values is reduced accordingly. For example, Tore-Supra¹² employs an ergodic divertor to widen the scrape-off layer (SOL) and thereby decreases the heat load on the divertor plates. To avoid reducing the peak temperature and density that can be

sustained in the core, the width of the non-ergodic island chains adjacent to the SOL should be kept comparable to or smaller than the boundary layer width about the island separatrix. Similarly, islands generated by error or applied helical fields must be kept small to avoid reducing the on-axis density and temperature for given edge values.

The present analysis of boundary-layer structure differs from a similar calculation by Fitzpatrick⁷ in two ways. First, we consider the collisionless, rather than the collisional, limit of transport along the field. Second, we do not impose boundary conditions on the separatrix *a priori* but determine the conditions self-consistently. Our procedure resolves details of boundary layer that affect the stability of islands in the core. Indeed, the stability calculation by Wilson *et al.*⁴ shows that a significant fraction of the ion polarization current originates inside the boundary layer, whose structure is therefore expected to influence the final stability criterion in a more complete analysis—a topic we leave to future work.

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FIGURE CAPTIONS

FIG. 1. The original island geometry (a) and its rearrangement (b). Note the regions I, below the island chain and outside its separatrix; II, inside the separatrix; and III, above the island chain and outside its separatrix. The two structures are physically equivalent provided the thick solid line in (b) is supposed to be impenetrable.