Nonlinear dynamics of feedback modulated magnetic islands in toroidal plasmas

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We present a comprehensive analysis of the dynamics of a helical magnetic island chain, embedded in a toroidal plasma, in the presence of an externally imposed, rotating, magnetic perturbation of the same helicity. Our calculations are carried out in the large aspect-ratio, zero-β, resistive magnetohydrodynamical (MHD) limit, and incorporate a *realistic* treatment of plasma viscosity. We find three regimes of operation, depending on the modulation frequency (*i.e.*, the difference in rotation frequency between the island chain and the external perturbation). For slowly modulated islands, the perturbed velocity profile extends across the whole plasma. For strongly modulated islands, the perturbed velocity profile is localized around the island chain, but remains much wider than the chain. Finally, for very strongly modulated islands, the perturbed velocity profile collapses to a boundary layer on the island separatrix, plus a residual profile which extends a few island widths beyond the separatrix. We obtain analytic expressions for the perturbed velocity profile, the island equation of motion, and the island width evolution equation in each of these three regimes. We find that the ion polarization correction to the island width evolution equation, which has previously been reported to be stabilizing, is, in fact, *destabilizing* in all three regimes.

I. INTRODUCTION

Recent experimental results strongly suggest that further progress in obtaining thermonuclear reactor grade plasmas in either tokamaks or reversed field pinches (RFPs) is dependent on the development of some reliable method for controlling the amplitudes of relatively low mode-number tearing modes, resonant within the plasma[1–5]. Tearing modes are naturally unstable in toroidal magnetic fusion devices[6]: they are driven by radial gradients in the plasma current density[7] and plasma pressure[8], and generally saturate at relatively low amplitudes (*i.e.*, $\tilde{B}/B \lesssim 1\%$)[9–11]. As the name suggests, "tearing modes" tear and reconnect magnetic field-lines to produce helical chains of magnetic islands inside the plasma. Such island chains degrade plasma confinement because both heat and particles are able to travel radially from one side of an island chain to the other by flowing along magnetic field-lines, which is a relatively fast process, instead of having to diffusing across magnetic flux-surfaces, which is a relatively slow process[12]. Moreover, *overlapping* island chains, such as are generally found inside the cores of RFP plasmas, give rise to the *ergodization* of the magnetic field, and the consequent destruction of ordered magnetic flux-surfaces[13,14]. In this case, the degradation in plasma confinement is particularly severe, since both heat and particles can travel radially over a large fraction of the plasma volume by flowing along field-lines[15].

In this paper, we investigate the active feedback control of tearing modes by means of externally applied, rotating, helical magnetic perturbations. Active control has already been implemented in a handful of tokamak experiments[16–18], with varying degrees of success. It is, admittedly, doubtful that an active magnetic feedback system would be feasible in a reactor environment. Nevertheless, active magnetic feedback represents our most *direct* way of probing the physics of tearing modes. We feel that a reliable method for limiting tearing mode amplitudes is only likely to become a reality once the physics of such modes in toroidal magnetic confinement devices is *fully understood*. Such an understanding is most likely to emerge from the interpretation of magnetic feedback data.

The aim of our study is to develop a set of ordinary differential equations which determine the time evolution of both the radial width and the rotation frequency of a helical magnetic island chain, embedded in a toroidal plasma, in the presence of an externally imposed, rotating, magnetic perturbation of the same helicity. We shall carry out our investigation within the context of zero-β, cylindrical, resistive magnetohydrodynamical (MHD) theory. Indeed, our analysis can be regarded as a direct extension of the well-known work of Rutherford[9,19]. It is undoubtably the case that finite-β effects[20,21], toroidal effects[8], and drift effects[22–24] play an important rôle in the physics of tearing modes in magnetic fusion devices. Nevertheless, the relatively straightforward resistive-MHD problem is worth studying since its solution may help guide our approach to the far more complicated finite-β/toroidal/drift problem.

It is, unfortunately, the case that, even within the restrictive context of zero-β, cylindrical, resistive-MHD theory, there is still considerable controversy regarding the form of the equations governing the dynamics of a magnetic

island chain in a toroidal plasma. The main point of contention is the effect of the ion polarization current (*i.e.*, plasma inertia) on island stability. It has been reported by many authors[23–27] that the ion polarization effect is *stabilizing* (taking the resistive-MHD limit, but assuming that the island chain rotates at a frequency other than the local $\boldsymbol{E}\wedge\boldsymbol{B}$ frequency at its rational surface). Indeed, the currently accepted explanation of the stability threshold for the β-driven tearing modes which limit plasma confinement in many long-pulse tokamak discharges[3] depends on a balance between the destabilizing effect of the perturbed bootstrap current[8] and the supposed stabilizing effect of the ion polarization current. We have previously published a paper[28] which demonstrates—admittedly, in a rather restrictive regime—that the ion polarization effect is, in fact, *destabilizing*. In this paper, we shall generalize our previous calculation so as to deal with virtually all regimes of interest. In each regime, the ion polarization effect is found to be robustly destabilizing. As mentioned in our previous publication, the essential error of earlier authors lies in the neglect of the current sheet which develops, due to viscosity, on the separatrix of a magnetic island chain surrounded by a strongly localized perturbed plasma velocity profile. We shall demonstrate that this current sheet is real, and that its neglect leads to highly unphysical consequences.

Previous investigations of the non-linear dynamics of tearing modes in the presence of rotating, resonant, helical magnetic perturbations[26,29,30] are marred by an excessively naive treatment of plasma viscosity: either this crucial effect is entirely neglected, or it is modeled in a crude and unsatisfactory manner. We present a comprehensive treatment of feedback modulated tearing mode dynamics which incorporates a far more realistic treatment of viscosity. This is achieved by writing the perturbed velocity profile as a separable form in time and space. We shall demonstrate the existence of three broad regimes of operation, depending on the modulation frequency. For a weakly modulated island chain, the perturbed velocity profile extends across the whole plasma. For a strongly modulated island chain, the perturbed velocity profile is localized around the chain, but remains much wider than the chain. Finally, for a very strongly modulated island chain, the perturbed velocity profile collapses to a boundary layer on the chain separatrix, plus a residual profile which extends a few island widths beyond the separatrix. We shall obtain analytic expressions for the perturbed velocity profile, the island equation of motion, and the island width modulation equation in each of these three regimes.

II. PRELIMINARY ANALYSIS

A. Plasma equilibrium

Consider a large aspect-ratio[31], zero-β[32], plasma equilibrium whose unperturbed magnetic flux-surfaces map out (almost) concentric circles in the poloidal plane. Such an equilibrium is well approximated as a periodic cylinder. Suppose that the minor radius of the plasma is a. Standard cylindrical polar coordinates (r,θ,z) are adopted. The system is assumed to be periodic in the z-direction, with periodicity length $2\pi R_0$, where R_0 is the simulated major radius of the plasma. It is convenient to define a simulated toroidal angle $\phi = z/R_0$.

The equilibrium magnetic field is written $\boldsymbol{B} = [0, B_\theta(r), B_\phi(r)]$, where $\nabla\wedge\boldsymbol{B} = \sigma(r)\,\boldsymbol{B}$.

B. Perturbed magnetic field

The magnetic perturbation associated with an m,n tearing mode (*i.e.*, a mode with m periods in the poloidal direction, and n periods in the toroidal direction) can be written

$$\boldsymbol{b}(\boldsymbol{r},t) = \boldsymbol{b}^{m,n}(r,t)\,\mathrm{e}^{\,\mathrm{i}\,\zeta}, \tag{1}$$

where $\zeta = m\,\theta - n\,\phi$ is a helical angle. In this paper, it is assumed that $m>0$ and $n\neq 0$. The linearized magnetic flux function $\psi(r,t)\equiv -\mathrm{i}\,r\,b_r^{m,n}$ satisfies Newcomb's equation[33],

$$\frac{d}{dr}\left[f^{m,n}\,\frac{d\psi}{dr}\right] - g^{m,n}\,\psi = 0, \tag{2}$$

where

$$f^{m,n}(r) = \frac{r}{m^2+n^2\epsilon^2}, \tag{3}$$

$$g^{m,n}(r) = \frac{1}{r} + \frac{r\,(n\epsilon\,B_\theta + m\,B_\phi)}{(m^2+n^2\epsilon^2)(m\,B_\theta - n\epsilon\,B_\phi)}\,\frac{d\sigma}{dr} + \frac{2\,m\,n\epsilon\,\sigma}{(m^2+n^2\epsilon^2)^2} - \frac{r\,\sigma^2}{m^2+n^2\epsilon^2}, \tag{4}$$

and $\epsilon = r/R_0$. As is well-known, Eq. (2) is *singular* at the m/n rational surface, minor radius r_s, which satisfies $F^{m,n}(r_s) = 0$, where $F^{m,n}(r) \equiv m\, B_\theta(r) - n\, \epsilon(r)\, B_\phi(r)$.

In the vacuum region ($\sigma = 0$) surrounding the plasma, the most general solution to Newcomb's equation takes the form $\psi = A\, i_m(n\epsilon) + B\, k_m(n\epsilon)$, where A, B are arbitrary constants, and

$$i_m(n\epsilon) = |n\epsilon|\, I_{m+1}(|n\epsilon|) + m\, I_m(|n\epsilon|), \tag{5}$$

$$k_m(n\epsilon) = -|n\epsilon|\, K_{m+1}(|n\epsilon|) + m\, K_m(|n\epsilon|). \tag{6}$$

Here, I_m, K_m represent standard modified Bessel functions.

C. Asymptotic matching

Suppose that the plasma is surrounded by a concentric, perfectly conducting shell, minor radius c, which contains thin gaps through which an externally generated magnetic perturbation leaks.

Let $\hat{\psi}_s(r, c)$ represent the normalized m, n tearing eigenfunction, calculated assuming the presence of a perfectly conducting shell (with no gaps) at $r = c$. In other words, $\hat{\psi}_s(r, c)$ is a *real*, continuous solution to Newcomb's equation, (2), which is well behaved as $r \to 0$, and satisfies $\hat{\psi}_s(r_s, c) = 1$ and $\hat{\psi}_s(c, c) = 0$. This prescription uniquely specifies $\hat{\psi}_s(r, c)$. It is easily demonstrated that $\hat{\psi}_s(r, c)$ is zero in the region $r > c$. In general, $\hat{\psi}_s(r, c)$ possesses gradient discontinuities at both $r = r_s$ and $r = c$. The real quantity

$$E(c) = \left[r\, \frac{d\hat{\psi}_s(r, c)}{dr} \right]_{r_s-}^{r_s+} \tag{7}$$

can be identified as the standard m, n tearing stability index[6], calculated assuming the presence of a perfectly conducting shell at minor radius c.

The quantity

$$\Psi_s(t) \equiv \psi(r_s, t) \tag{8}$$

represents the *reconnected magnetic flux* at the m, n rational surface. Likewise,

$$\Delta\Psi_s(t) \equiv \left[r\, \frac{d\psi}{dr} \right]_{r_s-}^{r_s+} \tag{9}$$

is a measure of the m, n helical current flowing in the vicinity of the rational surface. Note that both Ψ_s and $\Delta\Psi_s$ are complex quantities.

Suppose that, in the absence of plasma and the perfectly conducting shell, the externally generated magnetic perturbation is characterized by a magnetic flux function $\psi_{\rm ext}(r, \theta, \phi, t)$. The perfectly conducting shell (minor radius c) is assumed to possess *narrow* gaps which allow the perturbation to penetrate into the plasma. The m, n component of the perturbation filtering through these gaps is characterized by

$$\Psi_c(t) = \int\!\!\int_{\rm gaps} \psi_{\rm ext}(c, \theta, \phi, t)\, {\rm e}^{-{\rm i}\,\zeta}\, \frac{d\theta}{2\pi}\, \frac{d\phi}{2\pi}, \tag{10}$$

where the integral is taken over the angular extent of the gaps[34]. Note that Ψ_c is also a complex quantity.

Let

$$\Psi_s(t) = \hat{\Psi}_s(t)\, {\rm e}^{\,{\rm i}\,\varphi_s(t)}, \tag{11}$$

$$\Psi_c(t) = \hat{\Psi}_c(t)\, {\rm e}^{\,{\rm i}\,\varphi_c(t)}, \tag{12}$$

where $\hat{\Psi}_s$ and $\hat{\Psi}_c$ are both real. The helical phase φ_s of the reconnected flux at the rational surface is subject to the constraint

$$\frac{d\varphi_s}{dt} = -(\boldsymbol{V} \cdot \nabla\zeta)_{r_s}, \tag{13}$$

where $\boldsymbol{V}$ is the plasma velocity. This constraint—which is generally known as the "no slip" condition—follows from standard MHD theory[35], according to which the m, n tearing mode is convected by the plasma at its own rational surface.

Standard asymptotic matching[36] across the m, n rational surface yields

$$\Delta\Psi_s = E(c)\,\Psi_s + E_{sc}\,\Psi_c, \tag{14}$$

where

$$E_{sc} = \frac{\hat{\psi}_s(a,c)\,(m^2 + n^2\epsilon_s^2)}{k_m(n\epsilon_c)\,i_m(n\epsilon_a) - k_m(n\epsilon_a)\,i_m(n\epsilon_c)}. \tag{15}$$

Here, $\epsilon_s = r_s/R_0$, $\epsilon_a = a/R_0$, and $\epsilon_c = c/R_0$. Note that $E(c)$ and E_{sc} are both positive.

D. Island equations

Suppose that the radial width of the m, n island chain is small compared to the minor radius of the plasma. Let us assume the existence of an m, n helical quasi-equilibrium in the vicinity of the rational surface. The equations governing this quasi-equilibrium are as follows:

$$\boldsymbol{B}_\perp = \mathcal{C}\,\nabla\psi\wedge\hat{\boldsymbol{n}}, \tag{16}$$

$$\boldsymbol{V}_\perp = \mathcal{C}\,\nabla\chi\wedge\hat{\boldsymbol{n}}, \tag{17}$$

$$\mu_0\,\delta J = -\mathcal{C}\,\nabla^2\psi, \tag{18}$$

$$U = -\mathcal{C}\,\nabla^2\chi, \tag{19}$$

$$-\mathcal{C}\left(\frac{\partial\psi}{\partial t} + \boldsymbol{V}_\perp\cdot\nabla\psi\right) = \eta_s\,\delta J, \tag{20}$$

$$\rho_s\left(\frac{\partial U}{\partial t} + \boldsymbol{V}_\perp\cdot\nabla U\right) = \boldsymbol{B}_\perp\cdot\nabla\delta J + \mu_s\,\nabla^2 U. \tag{21}$$

Here, $\hat{\boldsymbol{n}} = \mathcal{C}\,(0, n\epsilon_s, m)$ is a unit vector pointing along the direction of the equilibrium magnetic field at the m, n rational surface, and $\mathcal{C} = (m^2 + n^2\epsilon_s^2)^{-1/2}$. The subscript $\perp$ indicates that the vector in question is perpendicular to $\hat{\boldsymbol{n}}$. Furthermore, $\psi(r,\zeta)$ is the magnetic flux function, $\chi(r,\zeta)$ is the velocity stream function, $\delta J(r,\zeta)$ is the parallel component of the perturbed current density due to the island chain, $U(r,\zeta)$ is the flow vorticity, and η_s, ρ_s, and μ_s, are the (constant) parallel electrical conductivity, mass density, and perpendicular viscosity, respectively, of the plasma in the vicinity of the rational surface.

E. Normalization scheme

Let us, first of all, transform into the *instantaneous rest frame* of the island chain. In this frame, we can write

$$\psi(r,\zeta) = -F_s'\,\frac{x^2}{2} + \hat{\Psi}_s(t)\,\cos\zeta, \tag{22}$$

where $x = (r - r_s)/r_s$, and $F_s' = (r^2\,dF^{m,n}/dr)_{r_s}$. In obtaining the above expression, we have made use of the well-known *constant-ψ approximation*[6]. The use of this approximation places some restrictions on the allowable amplitude and time variation of the applied external perturbation. However, unless the radial width of the island chain becomes extremely small, these restrictions are generally not particularly onerous[37].

Let $\hat{\psi} = -\psi/\hat{\Psi}_s$, $X = x/w$,

$$w = \frac{W}{4\,r_s} = \left(\frac{\hat{\Psi}_s}{F_s'}\right)^{1/2}, \tag{23}$$

and $\varsigma = \mathrm{sgn}(x)$. It follows that

$$\hat{\psi}(X,\zeta) = \frac{X^2}{2} - \cos\zeta. \tag{24}$$

As is easily demonstrated, the above magnetic flux function maps out a magnetic island chain, of full radial width W, centred on the rational surface ($X = 0$). The separatrix lies at $\hat{\psi} = 1$, the O-points are situated at $\hat{\psi} = -1$ and $\zeta = j\,2\pi$ (where j is an integer), and the X-points lie at $\hat{\psi} = 1$ and $\zeta = (2j-1)\,\pi$.

Let

$$\tau_H = \frac{r_s^{\,2}\,\sqrt{\mu_0\,\rho_s}}{F_s'}, \tag{25}$$

$$\tau_V = \frac{r_s^{\,2}\,\rho_s}{\mu_s}, \tag{26}$$

$$\tau_R = \frac{\mu_0\,r_s^{\,2}}{\eta_s}, \tag{27}$$

be the hydromagnetic, viscous diffusion, and resistive diffusion time-scales, respectively, calculated in the vicinity of the rational surface. Furthermore, let

$$\hat{t} = \Omega_s^{(0)}\,t, \tag{28}$$

$$\hat{\chi} = \frac{\chi}{\Omega_s^{(0)}\,r_s^{\,2}\,w}, \tag{29}$$

$$\hat{U} = -\frac{w\,U}{\mathcal{C}\,\Omega_s^{(0)}}, \tag{30}$$

$$\hat{J} = \frac{\mu_0\,r_s^{\,2}\,\delta J}{\mathcal{C}\,F_s'\,w}, \tag{31}$$

where $\Omega_s^{(0)}$ is the unperturbed helical phase velocity of the island chain. The normalized island equations take the form:

$$\frac{\partial^2\hat{\psi}}{\partial X^2} = w\,\hat{J}, \tag{32}$$

$$\frac{\partial^2\hat{\chi}}{\partial X^2} = \hat{U}, \tag{33}$$

$$\hat{J} = -(\Omega_s^{(0)}\,\tau_R\,w)\,\frac{d\ln\hat{\Psi}_s}{d\hat{t}}\,\cos\zeta + (\Omega_s^{(0)}\,\tau_R\,w)\left[\hat{\psi},\,\hat{\chi}\right], \tag{34}$$

$$\left[\hat{J},\,\hat{\psi}\right] = \frac{(\Omega_s^{(0)}\,\tau_H)^2}{w^3}\left(\frac{\partial\hat{U}}{\partial\hat{t}} - \frac{1}{\Omega_s^{(0)}\,\tau_V\,w^2}\frac{\partial^2\hat{U}}{\partial X^2}\right) + \frac{(\Omega_s^{(0)}\,\tau_H)^2}{w^3}\left[\hat{U},\,\hat{\chi}\right], \tag{35}$$

where

$$[f,g] = \frac{\partial f}{\partial X}\frac{\partial g}{\partial\zeta} - \frac{\partial f}{\partial\zeta}\frac{\partial g}{\partial X}. \tag{36}$$

Note that the functions $\hat{\psi}$, $\hat{\chi}$, $\hat{J}$, and $\hat{U}$ are designed to be $O(1)$ in the island region. In deriving the above equations, use has been made of the ordering $d\ln\hat{\Psi}_s/d\hat{t} \sim O[1/(\Omega_s^{(0)}\,\tau_R\,w)] \ll 1$.

F. Flux-surface average operator

The flux-surface average operator $\langle\cdots\rangle$ is defined

$$\langle f(\varsigma,\hat{\psi},\zeta)\rangle = \begin{cases} \oint \frac{f(\varsigma,\hat{\psi},\zeta)}{|X|}\frac{d\zeta}{2\pi} & \text{for } \hat{\psi}\geq 1 \\ \int_{-\zeta_0}^{\zeta_0} \frac{\left[f(\varsigma,\hat{\psi},\zeta)+f(-\varsigma,\hat{\psi},\zeta)\right]}{2\,|X|}\frac{d\zeta}{2\pi} & \text{for } \hat{\psi}<1 \end{cases}, \tag{37}$$

where $X(\varsigma, \hat{\psi}, \zeta_0) = 0$. It follows that

$$\left\langle \left[f,\, \hat{\psi} \right] \right\rangle = 0, \tag{38}$$

irrespective of the form of f.

III. CALCULATION OF PERTURBED VELOCITY PROFILE

A. Derivation of velocity equation

Equation (34) implies that

$$\hat{\chi} = \hat{\chi}(\hat{\psi}) + O\left(\frac{1}{\Omega_s^{(0)}\, \tau_R\, w} \right). \tag{39}$$

In other words, the velocity stream function is a *flux-surface function* provided that the island chain grows on a much longer time-scale than that associated with island rotation. Let us assume that this is the case. In writing the above equation, we are rejecting the possibility of strong resistive flows (*i.e.*, as strong as the flows associated with island rotation) inside the island separatrix, such as those considered by Fitzpatrick and Hender (1991)[25]. We believe that such flows are unphysical, since they would require the existence of unrealistically large electric fields and current densities inside the separatrix.

Let

$$M(\hat{\psi}, \hat{t}) = \frac{d\hat{\chi}}{d\hat{\psi}}. \tag{40}$$

In the region of the plasma well away from the island chain, it is easily demonstrated that

$$M\, X = -\frac{\boldsymbol{V}'\cdot\nabla\zeta}{\Omega_s^{(0)}}, \tag{41}$$

where $\boldsymbol{V}'$ is the plasma velocity in the chain's instantaneous rest frame.

The flux-surface average of Eq. (35) gives

$$\left\langle \frac{\partial^2 \hat{U}}{\partial X^2} \right\rangle - (\Omega_s^{(0)}\, \tau_V\, w^2) \left\langle \frac{\partial \hat{U}}{\partial \hat{t}} \right\rangle = 0. \tag{42}$$

It follows from Eq. (33) that

$$\hat{U} = X^2\, \frac{\partial M}{\partial \hat{\psi}} + M. \tag{43}$$

Hence, Eq. (42) yields the *velocity equation*:

$$\frac{\partial}{\partial \hat{\psi}} \left\{ \frac{\partial}{\partial \hat{\psi}} \left(\langle X^4 \rangle\, \frac{\partial M}{\partial \hat{\psi}} \right) - (\Omega_s^{(0)}\, \tau_V\, w^2)\, \langle X^2 \rangle\, \frac{\partial M}{\partial \hat{t}} \right\} = 0. \tag{44}$$

B. Localized solution of velocity equation

Let us search for solutions of the above velocity equation which correspond to perturbed velocity profiles which are *localized* in the vicinity of the magnetic island chain.

It is helpful to define the normalized helical phase velocity of the island chain,

$$\hat{v} = \frac{d\varphi_s}{d\hat{t}}, \tag{45}$$

the normalized helical phase velocity of the external perturbation,

$$\hat{v}_c = \frac{d\varphi_c}{d\hat{t}}, \tag{46}$$

and the quantity

$$\hat{v}' = \hat{v} - \hat{v}_c. \tag{47}$$

The latter quantity is the (normalized) *modulation frequency* of the electromagnetic torque exerted by the external perturbation on the plasma in the vicinity of the island chain.

Since the plasma in the vicinity of the island chain is subject to an external torque which modulates at (normalized) frequency $\hat{v}'$, it is reasonable to suppose that the perturbed plasma velocity profile also modulates at this frequency. Hence, we write the velocity function M in the *separable form*

$$M(\hat{\psi},\hat{t}) = M_1(\hat{\psi})\,\sin\left(\int_0^{\hat{t}}\hat{v}'\,d\hat{t}\right) + M_2(\hat{\psi})\,\cos\left(\int_0^{\hat{t}}\hat{v}'\,d\hat{t}\right). \tag{48}$$

This form is valid provided that the relative changes in both the modulation frequency and feedback amplitude per modulation cycle are small. There is a wide range of realistic feedback schemes for which this ordering is appropriate. We shall discuss schemes in which this ordering breaks down in a future publication.

For the case of a *localized* velocity profile, we can (without loss of generality) specify the following boundary conditions for the perturbed flow:

$$M_1\,X \to \mathcal{V} \qquad \text{as } |X| \to \infty, \tag{49}$$

$$M_2\,X \to 0 \qquad \text{as } |X| \to \infty. \tag{50}$$

Note that $\mathcal{V}$ is *constant* in time. Since the function $M(\hat{\psi},\hat{t})$ is clearly going to be *odd* about the rational surface, it follows that

$$M_1 = M_2 = 0 \qquad \text{for } \hat{\psi} < 1. \tag{51}$$

In other words, there is *zero* plasma flow inside the separatrix in the instantaneous rest frame of the island chain (since it is impossible to have a non-trivial, odd flux-surface function inside the separatrix).

For a localized perturbed velocity profile, the "no slip" condition (13), combined with Eqs. (41), (45), (49), and (50), yields

$$\hat{v} - 1 = -\mathcal{V}\,\sin\left(\int_0^{\hat{t}}\hat{v}'\,d\hat{t}\right). \tag{52}$$

Differentiating with respect to $\hat{t}$, we obtain

$$\frac{d\hat{v}}{d\hat{t}} = -\hat{v}'\,\mathcal{V}\,\cos\left(\int_0^{\hat{t}}\hat{v}'\,d\hat{t}\right), \tag{53}$$

and

$$\frac{d}{d\hat{t}}\left(\frac{1}{\hat{v}'}\frac{d\hat{v}}{d\hat{t}}\right) = -\hat{v}'\,(\hat{v}-1). \tag{54}$$

Integration of the velocity equation, (44), making use of Eqs. (48), (49), and (50), yields

$$\frac{d}{d\hat{\psi}}\left(\langle X^4\rangle\,\frac{dM_1}{d\hat{\psi}}\right) + \lambda\,\langle X^2\rangle\,M_2 = 0, \tag{55}$$

$$\frac{d}{d\hat{\psi}}\left(\langle X^4\rangle\,\frac{dM_2}{d\hat{\psi}}\right) - \lambda\,\langle X^2\rangle\,M_1 = -\varsigma\,\lambda\,\mathcal{V}, \tag{56}$$

where

$$\lambda = \hat{v}'\,\Omega_s^{(0)}\,\tau_V\,w^2. \tag{57}$$

Equations (55) and (56) can be solved analytically in two asymptotic limits. The first limit, $|\lambda| \ll 1$, corresponds to the case in which the localization scale-length of the perturbed velocity profile is much *larger* than the width of the island chain—this limit is referred to as the *weakly localized regime*. The second limit, $|\lambda| \gg 1$, corresponds to the case in which the localization scale-length of the perturbed velocity profile is much *smaller* than the width of the island chain—this limit is referred to as the *strongly localized regime*.

C. Weakly localized regime

Consider the solution of Eqs. (55) and (56) in the weakly localized regime, $|\lambda| \ll 1$. In the limit $|X| \gg 1$, Eqs. (55) and (56) reduce to

$$\frac{d^2(X\,M_1)}{dX^2} + \lambda\,X\,M_2 = 0, \tag{58}$$

$$\frac{d^2(X\,M_2)}{dX^2} - \lambda\,X\,M_1 = -\lambda\,\mathcal{V}. \tag{59}$$

The physical solutions of these equations are

$$X\,M_1 = \mathcal{V} + A\,\mathrm{sgn}(\lambda)\,\exp\left(-\sqrt{\frac{|\lambda|}{2}}\,|X|\right)\sin\left(\sqrt{\frac{|\lambda|}{2}}\,|X|\right) + B\,\exp\left(-\sqrt{\frac{|\lambda|}{2}}\,|X|\right)\cos\left(\sqrt{\frac{|\lambda|}{2}}\,|X|\right), \tag{60}$$

$$X\,M_2 = A\,\exp\left(-\sqrt{\frac{|\lambda|}{2}}\,|X|\right)\cos\left(\sqrt{\frac{|\lambda|}{2}}\,|X|\right) - B\,\mathrm{sgn}(\lambda)\,\exp\left(-\sqrt{\frac{|\lambda|}{2}}\,|X|\right)\sin\left(\sqrt{\frac{|\lambda|}{2}}\,|X|\right), \tag{61}$$

where A and B are arbitrary constants. In the asymptotic limit $1 \ll |X| \ll 1/\sqrt{|\lambda|}$, the above expressions reduce to

$$X\,M_1 \simeq \mathcal{V} + B + [A\,\mathrm{sgn}(\lambda) - B]\sqrt{\frac{|\lambda|}{2}}\,|X| + \cdots, \tag{62}$$

$$X\,M_2 \simeq A - [A + B\,\mathrm{sgn}(\lambda)]\sqrt{\frac{|\lambda|}{2}}\,|X| + \cdots. \tag{63}$$

In the limit $|X| \sim O(1)$, Eqs. (55) and (56) give

$$\frac{d}{d\hat{\psi}}\left(\langle X^4\rangle\,\frac{dM_1}{d\hat{\psi}}\right) \simeq 0, \tag{64}$$

$$\frac{d}{d\hat{\psi}}\left(\langle X^4\rangle\,\frac{dM_2}{d\hat{\psi}}\right) \simeq 0, \tag{65}$$

The physical solutions of these equations which satisfy the boundary conditions (51) are

$$M_1 = \varsigma\,\tilde{A}\,\mathcal{F}(\hat{\psi}), \tag{66}$$

$$M_2 = \varsigma\,\tilde{B}\,\mathcal{F}(\hat{\psi}), \tag{67}$$

where

$$\mathcal{F}(\hat{\psi}) = \begin{cases} 0 & \text{for } \hat{\psi} < 1, \\ 1 - \int_{\hat{\psi}}^{\infty} d\hat{\psi}/\langle X^4\rangle \Big/ \int_1^{\infty} d\hat{\psi}/\langle X^4\rangle & \text{for } \hat{\psi} \geq 1 \end{cases}, \tag{68}$$

and $\tilde{A}$, $\tilde{B}$ are arbitrary constants. In the limit $|X| \gg 1$, Eqs. (66) and (67) yield

$$X\,M_1 \simeq \tilde{A}\left[-\frac{1}{\mathcal{F}_\infty} + |X| + \cdots\right], \tag{69}$$

$$X\,M_2 \simeq \tilde{B}\left[-\frac{1}{\mathcal{F}_\infty} + |X| + \cdots\right], \tag{70}$$

where $\mathcal{F}_\infty = \int_1^\infty d\hat{\psi}/\langle X^4\rangle$.

Asymptotic matching between Eqs. (62), (63), (69), and (70) gives $A \sim O(\mathcal{V}\sqrt{|\lambda|})$, $B = -\mathcal{V}$, $\tilde{A} = \mathcal{V}\sqrt{|\lambda|/2}$, and $\tilde{B} = \mathrm{sgn}(\lambda)\,\mathcal{V}\sqrt{|\lambda|/2}$. Making use of Eqs. (52) and (53), we arrive at the following expression which specifies the perturbed velocity profile:

$$-X\,M = \left[1-\exp\left(-\sqrt{\frac{|\lambda|}{2}}\,|X|\right)\cos\left(\sqrt{\frac{|\lambda|}{2}}\,|X|\right)\right]\,\mathcal{F}(\hat{\psi})\,(\hat{v}-1)$$
$$+\exp\left(-\sqrt{\frac{|\lambda|}{2}}\,|X|\right)\sin\left(\sqrt{\frac{|\lambda|}{2}}\,|X|\right)\,\mathcal{F}(\hat{\psi})\,\frac{1}{|\hat{v}'|}\frac{d\hat{v}}{d\hat{t}}. \tag{71}$$

Note that the radial localization scale-length of the perturbed velocity profile is approximately $1/\sqrt{|\lambda|}$ times the width of the island chain.

D. Strongly localized regime

Consider the solution of Eqs. (55) and (56) in the strongly localized regime, $|\lambda| \gg 1$. Let

$$M_1 = \frac{\varsigma\,\mathcal{V}}{\langle X^2\rangle} + m_1, \tag{72}$$
$$M_2 = m_2, \tag{73}$$

where the solutions $m_1(\hat{\psi})$ and $m_2(\hat{\psi})$ are localized in a thin layer on the island chain separatrix. Of course, $M_1 = M_2 = 0$ for $\hat{\psi} < 1$. Note that $\lim_{\hat{\psi}\rightarrow 1}\langle X^2\rangle = 4/\pi$ and $\lim_{\hat{\psi}\rightarrow 1}\langle X^4\rangle = 32/3\pi$. It follows that the localized solutions satisfy

$$\frac{d^2 m_1}{d\hat{\psi}^2} + \frac{3\,\lambda}{8}\,m_2 = 0, \tag{74}$$
$$\frac{d^2 m_2}{d\hat{\psi}^2} - \frac{3\,\lambda}{8}\,m_1 = 0, \tag{75}$$

subject to the boundary conditions $m_1 \rightarrow -\varsigma\,\pi\,\mathcal{V}/4$ and $m_2 \rightarrow 0$ as $\hat{\psi}\rightarrow 1$, as well as physical boundary conditions as $\hat{\psi}\rightarrow\infty$. The physical solutions of the above equations are

$$m_1 = -\varsigma\,\frac{\pi\,\mathcal{V}}{4}\,\exp\left(-\sqrt{\frac{3\,|\lambda|}{16}}\,y\right)\,\cos\left(\sqrt{\frac{3\,|\lambda|}{16}}\,y\right), \tag{76}$$
$$m_2 = \varsigma\,\frac{\pi\,\mathcal{V}}{4}\,\mathrm{sgn}(\lambda)\,\exp\left(-\sqrt{\frac{3\,|\lambda|}{16}}\,y\right)\,\sin\left(\sqrt{\frac{3\,|\lambda|}{16}}\,y\right), \tag{77}$$

where $y = \hat{\psi} - 1$.

Making use of Eqs. (52) and (53), we arrive at the following expression which specifies the perturbed velocity profile:

$$M = -\varsigma\left\{\left[\frac{1}{\langle X^2\rangle} - \frac{\pi}{4}\,\exp\left(-\sqrt{\frac{3\,|\lambda|}{16}}\,y\right)\,\cos\left(\sqrt{\frac{3\,|\lambda|}{16}}\,y\right)\right]\,(\hat{v}-1)\right.$$
$$\left.+\frac{\pi}{4}\,\exp\left(-\sqrt{\frac{3\,|\lambda|}{16}}\,y\right)\,\sin\left(\sqrt{\frac{3\,|\lambda|}{16}}\,y\right)\,\frac{1}{|\hat{v}'|}\frac{d\hat{v}}{d\hat{t}}\right\}. \tag{78}$$

Of course, $M = 0$ for $\hat{\psi} < 1$. Note that the velocity profile consists of *two parts*. Firstly, a thin boundary layer on the separatrix, whose width is approximately $1/\sqrt{|\lambda|}$ times the width of the island chain. Secondly, a residual profile which extends a few island widths beyond the separatrix.

E. Non-localized regime

In the limit $|\lambda| \ll w^2$, the radial localization scale-length of the perturbed velocity profile becomes comparable with the minor radius of the plasma, and the weakly localized solution specified in Sect. III C breaks down. In this

new regime—which we refer to as the *non-localized regime*—the plasma is conveniently divided into two regions. The *inner region* consists of the plasma lying within a few island widths of the separatrix. The *outer region* consists of the remainder of the plasma.

Let

$$\Delta\hat{\Omega} = \frac{\boldsymbol{V}'\cdot\nabla\zeta}{\Omega_s^{(0)}} \tag{79}$$

represent the perturbed velocity profile in the outer region. This quantity satisfies the following equation[35],

$$\frac{\partial}{\partial\hat{r}}\left(\hat{r}\,\hat{\mu}\,\frac{\partial\Delta\hat{\Omega}}{\partial\hat{r}}\right) = \frac{\hat{r}\,\hat{\rho}}{\Omega_s^{(0)}\,\tau_V}\,\frac{\partial\Delta\hat{\Omega}}{\partial\hat{t}}. \tag{80}$$

Here, $\hat{r} = r/r_s$, $\hat{\mu} = \mu/\mu_s$, and $\hat{\rho} = \rho/\rho_s$, where $\mu(r)$ and $\rho(r)$ are the plasma viscosity and mass density profiles, respectively. Note that, in writing the above equation, we have implicitly assumed that the perturbed velocity in the outer region is predominately *toroidal* in nature (see Sect. IV E). The boundary conditions which must be satisfied by the solution of Eq. (80) are[35]

$$\frac{\partial\Delta\hat{\Omega}(0,\hat{t})}{\partial\hat{r}} = 0, \tag{81}$$

$$\Delta\hat{\Omega}(\hat{a},\hat{t}) = 0, \tag{82}$$

where $\hat{a} = a/r_s$.

In accordance with the analysis presented in Sect. III B, let us write the perturbed velocity profile in the outer region in the separable form

$$\Delta\hat{\Omega}(\hat{r},\hat{t}) = \Delta\hat{\Omega}_1(\hat{r})\,\sin\left(\int_0^{\hat{t}}\hat{v}'\,d\hat{t}\right) + \Delta\hat{\Omega}_2(\hat{r})\,\cos\left(\int_0^{\hat{t}}\hat{v}'\,d\hat{t}\right), \tag{83}$$

where

$$\Delta\hat{\Omega}_1(1)\,\sin\left(\int_0^{\hat{t}}\hat{v}'\,d\hat{t}\right) = \hat{v} - 1, \tag{84}$$

$$\Delta\hat{\Omega}_1(1)\,\cos\left(\int_0^{\hat{t}}\hat{v}'\,d\hat{t}\right) = \frac{1}{\hat{v}'}\frac{d\hat{v}}{d\hat{t}}, \tag{85}$$

$$\Delta\hat{\Omega}_2(1) = 0. \tag{86}$$

It follows that

$$\frac{d}{d\hat{r}}\left(\hat{r}\,\hat{\mu}\,\frac{d\Delta\hat{\Omega}_1}{d\hat{r}}\right) = -\frac{\lambda}{w^2}\,\hat{r}\,\hat{\rho}\,\Delta\hat{\Omega}_2, \tag{87}$$

$$\frac{d}{d\hat{r}}\left(\hat{r}\,\hat{\mu}\,\frac{d\Delta\hat{\Omega}_2}{d\hat{r}}\right) = \frac{\lambda}{w^2}\,\hat{r}\,\hat{\rho}\,\Delta\hat{\Omega}_1. \tag{88}$$

Assuming that $|\lambda| \ll w^2$, we can write

$$\Delta\hat{\Omega}_1(\hat{r}) = \Delta\hat{\Omega}_1(1)\,\mathcal{H}(\hat{r}) + O(|\lambda|/w^2)^2. \tag{89}$$

Here,

$$\mathcal{H}(\hat{r}) = \begin{cases} 1 & \text{for } \hat{r} < 1 \\ \int_{\hat{r}}^{\hat{a}} d\hat{r}\,/\hat{r}\hat{\mu}\,\Big/\,J & \text{for } 1 \le \hat{r} \le \hat{a} \end{cases}, \tag{90}$$

where $J = \int_1^{\hat{a}} d\hat{r}\,/\hat{r}\hat{\mu}$. We can also write

$$\Delta\hat{\Omega}_2(\hat{r}) = \frac{\lambda}{w^2}\,\Delta\hat{\Omega}_1(1)\left\{-\int_{\hat{r}}^{\hat{a}}\frac{d\hat{r}'}{\hat{r}'\,\hat{\mu}(\hat{r}')}\int_0^{\hat{r}'}\hat{r}''\,\hat{\rho}(\hat{r}'')\,\mathcal{H}(\hat{r}'')\,d\hat{r}'' + J\,J_1\,\mathcal{H}(\hat{r})\right\}, \tag{91}$$

where

$$J_1 = \int_0^{\hat{a}}\hat{r}\,\hat{\rho}(\hat{r})\,\mathcal{H}^2(\hat{r})\,d\hat{r}. \tag{92}$$

Hence, we arrive at the following expression which specifies the perturbed plasma velocity in the outer region:

$$\Delta\hat{\Omega}(\hat{r},\hat{t}) = \mathcal{H}(\hat{r})\,(\hat{v}-1) + \frac{\lambda}{w^2}\left\{-\int_{\hat{r}}^{\hat{a}}\frac{d\hat{r}'}{\hat{r}'\,\hat{\mu}(\hat{r}')}\int_0^{\hat{r}'}\hat{r}''\,\hat{\rho}(\hat{r}'')\,\mathcal{H}(\hat{r}'')\,d\hat{r}'' + J\,J_1\,\mathcal{H}(\hat{r})\right\}\frac{1}{\hat{v}'}\frac{d\hat{v}}{dt}. \tag{93}$$

In the inner region, the perturbed velocity profile is determined by the solution of

$$\frac{\partial}{\partial\hat{\psi}}\left(\langle X^4\rangle\frac{\partial M}{\partial\hat{\psi}}\right) \simeq 0. \tag{94}$$

In this region, Eq. (93) reduces to

$$M\,X = \Delta\hat{\Omega}(\hat{r},\hat{t}) - (\hat{v}-1) \simeq \frac{\partial\Delta\hat{\Omega}(1,\hat{t})}{\partial\hat{r}}\,w\,X. \tag{95}$$

The above expression yields the following boundary conditions to be satisfied by the solution of Eq. (94):

$$M\,X = \begin{cases} A & \text{for } X\to-\infty \\ A - \left\{(\hat{v}-1) + J\,J_1\,\frac{\lambda}{w^2}\,\frac{1}{\hat{v}'}\,\frac{d\hat{v}}{d\hat{t}}\right\}\frac{w\,X}{J} & \text{for } X\to+\infty \end{cases}. \tag{96}$$

Here, A is an unimportant constant. Hence, the appropriate solution to Eq. (94) is

$$M(\hat{\psi}) = A' - \varsigma\left\{(\hat{v}-1) + J\,J_1\,\frac{\lambda}{w^2}\,\frac{1}{\hat{v}'}\,\frac{d\hat{v}}{d\hat{t}}\right\}\frac{w}{2\,J}\,\mathcal{F}(\hat{\psi}), \tag{97}$$

where A' is another unimportant constant. The above expression characterizes the perturbed velocity profile in the island region. Note, however, that the perturbed velocity profile extends across the whole plasma in the non-localized regime.

F. General solution of velocity equation

It is convenient to write

$$M(\hat{\psi}) = M_+(\hat{\psi}) + \varsigma\,M_-(\hat{\psi}), \tag{98}$$

where M_+ represents the even (about the rational surface) component of M, whereas M_- represents the odd component. Note that M_+ is *constant* in the island region, whereas M_- is *zero* inside the island separatrix. It is also convenient to write

$$M_-(\hat{\psi}) = -\tilde{M}_1(\hat{\psi})\,(\hat{v}-1) - \tilde{M}_2(\hat{\psi})\,\frac{1}{|\hat{v}'|}\frac{d\hat{v}}{d\hat{t}}. \tag{99}$$

In the non-localized regime ($|\lambda| \ll w^2$):

$$\tilde{M}_1 = \frac{w}{2\,J}\,\mathcal{F}(\hat{\psi}), \tag{100}$$

$$\tilde{M}_2 = J\,J_1\,\frac{|\lambda|}{w^2}\,\frac{w}{2\,J}\,\mathcal{F}(\hat{\psi}). \tag{101}$$

In the weakly localized regime ($w^2 \ll |\lambda| \ll 1$):

$$\tilde{M}_1 = \frac{1}{|X|}\left[1 - \exp\left(-\sqrt{\frac{|\lambda|}{2}}\,|X|\right)\,\cos\left(\sqrt{\frac{|\lambda|}{2}}\,|X|\right)\right]\,\mathcal{F}(\hat{\psi}), \tag{102}$$

$$\tilde{M}_2 = \frac{1}{|X|}\,\exp\left(-\sqrt{\frac{|\lambda|}{2}}\,|X|\right)\,\sin\left(\sqrt{\frac{|\lambda|}{2}}\,|X|\right)\,\mathcal{F}(\hat{\psi}). \tag{103}$$

Finally, in the strongly localized regime ($|\lambda| \gg 1$):

$$\tilde{M}_1 = \frac{1}{\langle X^2\rangle} - \frac{\pi}{4}\,\exp\left(-\sqrt{\frac{3\,|\lambda|}{16}}\,(\hat{\psi}-1)\right)\,\cos\left(\sqrt{\frac{3\,|\lambda|}{16}}\,(\hat{\psi}-1)\right), \tag{104}$$

$$\tilde{M}_2 = \frac{\pi}{4}\,\exp\left(-\sqrt{\frac{3\,|\lambda|}{16}}\,(\hat{\psi}-1)\right)\,\sin\left(\sqrt{\frac{3\,|\lambda|}{16}}\,(\hat{\psi}-1)\right). \tag{105}$$

Of course, $\tilde{M}_1 = \tilde{M}_2 = 0$ for $\hat{\psi} < 1$.

Figures 1, 2, and 3 show typical perturbed velocity profiles calculated in the *non-localized*, *weakly localized*, and *strongly localized* regimes, respectively.

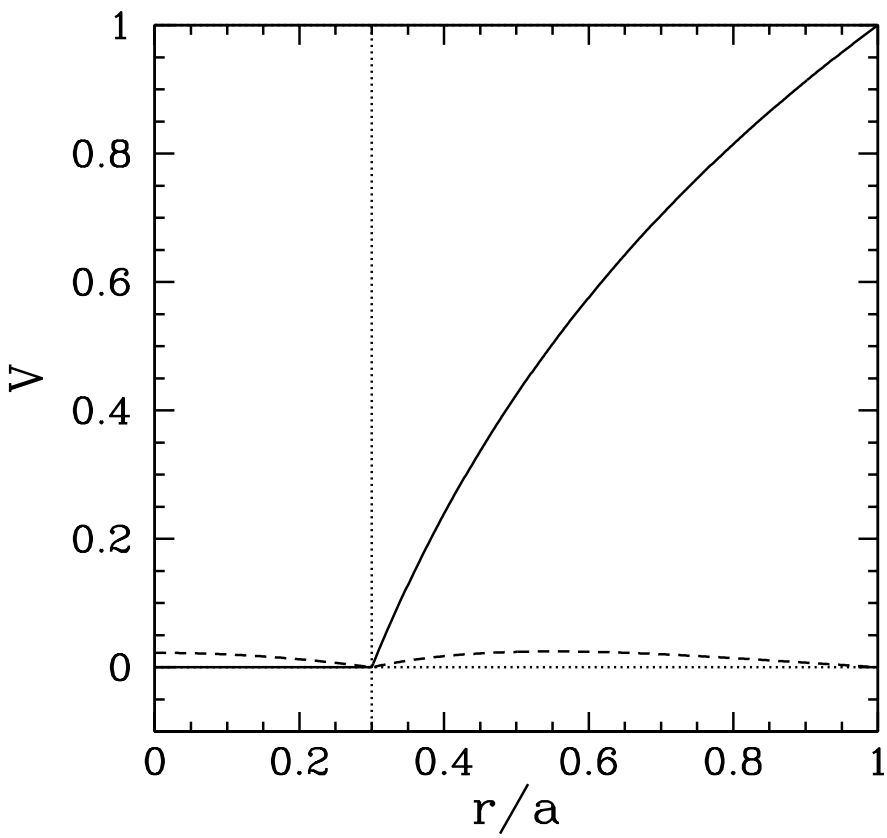

FIG. 1. *The normalized perturbed velocity profile of the plasma plotted as a function of radius, as seen in the instantaneous rest frame of the magnetic island chain. Here, the solid curve represents the component of the velocity profile which oscillates in phase with the island chain, whereas the dashed curve represents the component which oscillates in phase quadrature. The vertical dotted line indicates the radius of the rational surface ($r_s/a = 0.3$). In this example, $|\lambda|/w^2 = 0.09$, which corresponds to the* non-localized *regime.*

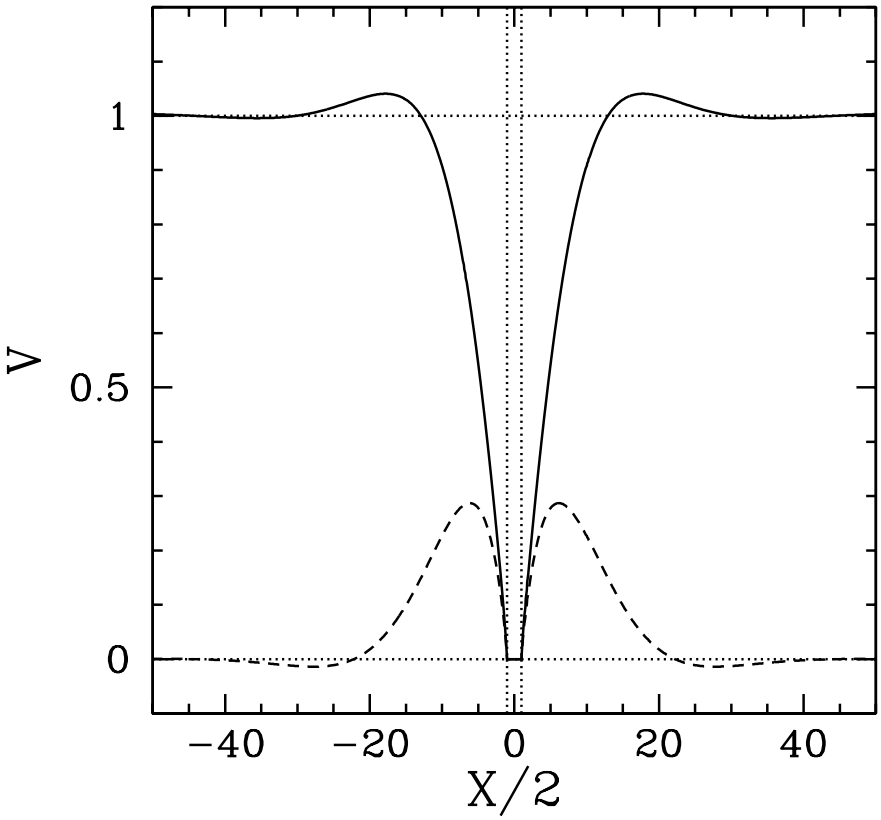

FIG. 2. *The normalized perturbed velocity profile of the plasma plotted as a function of normalized distance from the rational surface at the helical phase of an O-point, as seen in the instantaneous rest frame of the magnetic island chain. Here, the solid curve represents the component of the velocity profile which oscillates in phase with the island chain, whereas the dashed curve represents the component which oscillates in phase quadrature. The two vertical dotted lines indicate the location of the island chain separatrix. In this example,* $|\lambda| = 0.01$*, which corresponds to the* weakly localized *regime.*

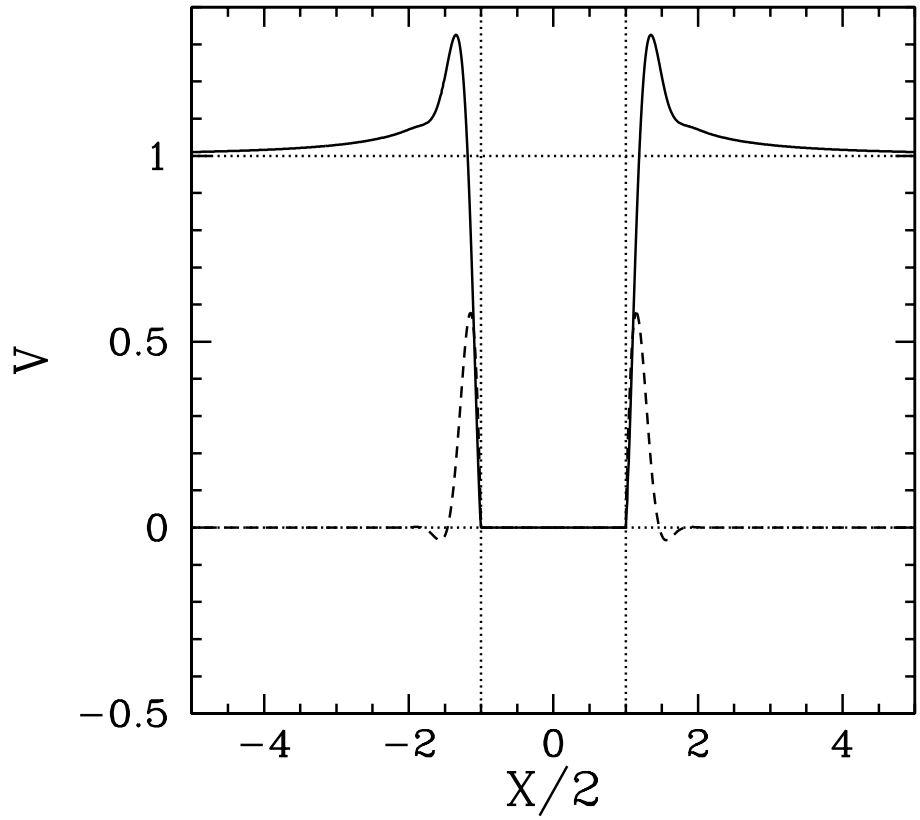

FIG. 3. *The normalized perturbed velocity profile of the plasma plotted as a function of normalized distance from the rational surface at the helical phase of an O-point, as seen in the instantaneous rest frame of the magnetic island chain. Here, the solid curve represents the component of the velocity profile which oscillates in phase with the island chain, whereas the dashed curve represents the component which oscillates in phase quadrature. The two vertical dotted lines indicate the location of the island chain separatrix. In this example,* $|\lambda| = 10$*, which corresponds to the* strongly localized *regime.*

IV. DERIVATION OF FEEDBACK EQUATIONS

A. Calculation of perturbed current density profile

Equation (35) can be integrated to give

$$\hat{J} = I(\hat{\psi}) + \frac{(\Omega_s^{(0)}\,\tau_H)^2}{w^3}\,M\,\hat{U} - \frac{(\Omega_s^{(0)}\,\tau_H)^2}{w^3}\int_0^{\zeta}\left(\frac{\partial\hat{U}}{\partial\hat{t}} - \frac{1}{\Omega_s^{(0)}\,\tau_V\,w^2}\,\frac{\partial^2\hat{U}}{\partial X^2}\right)\frac{d\zeta}{X}, \tag{106}$$

where $I(\hat{\psi})$ is an undetermined flux-surface function. Note that the final term in the above equation possesses the symmetry of $\sin\zeta$, and, therefore, flux-surface averages to zero. Hence, the flux-surface average of Eq. (106) yields

$$\langle\hat{J}\rangle = I\,\langle 1\rangle + \frac{(\Omega_s^{(0)}\,\tau_H)^2}{w^3}\,M\,\langle\hat{U}\rangle. \tag{107}$$

Now, the flux-surface average of Eq. (34) gives

$$\langle\hat{J}\rangle = -(\Omega_s^{(0)}\,\tau_R\,w)\,\frac{d\ln\hat{\Psi}_s}{d\hat{t}}\,\langle\cos\zeta\rangle. \tag{108}$$

Finally, Eq. (43) and Eqs. (106)–(108) can be combined to give the following expression which specifies the perturbed current density profile in the vicinity of the island chain:

$$\hat{J}(\hat{\psi},\zeta,\hat{t}) = -(\Omega_s^{(0)}\,\tau_R\,w)\,\frac{d\ln\hat{\Psi}_s}{d\hat{t}}\,\frac{\langle\cos\zeta\rangle}{\langle 1\rangle} + \frac{(\Omega_s^{(0)}\,\tau_H)^2}{w^3}\left(X^2 + \frac{\langle X^2\rangle}{\langle 1\rangle}\right) M\,\frac{\partial M}{\partial\hat{\psi}}$$
$$-\frac{(\Omega_s^{(0)}\,\tau_H)^2}{w^3}\int_0^{\zeta}\left(\frac{\partial\hat{U}}{\partial\hat{t}} - \frac{1}{\Omega_s^{(0)}\,\tau_V\,w^2}\,\frac{\partial^2\hat{U}}{\partial X^2}\right)\frac{d\zeta}{X}. \tag{109}$$

B. Asymptotic matching

It is easily demonstrated that

$$\mathrm{Re}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) = -4\int_{-1}^{\infty}\langle\hat{J}_+\,\cos\zeta\rangle\,d\hat{\psi}, \tag{110}$$

$$\mathrm{Im}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) = +4\int_{-1}^{\infty}\langle\hat{J}_+\,\sin\zeta\rangle\,d\hat{\psi}, \tag{111}$$

where $\hat{J}_+$ is the even (about the rational surface) component of $\hat{J}$. It follows from Eq. (109) that

$$\hat{J}_+(\hat{\psi},\zeta,\hat{t}) = -(\Omega_s^{(0)}\,\tau_R\,w)\,\frac{d\ln\hat{\Psi}_s}{d\hat{t}}\,\frac{\langle\cos\zeta\rangle}{\langle 1\rangle} + \frac{(\Omega_s^{(0)}\,\tau_H)^2}{w^3}\left(X^2 + \frac{\langle X^2\rangle}{\langle 1\rangle}\right) M_-\,\frac{\partial M_-}{\partial\hat{\psi}}$$
$$-\frac{(\Omega_s^{(0)}\,\tau_H)^2}{w^3}\int_0^{\zeta}\left(\frac{\partial\hat{U}_-}{\partial\hat{t}} - \frac{1}{\Omega_s^{(0)}\,\tau_V\,w^2}\,\frac{\partial^2\hat{U}_-}{\partial X^2}\right)\frac{d\zeta}{X}, \tag{112}$$

where

$$\hat{U}_- = X^2\,\frac{\partial M_-}{\partial\hat{\psi}} + M_- \tag{113}$$

Here, use has been made of the fact that M_+ is a constant in the island region.

C. Rutherford island equation

Equations (14), (99), (110), and (112) can be combined to give the familiar Rutherford island width evolution equation[9]:

$$I_1\,\Omega_s^{(0)}\,\tau_R\,\frac{d(4w)}{d\hat{t}} = E(c) + E_{sc}\,\frac{\hat{\Psi}_c}{\hat{\Psi}_s}\,\cos\varphi + \frac{(\Omega_s^{(0)}\,\tau_H)^2}{w^3}\left[I_{2(a)}\,(\hat{v}-1)^2 + 2\,I_{2(b)}\,\frac{(\hat{v}-1)}{|\hat{v}'|}\,\frac{d\hat{v}}{d\hat{t}} + I_{2(c)}\left(\frac{1}{\hat{v}'}\,\frac{d\hat{v}}{d\hat{t}}\right)^2\right], \tag{114}$$

where $\varphi = \varphi_s - \varphi_c$ is the helical phase of the island chain measured with respect to that of the external perturbation. Here,

$$I_1 = 2\int_{-1}^{\infty}\frac{\langle\cos\zeta\rangle^2}{\langle 1\rangle}\,d\hat{\psi} = 0.8227, \tag{115}$$

$$I_{2(a)} = 8\int_1^{\infty}\left(\langle\cos^2\zeta\rangle - \frac{\langle\cos\zeta\rangle^2}{\langle 1\rangle}\right)\tilde{M}_1\,\frac{d\tilde{M}_1}{d\hat{\psi}}\,d\hat{\psi}, \tag{116}$$

$$I_{2(b)} = 8\int_1^{\infty}\left(\langle\cos^2\zeta\rangle - \frac{\langle\cos\zeta\rangle^2}{\langle 1\rangle}\right)\frac{1}{2}\left(\tilde{M}_1\,\frac{d\tilde{M}_2}{d\hat{\psi}} + \tilde{M}_2\,\frac{d\tilde{M}_1}{d\hat{\psi}}\right)d\hat{\psi}, \tag{117}$$

$$I_{2(c)} = 8\int_1^{\infty}\left(\langle\cos^2\zeta\rangle - \frac{\langle\cos\zeta\rangle^2}{\langle 1\rangle}\right)\tilde{M}_2\,\frac{d\tilde{M}_2}{d\hat{\psi}}\,d\hat{\psi}. \tag{118}$$

According to Eq. (114), the radial width of the island chain evolves on a *resistive* time-scale. The first term on the right-hand side of Eq. (114) represents the linear driving term, the second term represents the modification to island stability due to the external perturbation, whereas the final term represents the inertial modification to island stability due to the effect of the ion polarization current.

D. Island equation of motion

Equations (111) and (112) yield

$$\mathrm{Im}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) = 4\int_{-1}^{\infty}\langle\hat{J}_+\,\sin\zeta\rangle\,d\hat{\psi} = -4\,\frac{(\Omega_s^{(0)}\,\tau_H)^2}{w^3}\int_{-1}^{\infty}\left\langle|X|\left(\frac{\partial\hat{U}_-}{\partial\hat{t}} - \frac{1}{\Omega_s^{(0)}\,\tau_V\,w^2}\frac{\partial^2\hat{U}_-}{\partial X^2}\right)\right\rangle d\hat{\psi}. \tag{119}$$

It follows from Eq. (113) that

$$\begin{aligned}\mathrm{Im}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) &= -4\,\frac{(\Omega_s^{(0)}\,\tau_H)^2}{w^3}\int_{-1}^{\infty}\left[\langle|X^3|\rangle\,\frac{\partial^2 M_-}{\partial\hat{\psi}\,\partial\hat{t}} + \langle|X|\rangle\,\frac{\partial M_-}{\partial\hat{t}}\right.\\ &\left.-\frac{1}{\Omega_s^{(0)}\,\tau_V\,w^2}\frac{\partial}{\partial\hat{\psi}}\left(\langle|X^5|\rangle\,\frac{\partial^2 M_-}{\partial\hat{\psi}^2} + 2\,\langle|X^3|\rangle\,\frac{\partial M_-}{\partial\hat{\psi}} - \langle|X|\rangle\,M_-\right)\right]d\hat{\psi}.\end{aligned} \tag{120}$$

Making use of the fact that $M_- = 0$ inside the island separatrix, the result

$$\lim_{|X|\to\infty}\left\{\langle|X^5|\rangle\,\frac{\partial^2 M_-}{\partial\hat{\psi}^2} + 2\,\langle|X^3|\rangle\,\frac{\partial M_-}{\partial\hat{\psi}} - \langle|X|\rangle\,M_-\right\} = X^2\,\frac{d}{dX}\left(\frac{1}{X}\,\frac{d(X\,M_-)}{dX}\right), \tag{121}$$

plus Eqs. (14), (54), (99), (111), and (120), we arrive at the following equation of motion for the island chain:

$$(I_{3(a)} + I_{3(b)})\,\frac{\Omega_s^{(0)}\,\tau_H^2}{w^5\,\tau_V}\,\frac{1}{|\hat{v}'|}\,\frac{d\hat{v}}{d\hat{t}} + (I_{4(a)} + I_{4(b)})\,\frac{\Omega_s^{(0)}\,\tau_H^2}{w^5\,\tau_V}\,(\hat{v}-1) + E_{sc}\,\frac{\hat{\Psi}_c}{\hat{\Psi}_s}\,\sin\varphi = 0. \tag{122}$$

Here,

$$I_{3(a)} = 4\,|\lambda|\int_{+1}^{\infty}\left[\langle|X^3|\rangle\frac{d\tilde{M}_1}{d\hat{\psi}} + \langle|X|\rangle\tilde{M}_1\right]d\hat{\psi}, \tag{123}$$

$$I_{4(a)} = -4\,|\lambda|\int_{+1}^{\infty}\left[\langle|X^3|\rangle\frac{d\tilde{M}_2}{d\hat{\psi}} + \langle|X|\rangle\tilde{M}_2\right]d\hat{\psi}, \tag{124}$$

$$I_{3(b)} = -4\,\lim_{|X|\to\infty}\left[X^2\,\frac{d}{dX}\left(\frac{1}{X}\,\frac{d(X\,\tilde{M}_2)}{dX}\right)\right], \tag{125}$$

$$I_{4(b)} = -4\,\lim_{|X|\to\infty}\left[X^2\,\frac{d}{dX}\left(\frac{1}{X}\,\frac{d(X\,\tilde{M}_1)}{dX}\right)\right]. \tag{126}$$

The first term in Eq. (122) represents the inertia of the plasma region which co-rotates with the magnetic island chain under the influence of viscosity, the second term represents the viscous torque exerted on this region by the remainder of the plasma, whereas the third term represents the electromagnetic torque exerted on the plasma in the island region by the external perturbation.

E. Neoclassical flow damping

There is one toroidal effect—namely, the neoclassical damping of poloidal plasma flow[38]—which is too important to be neglected in a study which intends to make a connection between theory and experiment. As is well known[26], poloidal flow damping couples perpendicular and parallel (to the equilibrium magnetic field) plasma flow, giving rise to an effective *enhancement* of ion inertia by a factor $(B_\phi/B_\theta)^2_{r_s}$. We can take this effect into account by making the transformation

$$\tau_H^2 \to \tau_H^2\left[1 + \left(\frac{B_\phi}{B_\theta}\right)^2\right]_{r_s} = \frac{\tau_H^2}{K_\phi}, \tag{127}$$

where

$$K_\phi = \frac{(n\epsilon_s)^2}{m^2 + (n\epsilon_s)^2}. \tag{128}$$

Note that the enhancement of ion inertia is generally very large for tokamaks [for which $m^2 \gg (n\epsilon_s)^2$], but only $O(1)$ for RFPs [for which $m^2 \sim (n\epsilon_s)^2$].

V. RESULTS

A. Critical modulation frequencies

The *non-localized* regime, discussed previously, corresponds to the *low modulation frequency* ordering

$$|v'|\,\tau_V \ll 1, \tag{129}$$

where v' is the (unnormalized) modulation frequency (*i.e.*, the instantaneous difference between the helical phase velocity of the island chain and that of the external perturbation) and τ_V is, roughly speaking, the momentum confinement time-scale [see Eq. (26)]. Likewise, the *weakly localized* regime corresponds to the *intermediate modulation frequency* ordering

$$1 \ll |v'|\,\tau_V \ll \left(\frac{r_s}{W}\right)^2 \tag{130}$$

where r_s is the radius of the rational surface and W is the island width. Finally, the *strongly localized* regime corresponds to the *high modulation frequency* ordering

$$\left(\frac{r_s}{W}\right)^2 \ll |v'|\,\tau_V. \tag{131}$$

In conventional, ohmically heated, toroidal magnetic fusion experiments the inequality (129) is only likely to be satisfied in situations where the magnetic island chain "locks" to the external perturbation (in which case, $|v'| \to 0$). In other words, the weakly localized regime probably only applies to phase-locked island chains. The inequality (131) is, in fact, very hard to satisfy in conventional toroidal magnetic fusion experiments, except in situations where the radial width of the island chain becomes very large (*i.e.*, $W \to r_s$). In other words, the strongly localized regime may be difficult to achieve in practice.

B. Non-localized regime

According to the analysis contained in Sects. III F and IV C–IV E, the (unnormalized) Rutherford equation takes the form

$$I_1\,\tau_R\,\frac{d(W/r_s)}{dt} = E(c) + E_{sc}\left(\frac{W}{W_c}\right)^2\cos\varphi + \frac{l_1}{8\sqrt{2}}\,E_{sc}^2\,K_\phi\,\frac{\tau_V^2}{\tau_H^2}\left(\frac{W}{4\,r_s}\right)^3\left(\frac{W_c}{4\,r_s}\right)^4\sin^2\varphi, \tag{132}$$

in the non-localized regime, whereas the (unnormalized) island equation of motion is written

$$J_1\,\tau_V^2\,\frac{dv}{dt} + \frac{\tau_V\,(v - v_0)}{J} + \frac{E_{sc}\,K_\phi}{2}\,\frac{\tau_V^2}{\tau_H^2}\left(\frac{W}{4\,r_s}\right)^2\left(\frac{W_c}{4\,r_s}\right)^2\sin\varphi = 0. \tag{133}$$

Here, the "vacuum island width" $W_c = 4\,r_s\,\sqrt{\hat{\Psi}_c/F_s'}$ is a convenient measure of the amplitude of the external perturbation, $v = d\varphi_s/dt$ is the instantaneous helical phase velocity of the magnetic island chain, and $v_0 = \Omega_s^{(0)}$ is the unperturbed phase velocity of the island chain. Furthermore,

$$l_1 = 4\sqrt{2}\int_1^\infty\left(\langle\cos^2\zeta\rangle - \frac{\langle\cos\zeta\rangle^2}{\langle 1\rangle}\right)\mathcal{F}\,\frac{d\mathcal{F}}{d\hat{\psi}}\,d\hat{\psi} = 0.3319. \tag{134}$$

Note that the final term in Eq. (132) is *positive*, indicating that the inertial modification to Rutherford's equation is *destabilizing* in the non-localized regime. This result is consistent with that reported in Waelbroeck and Fitzpatrick (1997)[28]. Equation (133) takes the form of a *pendulum equation*: the first term represents the inertia of the plasma which co-rotates with the island chain, the second term represents the viscous restoring torque acting on the chain, whereas the third term represents the electromagnetic torque exerted in the vicinity of the chain by the external perturbation.

C. Weakly localized regime

In the weakly localized regime, the Rutherford equation takes the form

$$I_1\,\tau_R\,\frac{d(W/r_s)}{dt} = E(c) + E_{sc}\left(\frac{W}{W_c}\right)^2\cos\varphi + \frac{l_1}{8\sqrt{2}}\,E_{sc}^2\,K_\phi\,\frac{\tau_V^2}{\tau_H^2}\left(\frac{W}{4\,r_s}\right)^3\left(\frac{W_c}{4\,r_s}\right)^4\sin^2\varphi, \tag{135}$$

whereas the island equation of motion is written

$$\sqrt{\frac{2}{|v'|\,\tau_V}}\,\tau_V^2\,\frac{dv}{dt} + \sqrt{2\,|v'|\,\tau_V}\,\tau_V\,(v-v_0) + \frac{E_{sc}\,K_\phi}{2}\,\frac{\tau_V^2}{\tau_H^2}\left(\frac{W}{4\,r_s}\right)^2\left(\frac{W_c}{4\,r_s}\right)^2\sin\varphi = 0. \tag{136}$$

Here, $v' = d\varphi/dt$ is the modulation frequency (*i.e.*, the instantaneous difference between the helical phase velocity of the island chain and that of the external perturbation). Note that the inertial modification to Rutherford's equation is again *destabilizing*. Moreover, the island equation of motion again takes the form of a pendulum equation. However, the inertial term [*i.e.*, the first term in Eq. (136)] is smaller than previously, since the region of the plasma which co-rotates with the island chain has shrunk in going from the non-localized to the weakly localized regime. Likewise, the viscous restoring term (*i.e.*, the second term) is larger than previously, because the radial velocity scale-length has decreased in going from the non-localized to the weakly localized regime.

D. Strongly localized regime

In the strongly localized regime, the Rutherford equation takes the form

$$I_1\,\tau_R\,\frac{d(W/r_s)}{dt} = E(c) + E_{sc}\left(\frac{W}{W_c}\right)^2\cos\varphi + 2\sqrt{2}\,l_2\left(\frac{4\,r_s}{W}\right)^3\frac{(v-v_0)^2\,\tau_H^2}{K_\phi}, \tag{137}$$

whereas the island equation of motion is written

$$l_3\left(\frac{W}{4\,r_s}\right)\tau_V^2\,\frac{dv}{dt} + l_4\,\sqrt{2\,|v'|\,\tau_V}\,\tau_V\,(v-v_0) + \frac{E_{sc}\,K_\phi}{2}\,\frac{\tau_V^2}{\tau_H^2}\left(\frac{W}{4\,r_s}\right)^2\left(\frac{W_c}{4\,r_s}\right)^2\sin\varphi = 0. \tag{138}$$

Here,

$$l_2 = \frac{\pi}{3\sqrt{2}} + 2\sqrt{2}\int_1^\infty\left(\langle\cos^2\zeta\rangle - \frac{\langle\cos\zeta\rangle^2}{\langle 1\rangle}\right)\frac{d}{d\hat{\psi}}\left(\frac{1}{\langle X^2\rangle}\right)\frac{d\hat{\psi}}{\langle X^2\rangle} = 0.7405 - 0.2503 = 0.4875, \tag{139}$$

$$l_3 = \pi + 2\int_1^\infty\left[\langle|X^3|\rangle\,\frac{d}{d\hat{\psi}} + \langle|X|\rangle\right]\frac{d\hat{\psi}}{\langle X^2\rangle} = 3.1415 - 0.3840 = 2.758, \tag{140}$$

$$l_4 = \pi\,\sqrt{\frac{8}{3}} = 5.130. \tag{141}$$

Note that, in Eqs. (139) and (140), the first term on the right-hand side represents the contribution of the boundary layer on the island separatrix, whereas the second term represents the contribution of the residual perturbed velocity profile beyond the separatrix. It can be seen that, in both cases, the boundary layer contribution is *dominant*, and, moreover, determines the sign of the integral. Note that the inertial modification to Rutherford's equation is again *destabilizing*. Furthermore, the island equation of motion again takes the form of a pendulum equation. The shrinkage of the region of the plasma which co-rotates with the island chain, in going from the weakly to the strongly localized regime, has the effect of reducing the magnitude of the inertia term, and enhancing the magnitude of the viscous restoring term, in the island equation of motion.

All previous studies of the influence of the ion polarization current on island stability[23–27] [with the exception of Waelbroeck and Fitzpatrick (1997)[28]] were performed in the strongly localized regime. As mentioned previously, these studies unanimously declare the ion polarization effect to be stabilizing. However, in all cases, this conclusion was arrived at after *neglecting* the effect of the boundary layer on the separatrix. The neglect of the separatrix boundary layer has three main consequences. Firstly, the ion polarization correction to Rutherford's equation is converted from a destabilizing effect into a stabilizing effect. Secondly, the form of the viscous restoring term in the island equation

of motion is modified: in the absence of the boundary layer this term is inversely proportional to the island width W, whereas in the presence of the boundary layer this term is inversely proportional to the width of the layer, and has no explicit dependence on W. Thirdly, as is clear from Eq. (140), if the inertial term in the island equation of motion is calculated *self-consistently*, then this term becomes *negative* in the absence of the boundary layer.

We believe that the neglect of the separatrix boundary layer in the strongly localized regime is *completely unjustifiable*. After all, this layer is resolved in our analysis, and develops for perfectly understandable physical reasons. Moreover, as explained above, if the boundary layer is neglected then the island chain obeys a pendulum-like equation of motion with a *negative inertia*. Of course, such an equation would predict bizarre, and manifestly unphysical, island dynamics.

VI. SUMMARY

We have analyzed the dynamics of a helical magnetic island chain—embedded in a large aspect-ratio, zero-β, toroidal plasma—in the presence of an externally imposed, rotating, magnetic perturbation of the same helicity. Our study is performed within the context of standard, zero-β, resistive-MHD theory. However, unlike previous studies, we have made no explicit tokamak orderings in our calculations, so our results apply to RFPs just as well as tokamaks. Furthermore, we have—for the first time—incorporated a *realistic* treatment of plasma viscosity into this problem. This is achieved by writing the perturbed velocity profile as a separable form in time and space—see Sect. III B.

We have demonstrated the existence of *three* separate asymptotic regimes, depending on the size of the *modulation frequency*: *i.e.*, the instantaneous difference between the helical phase velocity of the island chain and that of the external perturbation. As specified in Sect. V A, for low modulation frequencies, the perturbed velocity profile extends over the whole plasma—we term this the *non-localized regime*. For intermediate modulation frequencies, the perturbed velocity profile is localized around the island chain, but its radial width remains much larger than that of the chain—we term this the *weakly localized regime*. Finally, for high modulation frequencies, the perturbed velocity profile collapses to a boundary layer on the island chain separatrix, plus a residual profile which extends a few island widths beyond the separatrix—we term this the *strongly localized regime*. Analytic expressions for the perturbed velocity profiles in each of these three regimes are given in Sect. III F.

We have also derived the forms of the Rutherford island width evolution equation and the island equation of motion in each of the aforementioned asymptotic regimes—see Sects. V B–V D. We find that the inertial correction to the Rutherford island equation, due to the effect of the ion polarization current, is robustly *destabilizing* in all three regimes. Previous reports that this effect is stabilizing can all be traced back to the erroneous neglect of the influence of the separatrix boundary layer in the strongly localized regime. We also find that the island equation of motion can be written as a simple pendulum equation in all three regimes. Of course, the coefficients in front of the inertial and viscous terms vary from regime to regime.

Our results have implications for tokamak physics, since the currently accepted explanation of the stability threshold for the β-driven tearing modes which limit plasma confinement in many long-pulse tokamak discharges depends on a balance between the destabilizing effect of the perturbed bootstrap current and the supposed stabilizing effect of the ion polarization current. Clearly, since the ion polarization effect is, in fact, destabilizing, the validity of this explanation must, henceforth, be regarded as somewhat questionable.

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