Stability Properties of High-Beta Geotail Flux Tubes

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Abstract

Kinetic theory is used to investigate the stability of ballooning-interchange modes in the high pressure geotail plasma. A variational form of the stability problem is used to compare new kinetic stability results with earlier MHD, FAST-MHD and Kruskal-Oberman stability results (Kruskal and Oberman, 1958). Two types of drift modes are analyzed. A kinetic ion pressure gradient drift wave with a frequency given by the ion diamagnetic drift frequency ω_{*pi}, and a very low frequency mode $|\omega| \ll \omega_{*pi}, \omega_{Di}$ that is often called a convective cell or the trapped particle mode. For these slow modes a general procedure for solving the stability problem in a $1/\beta$ expansion for the minimizing $\delta B_\|$ is carried out to derive an integral-differential equation for the (valid) displacement field ξ^ψ for flux tube displacement. The plasma energy released by these modes is estimated in the nonlinear state. The role of these instabilities in the substorm dynamics is assessed in the substorm scenarios described in Maynard *et al.* (1996).

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1 Introduction

A key component of quantitative modeling of magnetic substorms is to understand the stability of the stretched nightside geomagnetic flux tubes. The plasma is a hot ion plasma with a ratio $\beta = 2\mu_0 p/B^2$ of plasma pressure to magnetic pressure that varies strongly with position reaching values greater than unity at the equatorial plane. The plasma pressure is confined by the magnetic field loops which have a curvature vector $\boldsymbol{\kappa} = (\mathbf{b}\cdot\nabla)\mathbf{b}$ that is strongly peaked at the equatorial plane where the Earthward pressure gradient dp/dx is also a maximum. Under conditions whose details are still strongly debated in the literature, the product of the pressure gradient and the curvature allow a spontaneous local interchange of flux tubes to lower the systems energy $\delta W < 0$. In such regions, an initial disturbance $\boldsymbol{\xi}$ with a strong East-West variation described by numbers $k_y \gg k_x \gg k_\parallel$ grows exponentially at the rate of the fast MHD interchange growth rate $\gamma_{\rm MHD} = (dp/\rho R_c dx)^{1/2} = v_i/(L_p R_c)^{1/2}$. Here L_p is the pressure gradient scale length and R_c^{-1} is the (equitorial) value of $\max|\boldsymbol{\kappa}|$, and $v_i = (T_i/m_i)^{1/2}$ is the ion thermal velocity. The coordinates are the GSM orthogonal x, y, z coordinates which are centered on the Earth with $\widehat{\mathbf{x}}$ directed toward the sun, $\widehat{\mathbf{y}}$ in the Earth's elliptic plane, and $\widehat{\mathbf{z}}$ in the northward direction in the plane defined by the Earth magnetic dipole axis and $\widehat{x}$. The GSM coordinates are approximate for geotail physics where the solar wind controls the direction of currents and pressure gradients rather than the near-Earth dipole field. There are two immediately apparent conditions for the growth to occur: the interchange energy released must exceed (i) the energy involved in compressing the plasma $\delta W_{\rm comp} \propto pV(\kappa\xi_r)^2 > 0$ and (ii) the energy increase resulting from disturbing and bending the magnetic field lines which are $\delta B_\parallel^2/2\mu_0$ and $\delta B_\perp^2/2\mu_0$ respectively. Here $V = \int ds/B$ is the volume of the flux tube. In Horton *et al.* (1999) the result of the stability analysis is that the range of β between β_1 and β_2 is unstable with β_1 set by line bending stabilization and β_2 by the compressional energy. The result is shown schematically in Fig. 1.

These precise stability conditions derived from the constraints are complex and depend greatly on the dynamical description of the plasma. For fast modes $|\omega| > \omega_{*pi}, \omega_{Di}, \omega_{bi}$ the MHD description is adequate and the results of Lee and Wolf (1992) and Lee (1998, 1999) and others apply and are reviewed in Sec. 2. Typically the growth rate computed from the MHD theory is not sufficiently fast to justify the MHD description particularly at realistic values of k_x, k_y and $k_\|$. Some recent works Sundaram and Fairfield (1997), Cheng and Lui (1998), Horton *et al.* (1999b) emphasize this same point that a kinetic description of the geotail stability problem is required for substorm stability theory.

There is considerable observational evidence to suggest that in the early stage of substorm development there are westward propagating magnetic oscillations on auroral field lines where the equatorial $\beta_n = 2\mu_0 p/B_n^2$ is in the transitional range between the dipole-dominated potential-like field ($\mathbf{B}_{dp} = -\nabla\Phi_{dp}$) and the high beta geotail plasma field in which $\mu_0 j_y \cong \partial B_x/\partial z \gg \partial B_z/\partial x$. Horton *et al.* (1999) conclude that fast-growing interchange-ballooning fluctuations that satisfy the validity conditions for the hydrodynamic modes arise only in this transitional region. While Lee (1999) reports finding small, negative values of the MHD variational energy $\delta W^{\rm MHD}$ throughout the geotail (with the condition of $\mathbf{B}\cdot\nabla(\nabla\cdot\boldsymbol{\xi}) = 0$ imposed), the values of the e-folding rate from the $\delta W^{\rm MHD} < 0$ calculation are too slow ($> 100\,$s) to be valid within the MHD model. Thus, we essentially disagree with his conclusion that the geotail is MHD unstable for $\beta > \beta_{cr} \gg 1$.

In contrast with Fig. 4 in Lee (1999) that shows the $\delta W^{\rm MHD}$ (FAST)> 0 for the fast MHD model given in Horton *et al.* (1999), we show here in the corresponding Fig. 2 that, within the framework of the hydrodynamic approximation, there is a window of instability for $\beta_1 < \beta < \beta_2 \sim 1-3$ where MHD is valid. For $\beta > \beta_{\rm cr} \gg 1$ the small $\delta W^{\rm MHD}$ values show that a kinetic variational principle must be used to determine the stability. The most serious limitations on the hydrodynamic model is the neglect of (1) the divergence of the thermal flux and (2) the role of the charge separation arising from the divergence of the off-diagonal momentum stress tensor. With respect to global substorm dynamics both these kinetic effects are analyzed and included in the low-dimensional simple global modeling procedure in the WINDMI substorm model of Horton and Doxas (1996, 1998). We return to this discussion after presenting our new kinetic stability theory formulas.

The theory developed here provides theoretical support for the secnarios of substorm dynamics developed in works such as Maynard *et al.* (1996) and by Frank *et al.* (1998). The drift waves driven by the ion pressure gradient form the precursor Pc5 and then Pi 2 oscillations well in advance of the sudden auroral brightening in the scenarios. Integraion of this microscopic description with global M-I coupling models will provide new quantitative models of substorm dynamics.

Here we develop and analyze the kinetic stability theory of the finite β geotail flux tubes. In particular, we analyze two regimes in detail. The higher frequency modes are in the drift wave regime where $k_\| v_i \lesssim \omega_{bi} < \omega < \omega_{be}$ so that the ions have a local kinetic response and the electrons have a bounce average response to the fluctuations. In this frequency interval the dominant mode is the kinetic ion drift wave with $\omega_k \simeq \omega_{*pi} = k_y dp_i/en_i dA = k_y T_i/eB_n L_p$ that propagates westward and resonates with the guiding center drift velocity of ions. Here $A(x,z)$ is the equilibrium poloidal flux function giving $\mathbf{B} = \nabla \times (A\hat{\mathbf{y}})$ for the local field. As the growth rate $\gamma_k(t)$ increases, this mode continuously deforms into the MHD ballooning/interchange mode with small $E_\| \cong (v_A^2 k_\|^2 k_\perp^2 \rho_s^2/\omega_*^2)E_{k_\perp}$. The work here presents the most detailed analysis of the full compressional energy for this mode and the β-dependence of the growth rate.

The second kinetic mode is the $|\omega| < \omega_{Di}, \omega_{bi}$ low frequency (LF) mode that is called either a convective cell or a trapped particle mode in the literature. Here we show that the compressional stabilization term dominates the energy release through the $\delta B_\|$ perturbation. For high plasma pressure we introduce a new expansion in powers of $1/\beta$ for finding the minimum kinetic δW. The result gives a new, analytic theory for the high beta stability of these low frequency disturbances. Due to their low frequency and relatively large scale $k_y \rho_i \ll 1$, these modes can release substantial amounts of energy when unstable. When the modes are neutral, or weakly damped, they can be driven up by nonlinear coupling to the higher frequency drift wave instabilities. Such low frequency modes are thought to be a mechanism for producing Bohm scaling of thermal energy transport in laboratory confinement devices. To our knowledge, this work is the first study of such kinetic modes for the high β plasma with strongly curved field loops.

Since the remainder of the paper is rather technical, we briefly review some features of the low frequency waves in the kinetic, high beta, collisionless plasma in this paragraph. The local kinetic modification of the three MHD waves is determined by $\omega/k_\parallel v_e$ and $\omega/k_\parallel v_i$. Recall the three modes $\omega^2 = k_\parallel^2 v_A^2$ with E_y dominant and $\omega_\pm^2 = k^2\{(v_A^2+v_s^2) \pm [(v_A^2+v_s^2)^2 - 4v_A^2 v_s^2 k_\parallel^2/k^2]^{1/2}\}$ with $E_x, E_\parallel$ dominant for $\mathbf{k} = (0, k_y, k_\parallel)$. The first Alfvén mode is an ordinary wave in the limit of a weak magnetic field ($1/\beta \to 0$). For $v_i < \omega/k_\parallel = v_A < v_e$ the Alfvén wave has a weak damping rate $\gamma_A/\omega = -\frac{1}{2}(\pi T_i/T_e)^{1/2}(m_e/m_i)(v_e/v_A)(\omega/\omega_{ci})^2(\tan^2\theta + \text{cotan}^2\theta)$ where $k_\parallel = k\cos\theta$. The polarization vector for $\cos\theta \ll 1$ is

$$\mathbf{e}_A = \left(\frac{i\omega\cos^2\theta}{\omega_{ci}},\ 1,\ -\frac{\omega^2 v_s^2}{\omega_{ci}^2 v_A^2}\right). \tag{1}$$

The kinetic modification to the second set of modes for $\cos\theta \ll v_A/v_e$ the magnetoacoustic wave is $\omega^2 = k^2 v_a^2 + 2(1+T_e/T_i)(k^2 v_i^2)$ which gives the kinetic value of the sound speed $v_s^2 = 2(1+T_i/T_e)(T_i/m_i)$ for these waves. The polarization vector for the wave is

$$\mathbf{e}_M = \left(1,\ \frac{i\omega}{\omega_{ci}},\ \frac{-iv_s^2\omega}{v_A^2\omega_{ci}}\cos\theta\right). \tag{2}$$

The damping of this wave is also very weak. In the limit $B \to 0(1/\beta \to 0)$ this mode becomes the extraordinary wave.

Due to the rapid change in direction with position $\mathbf{x}$ of the vector $\mathbf{b}(\mathbf{x}) = \mathbf{B}/B$ in the geomagnetic tail and the other inhomogeneities leading to the diamagnetic drift frequencies $\omega_{*pi}, \omega_{*e}$ and to $\omega_{\nabla B}$ and ω_*, the two modes in Eqs. (1)-(2) are coupled. Thus, we must present a full, symmetric 3×3 matrix for the waves in the geotail plasma. The six complex matrix elements determine the waves and their polarizations. We solve the full matrix locally, numerically, and take various analytic limits to recover well-known simplified descriptions.

Before presenting the details of the full 3×3 matrix, let us recall the low frequency complex dielectric functions that give the dominant contributions to the wave matrix for $k_y\rho_i \ll 1$ waves.

They are the kinetic cross-field polarization drift dielectric (in mks units)

$$\epsilon_\perp \cong \sum_i \frac{m_i n_i}{B^2}\left(1-\frac{\omega_{*pi}}{\omega}\right) \tag{3}$$

with ω_{*pi} the ion diamagnetic drift (westward) frequency and the sum is overall ion components of the plasma. The parallel dielectric function is dominated by the electron current and is

$$\epsilon_\parallel \cong 1-\frac{\omega_{pe}^2}{2k_\parallel^2 v_e^2}\left(1-\frac{\omega_{*e}}{\omega}\right) Z'\left(\frac{\omega}{k_\parallel v_e}\right) \tag{4}$$

where Z' is the standard plasma dispersion function. For $k_\parallel v_e/\omega \to 0$ there are $\epsilon_\parallel \cong 1-\omega_{pe}^2/\omega^2 = 0$ plasma waves. For the slower waves of interest here $k_\parallel v_e \gg |\omega|$, the parallel dielectric function becomes

$$\epsilon_\parallel \simeq \left(\frac{\omega_{pe}^2}{k_\parallel^2 v_e^2}\right)\left(1-\frac{\omega_{*e}}{\omega}\right)\left(1+i\sqrt{\frac{\pi}{2}}\,\frac{\omega}{|k_\parallel| v_e}\right). \tag{5}$$

The approximate dispersion relation from the full determinant $\|k^2\delta_{ij}-k_ik_j-\omega^2\mu_0\epsilon_{ij}\|=0$ approximately separates into $\left(k_y^2\epsilon_\perp+k_\parallel^2\epsilon_\parallel-\omega^2\mu_0\epsilon_\perp\epsilon_\parallel\right)=0$ and $k_\parallel^2=\omega^2\mu_0\epsilon_\perp$ modes. The result is that there are modes with

$$\omega^2-\omega\omega_{*pi}-k_\parallel^2 v_A^2=0, \tag{6}$$

which at high β has an ion diamagnetic drift wave

$$\omega=\omega_{*pi}+\frac{k_\parallel^2 v_i^2}{\omega_{*i}\,\beta}\cong\omega_{*pi} \tag{7}$$

with a small $E_\parallel \simeq k_\perp^2\rho_s^2(k_\parallel^2 v_A^2/\omega_*^2)E_y$. The drift wave shear Alfvén mode has $\omega^2(1+k_\perp^2\rho_s^2)-\omega\omega_{*e}-k_\parallel^2 v_A^2 k_\perp^2\rho_s^2=0$ with $E_\parallel \cong i\omega\delta A_\parallel \gg ik_\parallel\phi$ and becomes the low frequency convective cell for $T_e/T_i \ll 1$ and $\beta \gg 1$. Thus, there are modes: one with $\omega\cong\omega_{*pi}$ and small $E_\parallel$ and another with $\delta j_\parallel \cong 0$ and $E_\parallel \neq 0$ at $\omega \simeq \omega_{*e}$ where at high β the $E_\parallel$ is dominated by the induction electric field from $\delta B_\perp$.

The symmetry of the full 3×3 wave matrix arises from the Hamiltonian structure of the wave-particle system in the phase space. A consequence of these symmetries is that there is a variation form of the perturbed energy $\mathcal{L}(\mathbf{U})$. Here we use the symbol $\mathcal{L}$ to reserve δW for the conventional potential energy release used later in the article. Here $\mathbf{U}$ represents any number of choices of three potentials used to represent the perturbed electromagnetic fields. The different choices are related by gauge transformations. One choice of the three potentials is $\mathbf{U} = (\phi, \psi, \delta B_\parallel)$ with $E_\parallel = -ik_\parallel(\phi - \psi)$ used in Horton *et al.* (1985). Here ψ is the back *emf* in volts from the induction electric field from $\partial B/\partial t$ for a Faraday loop going in the equitorial plane from $y = 0$ to y and up the field line to the field point x, y, z. A second closely-related choice of fields $\mathbf{U} = (\xi^\psi, \psi, Q_L)$ is used for the variational quadratic form Sec. 2 to show the relation to the MHD δW. This form of the potentials is convenient for passing to the limit of the Kruskal-Oberman stability test and to the ideal MHD stability limit.

2 Electromagnetic Modes—Full Kinetic Mode Equation

Here we use the field representation of $\mathbf{E}_\perp = -\nabla_\perp\phi, E_\parallel = -\nabla_\parallel(\phi - \psi)$, and $\delta\mathbf{B} = \widehat{\mathbf{b}}_0 \times \nabla\psi + \widehat{\mathbf{b}}_0 \delta B_\parallel$ following Horton *et al.* (1985). The wave matrix is a complex symmetric one due to the time-reversed symmetry and parity symmetry of the underlying equations. The details are given in Horton *et al.* (1985) and in the fluid description in Horton *et al.* (1983a,b). Here we summarize the key results.

2.1 Kinetic dispersion relation

The wave fluctuations satisfy

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ \delta B_\parallel \end{bmatrix} = 0 \tag{8}$$

where the six complex response function are

$$a = -1 + \frac{T_e}{T_i}(P - 1)$$

$$b = 1 - \frac{\omega_*}{\omega}$$

$$c = Q \tag{9}$$

$$d = \frac{k_\perp^2 \rho_s^2 \omega_A^2}{\omega^2} - \left(1 - \frac{\omega_*}{\omega}\right) + \frac{\omega_{De}}{\omega}\left(1 - \frac{\omega_{*pe}}{\omega}\right)$$

$$e = -\left(1 - \frac{\omega_{*p_e}}{\omega}\right)$$

$$f = \frac{2}{\beta_e} + R.$$

Here the local ($|\omega| \gg k_\parallel v_i$) ion kinetic response function are P, Q, R given by

$$P = \left\langle \frac{\omega - \omega_{*i}(\epsilon)}{\omega - \omega_{Di}} J_0^2 \right\rangle$$

$$R = \left\langle \frac{\omega - \omega_{*i}(\epsilon)}{\omega - \omega_{Di}} \cdot \frac{m_i v_\perp^2}{b_i T_i} J_1^2 \right\rangle$$

$$Q = \left\langle \frac{\omega - \omega_{*i}(\epsilon)}{\omega - \omega_{Di}} \left(\frac{m_i}{b_i T_i}\right)^{1/2} v_\perp J_0 J_1 \right\rangle$$

Let us consider the well-known limits. For low β plasma $f \gg 1$ and determinant D of Eq. (8) is

$$D = (ad - b^2) f \simeq 0. \tag{10}$$

For this system the MHD modes have $a \simeq -b$ and $d \cong -b$. Equation (10) for $f \neq 0$ gives the kinetically modified MHD modes

$$\omega^2 - \omega\omega_{*pi} - k_\parallel^2 v_A^2 + \gamma_{\mathrm{MHD}}^2 = 0$$

and the electron drift mode

$$\omega^2 (1 + k_\perp^2 \rho_s^2) - \omega\omega_{*e} - k_\parallel^2 v_A^2 k_\perp^2 \rho_s^2 = 0. \tag{11}$$

The details of the fluid reductions are given in Horton *et al.* (1985).

2.2 Bounce-Averaged Electrons

Small pitch angle electrons are either lost to the atmosphere or take such a long path that they return out of phase with wave due to fluctuations in the intervening medium. We let f_t = fraction of trapped electrons in the flux tube. The precise value of f_t depends on how long a path length is allowed for coherent return and the steepness of the pitch angle gradient of the electron distribution function near the loss one pitch angle.

1. Lost electrons have only the local adiabatic perturbed velocity distribution

$$\delta f_e = \left[\frac{e\phi}{T_e} - \left(1 - \frac{\omega_{*te}}{\omega}\right)\frac{e\psi}{T_e}\right] F_e$$

   for $\lambda < \lambda_{\text{crit}}$. Here λ_{crit} is the critical value of the $\sin^2\alpha$ for pitch angle α defined by the velocity vector at the magnetic equitorial plane.

2. Trapped electrons have both an adiabatic and a nonlocal bounce-averaged response:

$$\delta f_e = \left[\frac{e\phi}{T_e} - \left(1 - \frac{\omega_{*te}}{\omega}\right)\frac{e\psi}{T_e} - \left(\frac{\omega - \omega_{*e}}{\omega - \overline{\omega}_{De}}\right)\overline{K}_e\right] F_e \tag{12}$$

   where

$$\overline{K}_e = \frac{1}{\tau}\oint \frac{ds}{|v_\parallel|}\left[\phi - \left(1 - \frac{\omega_{De}}{\omega}\right)\psi - \frac{v_\perp^2}{v_e^2}\frac{\delta B_\parallel}{B}\right]. \tag{13}$$

   The electron density fluctuation is found to be

$$\frac{\delta n_e}{n_e} = \frac{e\phi}{T_e} - \left(1 - \frac{\omega_{*e}}{\omega}\right)\frac{e\psi}{T_e} - f_t \int_{t_r} d^3v\, F_e \tag{14}$$

$$\left[\frac{\omega - \omega_{*te}}{\omega - \overline{\omega}_{De}}\frac{1}{\tau}\oint \frac{ds}{v_\parallel}\left(\overline{\phi} - \overline{\left(1 - \frac{\omega_{De}}{\omega}\right)\psi} - \overline{\frac{v_\perp^2}{v_e^2}\frac{\delta B_\parallel}{B}}\right)\right]$$

   where the overbar denotes the bounce averaging $\overline{\phi} = \tau^{-1}\oint ds/|v_\parallel|\phi$. We use the angular brackets to denote the integral over velocity space $\langle F\rangle = \int d^3vF = 2\pi\int \frac{d\mathcal{E}d\mu}{|v_\parallel|}F$.

2.3 Quasineutrality Condition

From the electron distribution function and δn_e in Eq. (14) we compute for quasineutrality $\delta\widetilde{n}_e = \delta\widetilde{n}_i$ that

$$(AU)_1 = \widehat{a}\phi + \widehat{b}\psi + \widehat{c}\delta B_\parallel = 0,$$

where the operators $\widehat{a}, \widehat{b}, \widehat{c}$ are

$$\widehat{a}\phi = \{-1+\tau(P-1)\}\,\phi + f_t\left\langle \left(\frac{\omega-\omega_{*te}}{\omega-\overline{\omega}_{De}}\right)\overline{\phi}\right\rangle$$

$$\widehat{b}\psi = \left(1-\frac{\omega_{*e}}{\omega}\right)\psi - f_t\left\langle \left(\frac{\omega-\omega_{*te}}{\omega-\overline{\omega}_{De}}\right)\overline{\left(1-\frac{\omega_{De}}{\omega}\right)\psi}\right\rangle \tag{15}$$

$$\widehat{c}B_\parallel = QB_\parallel - f_t\left\langle \frac{(\omega-\omega_{*te})}{(\omega-\overline{\omega}_{De})}\,\overline{\frac{v_\perp^2}{v_e^2}\,B_\parallel}\right\rangle$$

and we suppress the detailed expression for nonlocal bouncing averaging operator in Eqs. (15) and (16).

Now we absorb the effect of the bounce averaging operator into the factor f_t to force the system to be that of an algebraic system of equations. For $|\omega| \gg \overline{\omega}_{De}$ we get the simplified response

$$a \cong -1 + f_t - f_t\,\frac{\omega_{*e}}{\omega} - f_t\,\frac{\omega_{*pe}\,\omega_{De}}{\omega^2} + \tau(P-1)$$

$$b \cong 1 - \frac{\omega_{*e}}{\omega} - f_t\left(1-\frac{\omega_{*e}}{\omega}\right) \tag{16}$$

$$c \cong Q - f_t\left(1-\frac{\omega_{*pe}}{\omega}\right).$$

2.4 Parallel Component of Ampére's Law

For $\omega^2 \gg \omega_s^2$ we obtain from Ampére's law, $\nabla_\perp^2 A_\parallel = -\mu_0\delta j_\parallel$, the following law, integral-differential equation:

$$\frac{\rho_s^2 v_A^2}{\omega^2}\,\frac{\partial}{\partial s}\,\nabla_\perp^2\,\frac{\partial\psi}{\partial s} = \left[-1+\frac{\omega_{*e}}{\omega} + f_t\left\langle\left(1-\frac{\omega_{De}}{\omega}\right)g_e\right\rangle\right]\phi$$
$$+\left[1-\frac{\omega_{*e}}{\omega} - \left(1-\frac{\omega_{*pe}}{\omega}\right)\frac{\omega_{De}}{\omega} - f_t\left\langle\left(1-\frac{\omega_{De}}{\omega}\right)g_e\overline{\left(1-\frac{\omega_{De}}{\omega}\right)}\right\rangle\right]\psi$$
$$+\left[1-\frac{\omega_{*pe}}{\omega} + f_t\overline{\left(1-\frac{\omega_{De}}{\omega}\right)\frac{v_\perp^2}{v_e^2}}\;\right]B_\parallel \tag{17}$$

where $g_e = \omega/(\omega-\overline{\omega}_{De}+i^{0t})$.

Gives the field equation

$$(AU)_2 = \widehat{b}\phi + \widehat{d}\psi + \widehat{e}B_\parallel = 0 \tag{18}$$

where $\widehat{b}$ is the same operator as in Eq. (15) and the new operators are

$$\widehat{d}\psi = \left\{k_\perp^2\rho_s^2\,\frac{\omega_A^2}{\omega^2} - \left(1-\frac{\omega_*}{\omega}\right) + \left(1-\frac{\omega_{*pe}}{\omega}\right)\frac{\omega_{De}}{\omega}\right\}\psi \tag{19}$$

$$- f_t\left\langle\left(1-\frac{\omega_{*te}}{\omega-\overline{\omega}_{De}}\right)\left(1-\frac{\omega_{De}}{\omega}\right)\overline{\left(1-\frac{\omega_{De}}{\omega}\right)\psi}\right\rangle$$

$$\widehat{e}B_\| = \left(1-\frac{\omega_{*pe}}{\omega}\right)B_\| + f_t\left\langle\frac{(\omega_{*te}}{(\omega-\overline{\omega}_{De})}\left(1-\frac{\omega_{De}}{\omega}\right)\,\overline{\frac{v_\perp^2}{v_e^2}\,B_\|}\right\rangle.$$

In Eq. (19) $k_\perp^2\widehat{\omega}_A^2 = -v_A^2 k_\perp^2(s)\partial_s$ is the line-bending operator and $\omega_e\ f_t$ is to be understood as the bounce averaging operator.

2.5 Radial Component of Ampére's Law

Radial (∇A) component of Ampére's law is:

$$(\widehat{\mathbf{e}}_y\cdot\nabla)\delta B_\| = \mu_0\delta J_\psi = \mu_0\left(\delta J_\psi^e + \delta J_\psi^i\right) \tag{20}$$

$$\mu_0\delta J_\psi^e = \frac{ik_\theta\mu_0 neT_e}{m_i\Omega_i}\left[\left(1-\frac{\omega_{*pe}}{\omega}\right)\psi + f_t\langle\overline{g}_e\overline{K}_e\rangle\right]$$

$$\mu_0\delta J_\psi^i = -\frac{ik_\theta\mu_0 n_0 eT_i}{m_i\Omega_i}\left[\tau Q\phi + RB_\|\right]. \tag{21}$$

Thus we get the third field equation

$$(AU)_3 = \widehat{c}\phi + \widehat{e}\psi + \widehat{f}B_\| = 0 \tag{22}$$

where the $\widehat{f}$ operator is derived from

$$\frac{\delta B_\|}{B} = B_\| = \frac{\beta_e}{2}\left[\left(1-\frac{\omega_{*pe}}{\omega}\right)\psi + f_t\left\langle\frac{mv_\perp^2}{2T_e}\,\overline{g}_e\overline{K}_e\right\rangle\right] - \frac{\beta_i}{2}\left(\tau Q\phi + R\frac{\delta B_\|}{B}\right) \tag{23}$$

giving

$$\widehat{f}B_\| = \left(1+\frac{\beta_i}{2}\,R\right)B_\| + \frac{\beta_e}{2}\,f_t\left\langle\left(\frac{\omega-\omega_{*te}}{\omega-\overline{\omega}_{De}}\right)\frac{v_\perp^2}{v_e^2}\,\overline{\frac{v_\perp^2}{v_e^2}\,B_\|}\right\rangle.$$

2.6 Reduced 2×2 Matrix Equations and Compressional Effects

The motions induced by ϕ and ψ produce a compressional change in the magnetic field $\delta B_\parallel$ given by

$$\frac{\delta B_\parallel}{B} = -\frac{1}{f}(c\phi + e\psi) \tag{24}$$

where properly $1/f$ is the inverse f^{-1} of the complicated integral operator in Eq. (23). The compressional change in the magnetic field is dictated by Amperé's law from Eq. (20) with the currents flow across the magnetic field lines in the $\widehat{x} \propto \nabla A$ direction. Substituting Eq. (24) into Eqs. (21) and (22) gives the reduced 2×2 symmetric matrix

$$\begin{bmatrix} a - \dfrac{c^2}{f} & b - \dfrac{ce}{f} \\ b - \dfrac{ce}{f} & d - \dfrac{e^2}{f} \end{bmatrix} \begin{bmatrix} \phi \\ \\ \psi \end{bmatrix} = 0 \tag{25}$$

with dispersion relation from the determinant

$$D_k(\omega, \mu) = ad - b^2 - \frac{1}{f}\left(c^2 d - 2bce + e^2 a\right) = 0. \tag{26}$$

The compressional MHD limit is $a \simeq -b \simeq d$ and reduces Eq. (26) to

$$ad - b^2 - \frac{d}{f}(c+e)^2 = 0. \tag{27}$$

Dispersion relation (27) will be found again in the variation equations with $E_\parallel = 0$ in Sec. 3. Now we calculate the roots of Eq. (26) and (27).

2.7 Compressional terms at finite plasma pressure

To find analytically the kinetic ballooning-interchange drift mode and connection with the variational formulas we take $\eta_e = 0$ and note $a \simeq -b \simeq d \simeq e$ for the dominant terms owing to the small $E_\parallel$. Then the full determinant D factors as

$$ad - b^2 + \frac{\tau(1 - \omega_*/\omega)(Q - (1 - \omega_*/\omega))^2}{\dfrac{2}{\beta_e} + R} = 0 \tag{28}$$

as shown by Eq. (27). Here $\omega_{*e} = \omega_*$ for $\eta_e = 0$. The last term in Eq. (28) gives the kinetic compressional response. For the near-to-MHD regime the response function ϕ in Eq. (9) reduces to

$$Q - 1 + \frac{\omega_*}{\omega} \simeq -\frac{\omega_{*pi} - \omega_*}{\omega} = -\frac{\omega_{*p}}{\omega} + i\Delta_Q \tag{29}$$

with ω_{*p} having the total pressure gradient and a small resonant part $i\Delta_Q$ from wave-particle scattering. A reasonable approximation for R in the region $\omega\omega_{Di} > 0$ is

$$R \cong c_0 \frac{[\omega - \omega_{*i}(1 + 2\eta_i)]}{\omega - \omega_{Di} + ic_1|\omega_{Di}|} \simeq 1 - \frac{\omega_{*i}(1 + 2\eta_i)}{\omega} - i\Delta_R \tag{30}$$

where c_0 and c_1 are positive fitting parameters of order $c_0 \simeq 2, c_1 \simeq 0.1$.

The resonant modes in the high-β region have

$$\omega = \omega_0 + i\gamma \simeq \omega_{*i}(1 + \eta_i) + i\gamma_k \tag{31}$$

which is a westward propagating drift wave. The Taylor series expansion of the dominant terms in Eq. (28) gives

$$\begin{aligned} ad - b^2 \simeq & \left(\frac{\omega_*}{\omega} - 1\right) \left[\frac{\omega_A^2}{\omega_0^2}\, b - b\left(1 - \frac{\omega_{*pi}}{\omega_0}\right) - \frac{\omega_*\omega_D}{\omega_0^2}\right] \\ & + \left(\frac{\omega_*}{\omega_0} - 1\right) \frac{i\gamma}{\omega_0} \left[\frac{-2\omega_A^2}{\omega_0^2}\, b - \frac{b\omega_{*pi}}{\omega_0} + \frac{2\omega_*\omega_D}{\omega_0^2}\right] \end{aligned} \tag{32}$$

for $\omega = \omega_0 + i\gamma$ with $|\gamma| \ll \omega_0$. Thus, the growth rate γ is determined by

$$\frac{i\gamma}{\omega_0} \left[\frac{2\omega_A^2}{\omega_0^2} + \frac{\omega_{*pi}}{\omega_0}\right] b + \frac{T_e}{T_i} \frac{(\omega_{*p}/\omega_0 - i\Delta_Q)^2}{c_0[1 - \dfrac{\omega_{*i}(1 + 2\eta_i)}{\omega_0} - i\Delta_R]} = 0 \tag{33}$$

for $\Delta_Q \sim \Delta_R \ll 1$. Since $\omega_{*p}/\omega_0 > 1$ the significant resonant contribution comes from the denominator. Thus, we obtain the growth rate formulas

$$\frac{i\gamma}{\omega_0} \left(\frac{2\omega_A^2}{\omega_{*p}^2} + 1\right) b + \frac{T_e}{T_i} \frac{(\omega_*/\omega_0)^2 \left[\dfrac{-\eta_i}{1 + \eta_i} + i\Delta_R\right]}{c_0 \left[\left(\dfrac{\eta_i}{1 + \eta_i}\right)^2 + \Delta_R^2\right]} = 0 \tag{34}$$

with $\gamma > 0$ for $-\Delta_R \equiv \mathrm{Im}(R) > 0$.

In Fig. 3 we show the root of Eq. (26) compared with the approximate formulas in Eq. (31). In frame 3a we show that there is a maximum growth rate at $k \cong 0.5$. In frame 3b we show the variation of $\gamma(\beta)$ for $k = 0.3$ near the maximum growth rate. Here $k = k_y \rho_i$ and ω, γ are in units of v_i/L_n.

2.8 Trapped Electron Effect on MHD Stability

Here we explain the relationship between our theory and that of Cheng (1999). Let us remind the reader of Cheng's notation for the trapped electron (t) and untrapped (u) fractional electron densities by

$$\begin{aligned} \frac{N_{et}(s)}{N_e} &= \left(1 - \frac{B(s)}{B_{\max}}\right)^{1/2} \\ \frac{N_{eu}(s)}{N_e} &= 1 - \left(1 - \frac{B(s)}{B_{\max}}\right)^{1/2} . \end{aligned} \tag{35}$$

In the CPS where $B(s) \ll B_{\max}$ then $N_{eu}(s)/N_e \simeq B(s)/2B_{\max} = 1 - f_t \ll 1$. Now away from $B_{\max}$ the perturbed electrons have the small untrapped density

$$\delta n_{eu} = \frac{eN_{eu}}{T_e} \left[\frac{\omega_{*e}}{\omega} \phi + \left(1 - \frac{\omega_{*e}}{\omega}\right)(\phi - \psi) \right], \tag{36}$$

and the large, trapped density perturbation

$$\delta n_{et} = \frac{eN_{et}}{T_e} \left[\frac{\omega_{*e}}{\omega} \phi + \left(1 - \frac{\omega_{*}}{\omega}\right) f_t \left[\phi - \psi - \left\langle \frac{\overline{(\omega - \omega_{De})(\phi - \psi)}}{(\omega - \overline{\omega}_{De})} \right\rangle \right] + \delta \widehat{n}_{et} \right] \tag{37}$$

where

$$\delta \widehat{n}_{et} = - \int \frac{d^3v\, F_e}{T_e} \left(\frac{\omega - \omega_{*te}}{\omega - \overline{\omega}_{De}} \right) \left(\frac{\overline{\omega_{De}}}{\omega}\, \phi + \overline{\frac{v_\perp^2}{2\omega_{ce}}\, \delta B_\parallel} \right).$$

The perturbed ion density, following Cheng (1999), is

$$\begin{aligned} \delta n_i \cong & -\frac{eN_i}{T_i} \left[\frac{\omega_{*i}}{\omega} + \left(1 - \frac{\omega_{*pi}}{\omega}\right)\left(1 - \overline{J_0^2}\right) \right] \phi \\ & + \frac{e}{T_i} \int d^3v\, F_i \left(\frac{\omega - \omega_{*t}}{\omega - \omega_{di}} \right) \left(\frac{\omega_{di}}{\omega}\, J_0 \phi + \frac{v_\parallel J_1 \delta B_\parallel}{k_\perp} \right). \end{aligned} \tag{38}$$

The quasineutrality condition yields

$$\left(\frac{N_{eu} + N_{et}\Delta}{N_e} \right) (\phi - \psi) = -\frac{T_e}{T_i} \left(\frac{\omega - \omega_{*pi}}{\omega - \omega_{*e}} \right) (1 - \Gamma_0)\phi + \frac{T_e}{eN_e} \left(\delta \widehat{n}_i - \delta \widehat{n}_e \right).$$

where $\Gamma_0 = I_0(b_i)\exp(-b_i)$ with $b_i = k_\perp^2 \rho_i^2$ and $b = T_e b_i / T_i$.

Using these results in the parallel component of Amperé's law one obtains

$$\nabla_\parallel \left(k_\perp^2 \nabla_\parallel \psi \right) + \frac{\omega(\omega - \omega_{*pi})}{v_A^2} \frac{1-\Gamma_0}{\rho_i^2} \phi + \left(\frac{\mathbf{B} \cdot \boldsymbol{\kappa} \times \mathbf{k}}{B^4} \right) (2\mathbf{k}_\perp \cdot \mathbf{B} \times \nabla p) \, \phi - \frac{\mathbf{B} \cdot \boldsymbol{\kappa} \times \mathbf{k}_\perp}{B^2} \omega \sum_j \delta \widehat{p}_j = 0 \tag{39}$$

where

$$\delta \widehat{p}_j = m_j \int d^3 v \left(\frac{1}{2} v_\perp^2 + v_\parallel^2 \right) F_M g_j \left[\left(1 - \frac{\omega_{*ti}}{\omega} \right) \phi \left(1 - J_0^2 \right) + g_j \, J_0 \psi \right]$$

and quasineutrality relates ψ to ϕ.

The approximate solution reduces to $\omega = \omega_{*pi}/2 + i\gamma_k^{\text{Cheng}}$ with a stabilizing factor $S \gg 1$.

$$\gamma_k^{\text{Cheng}} \cong \left(\gamma_{\text{MHD}}^2 - k_\parallel^2 v_A^2 S \right)^{1/2} \tag{40}$$

from the trapped electron's "stiffening" the magnetic field lines.

We differ with Cheng due to our inclusion of $\delta B_\parallel$ as the key first order feature of the problem. The magnetic compression in $\delta B_\parallel$ effect provides a large stabilizing influence over the modes where $\beta > \beta_{\text{cr}}$ that Cheng views as strongly unstable. We find that the high modes have a weak resonant instability at high β given approximately by Eq. (34).

2.9 Finite $E_\parallel$ in the low-β_i region

As emphasized by Cheng (1999) the kinetic theory produces a finite parallel electric field from charge separation effects. The result is expressed as the polarization relation

$$\frac{E_\parallel}{(-ik_\parallel \phi)} = \frac{\phi - \psi}{\phi} \tag{41}$$

and follows from the 3×3 matrix in the general case. It is useful to work out analytically the low-β, small $b_i = k_\perp^2 \rho_i^2$ limit of the dispersion relation, however. We show that the effect is to increase the critical β_1 for the onset of modes with finite b_i. For $b_i \ll 1$ the effect is weak, however. Thus, for the large-scale kinetic modes the more important modification is from the fluctuation-ion drift resonances than from the finite $E_\parallel$. For the effect on the electrons the $E_\parallel$ is large and their motion alone would drive $\psi \to \phi$ producing $E_\parallel = 0$, as we shall see now from quasineutrality.

From Eq. (14) for the perturbed electron density we get

$$\frac{\widetilde{n}_e}{N_e} = \frac{e\phi}{T_e} - \left(1 - \frac{\omega_{*e}}{\omega}\right)\frac{e\psi}{T_e} - f_t\left(1 - \frac{\omega_{*e}}{\omega}\right)\left[\left(\frac{e\phi}{T_e} - \frac{e\psi}{T_e}\right) + \left(\frac{\omega_{De}}{\omega}\right)\frac{e\psi}{T_e}\right]. \tag{42}$$

For $E_\parallel = 0$ the equation gives $\widetilde{n}_e = (\omega_{*e}/\omega)(N_e e\phi/T_e)$ and for $\psi = 0$ the adiabatic electrons and trapped fraction with $(1 - \omega_{*e}/\omega)(e\phi/T_e)$ response. For the ions the well-known drift wave frequency response is electrostatic

$$\frac{\widetilde{n}_i}{N_i} = \frac{e\phi}{T_i}\left[-\frac{\omega_{*i}}{\omega} - b_i\left(1 - \frac{\omega_{*pi}}{\omega}\right)\right].$$

Thus, quasineutrality ($\widetilde{n}_i = \widetilde{n}_e$) gives the result that

$$\left(1 - \frac{\omega_{*e}}{\omega}\right)(1 - f_t)(\psi - \phi) = \frac{T_e}{T_i}\, b_i\left(1 - \frac{\omega_{*pi}}{\omega}\right)\phi \tag{43}$$

showing that the FLR ion polarization current in the right-hand side of Eq. (43) produces the charge separation driving the $E_\parallel$.

Now recall that f_t is a short hand for the integral over the phase space of the bounce averaging operator defined in Eq. (13). So if we define the (unknown) eigenvalue λ_b of the operator

$$\widehat{\mathcal{L}} \equiv \left\langle \frac{1}{\tau}\int \frac{ds(\cdots)}{|v_\parallel(\mathcal{E},\mu)|}\right\rangle \to \lambda_b$$

acting on the ϕ and the ψ (and these two functions have different degrees of localization in the general case so that $\lambda_{b\phi} \neq \lambda_{b\psi}$), then we have the replacement that $f_t \to \lambda_b$. This distinction is an important one since f_t is near unity for a high mirror ration $B_{\max}/B_{\min} > 10$ while λ_b need not be so close to unity. Unfortunately, the calculation of $\lambda_{b\phi}$ and $\lambda_{b\psi}$ is a numerical problem beyond the scope of the present work. We know from the properties of the bounce averaging operator that $0 \le \lambda_b \le 1$.

From Eq. (43) we may express the polarization as

$$\phi = \frac{\psi}{1 + \dfrac{b(1 - \omega_{*pi}/\omega)}{(1 - f_t)(1 - \omega_{*e}/\omega)}} \tag{44}$$

where $b = b_i T_e/T_i$.

Polarization relation (44) is convenient for seeing that modes with $\omega \approx \omega_{*e}$ (drift modes propagating eastward) can be large $\widetilde{E}_\parallel$ modes with $\widetilde{j}_\parallel \approx 0$. In contrast drift modes propagating westward ($\omega \approx \omega_{*pi}$) have small $\widetilde{E}_\parallel$ compared with either $-ik_\parallel \phi$ or$-\partial \widetilde{A}_\parallel/\partial t$. That is, the inductive $E_\parallel(\psi)$ nearly cancels the charge separation $E_\parallel(\phi)$, $|\psi - \phi|/|\psi| \ll 1$.

Cheng (1999) argues that the factor

$$S = 1 + \frac{b(\omega - \omega_{*pi})}{(1 - f_t)(\omega - \omega_{*e})} \tag{45}$$

from the denominator of Eq. (45) is large and this gives an increase of the Alfvén wave line bending stabilization. We argue that $f_t \to \lambda_b$ with λ_b not so close to unity which makes it difficult for the S-factor to be much larger than $S_{\max} \lesssim 2$. Thus, we find that the $\delta B_\parallel$-effects and the wave-particle resonances are the dominant effects determining the onset of the westward propagating drift waves.

3 Variational Forms for the System's Energy

For understanding the relationship to ideal MHD stability theory, a convenient gauge is to express the perturbed fields $\delta\mathbf{E}$, $\delta\mathbf{B}$ in terms of the perpendicular component of the displacement vector $\boldsymbol{\xi}\ \exp(iky - i\omega t)$ and the electrostatic potential $\phi\ \exp(iky - i\omega t)$. The perpendicular component of the vector potential $\mathbf{A}_\perp$ is related to $\boldsymbol{\xi}$ by $\mathbf{A}_\perp = \boldsymbol{\xi} \times \mathbf{B}$, where $\delta\mathbf{B} = \nabla \times \mathbf{A}_\perp$ and $\delta\mathbf{E} = -\nabla\phi - \partial_t\,\mathbf{A}_\perp$.

The variational quadratic form $\mathcal{L}$ for the dynamics of the perturbed fields in these fields is

$$\begin{aligned}\mathcal{L}(\boldsymbol{\xi}, \phi) = \int d^3r \left[-\frac{mN\omega^2}{B^2}\,\boldsymbol{\xi}\cdot\boldsymbol{\xi} + \frac{1}{4\pi}\mathbf{Q}_\perp \cdot \mathbf{Q}_\perp + \frac{1}{4\pi} Q_L Q_L - 2\boldsymbol{\xi}\cdot\nabla p(\psi)\,\boldsymbol{\xi}\cdot\boldsymbol{\kappa} \right] \\ + \sum \int d^3r\, d^3v \frac{\partial F_0}{\partial \mathcal{E}} \left\{ e^2 \phi\phi - \frac{\omega - \omega^*}{\omega - \overline{\omega}_D} \overline{K}\overline{K} \right\} = 0\end{aligned} \tag{46}$$

where $\mathbf{Q}$ is the perturbed magnetic field,

$$\mathbf{Q} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) \tag{47}$$

and

$$\mathbf{Q}_\perp = \mathbf{b} \times (\mathbf{Q} \times \mathbf{b}) \qquad \mathbf{Q}_\parallel = \mathbf{b}\cdot\mathbf{Q}\ \ Q_L = \mathbf{Q}_\parallel + \boldsymbol{\xi}\cdot\nabla B - B\boldsymbol{\xi}\cdot\boldsymbol{\kappa} = -B(\nabla\cdot\boldsymbol{\xi} + 2\boldsymbol{\xi}\cdot\boldsymbol{\kappa})$$

in Eq. (36), where $\mu B = mv_\perp^2/2$ and $\mathcal{E} = mv^2/2$. The bounce frequencies ω_b of the ions and electrons are assumed to be large compared to $\omega, \omega_*, \omega_D$. The various quantities for $\mathcal{L}$ are as follows:

$$K = e\phi + \mu Q_L + (2\mathcal{E} - \mu B)\,\boldsymbol{\xi}\cdot\boldsymbol{\kappa}$$

$$\overline{K}(\mathcal{E},\mu) = \frac{\displaystyle\int \frac{ds}{v_\parallel} K(\mathcal{E},\mu,s)}{\displaystyle\int \frac{ds}{v_\parallel}}$$

$$\langle(..)\rangle = \int d^3r \int d^3v\ (..)$$

$$\boldsymbol{\kappa} = \frac{\kappa}{B}\nabla\psi$$

$$\omega_* = -\frac{k_y}{eB}\frac{\nabla y\cdot\mathbf{b}\times\nabla F_0}{\dfrac{\partial F_0}{\partial\mathcal{E}}}$$

$$\omega_D(\mathcal{E},\mu) = \frac{k_y}{eB}\left(\mu\nabla y\cdot\mathbf{b}\times\nabla B + 2(\mathcal{E}-\mu B)\nabla y\cdot\mathbf{b}\times\boldsymbol{\kappa}\right) \tag{48}$$

where $\mathbf{b}$ is the unit magnetic field vector, $\boldsymbol{\kappa}$ the field line curvature, $p(\psi)$ the plasma pressure, $F_0(\mathcal{E})$ the equilibrium particle distribution function.

We use curvilinear coordinates ψ, y, s, where s is the coordinate measuring distance along the field line. Let the vector potential be expressed in terms of the field components A_ψ and χ as follows

$$\mathbf{A}_\perp = A_\psi\nabla\psi + B\nabla_\perp\frac{\chi}{B}. \tag{49}$$

Then, we have,

$$\boldsymbol{\xi} = \frac{A_\psi}{B}\mathbf{b}\times\nabla\psi + \mathbf{b}\times\nabla\frac{\chi}{B}$$

$$\nabla\cdot\boldsymbol{\xi} = ikA_\psi - \boldsymbol{\xi}\cdot\boldsymbol{\kappa}ikA_\psi + ik\frac{\kappa}{B}\chi$$

$$\mathbf{Q}_\perp = \mathbf{b}\times\nabla\psi\ \mathbf{b}\cdot\nabla\left(A_\psi + B\frac{\partial}{\partial\psi}\frac{\chi}{B}\right) + ik\ \mathbf{b}\times\nabla y\ \mathbf{b}\cdot\nabla\chi. \tag{50}$$

We find it convenient to take ϕ, Q_L and $\xi^\psi = \boldsymbol{\xi}\cdot\nabla\psi = -ik\chi$ as our field variables instead of ϕ, A_ψ and χ, and to express all quantities in terms of ϕ, Q_L and ξ^ψ:

$$A_\psi = \frac{i}{k}\left(\frac{Q_L}{B} + \frac{\kappa}{B}\xi^\psi\right)$$

$$\chi = \frac{i}{k}\xi^\psi \tag{51}$$

In the limit of $k \gg \kappa$, the quadratic form can be approximated by:

$$\mathcal{L}(\boldsymbol{\xi},\phi) = \int \frac{ds}{B}\left[-\frac{mN\omega^2}{B^2}\,\xi^\psi\xi^\psi + \frac{1}{4\pi}\frac{\partial\xi^\psi}{\partial s}\frac{\partial\xi^\psi}{\partial s} + \frac{1}{4\pi}Q_LQ_L - 2\frac{\kappa}{B}\,\frac{\partial p}{\partial\psi}\xi^\psi\xi^\psi\right]$$
$$+\sum\int\frac{ds}{B}\int d^3v\frac{\partial F_0}{\partial\mathcal{E}}\left\{e^2\phi\phi - \frac{\omega-\omega^*}{\omega-\overline{\omega}_D}\overline{K}\overline{K}\right\} = 0 \tag{52}$$

where

$$K = e\phi + \mu Q_L + (2\mathcal{E}-\mu B)\frac{\kappa}{B}\xi^\psi. \tag{53}$$

3.1 Equivalence of Variation Equations and 3×3 Matrix

Quadratic form $\mathcal{L}$ (52) can be re-expressed in the potentials

$$\mathbf{U} = \left(\phi, A_\parallel = -\frac{ic}{\omega}\,\mathbf{b}\cdot\nabla\psi,\ B_\parallel\right)$$

that were used in Sec. 2 and in Horton *et al.* (1985). In terms of these potentials $(\psi,\phi,\delta B_\parallel = B_\parallel)$ the $\mathcal{L}$ function in the high bounce frequency limit becomes the quadratic form

$$\int\frac{ds}{B}\left[\frac{c^2\,k_\perp^2}{\omega^2}(\mathbf{b}\cdot\nabla\psi)^2 + B_\parallel^2\right] + \sum\int\frac{ds}{B}\,4\pi\int d^3v\biggl[-\frac{q^2\phi^2}{T}\,F_0 + 2\left(1-\frac{\omega_*}{\omega}\right)\frac{q\psi}{T}(q\phi+\mu B_\parallel^2)$$
$$-\frac{(\omega-\omega_*)(\omega-\omega_D)}{\omega^2}\,\frac{q^2\psi^2}{T}\,F_0\biggr]$$
$$+\sum\int\frac{ds}{B}\,4\pi\int\frac{d^3vF_0}{T}\,\frac{(\omega-\omega_*)}{(\omega-\overline{\omega}_D)}\left\{q\overline{\phi}+\mu\overline{B}_\parallel - \overline{\left(1-\frac{\omega_D}{\omega}\right)q\lambda\psi}\right\}^2 = 0. \tag{54}$$

We are able to evaluate the velocity integrals for F_0 Maxwellian. Using quasineutrality, $(\sum \frac{F_0q^2\omega_*}{T} = 0)$, we have from Eq. (54)

$$\sum\int\frac{ds}{B}\,4\pi\int d^3v\,\frac{F_0}{T}\left[-q^2\phi^2 + 2\left(1-\frac{\omega_*}{\omega}\right)q\psi(q_0+\mu B_\parallel) - \left(1-\frac{\omega_*}{\omega}\right)\left(1-\frac{\omega_D}{\omega}\right)q^2\psi^2\right]$$

$$= \sum \int \frac{ds}{B}\, 4\pi \Bigg[-q^2 \frac{N_0}{T} (\phi - \psi)^2 - 2\psi B_\parallel \frac{c}{\omega B^2} \mathbf{k}_\perp \cdot \mathbf{b} \times \nabla p + \psi^2 \frac{c^2}{B^\psi \omega^2}\, 4\pi (\mathbf{k}_\perp \cdot \mathbf{b} \times \nabla p)^2$$

$$- \frac{c^2}{B^\psi \omega^2}\, \psi^2 \mathbf{k}_\perp \cdot \mathbf{b} \times \nabla p\ \mathbf{k}_\perp \cdot \mathbf{b} \times \nabla (B^2 + 4\pi p) \Bigg].$$

then

$$p = \sum NT$$

$$\mathbf{b} \times (\mathbf{b} \cdot \nabla \mathbf{b}) = \frac{1}{2B^2}\, \mathbf{b} \times \nabla (B^2 + 8\pi p) \tag{55}$$

$$\mathbf{b} \times \boldsymbol{\kappa} = \frac{1}{b}\, \mathbf{b} \times \nabla B + \frac{4\pi}{B^2}\, \mathbf{b} \times \nabla p.$$

Thus, the quadratic form is:

$$\int \frac{ds}{B} \Bigg[\frac{c^2}{\omega^2}\, k_\perp^2 (\mathbf{b} \cdot \nabla \psi)^2 + B^2)_\parallel - 4\pi \sum \frac{N_0 q^2}{T} (\phi - \psi)^2$$

$$-2\psi B_\parallel \frac{4\pi c}{\omega B^2}\, \mathbf{k}_\perp \mathbf{b} \times \nabla p + \psi^2 \left(\frac{4\pi c}{\omega B^2} \right)^2 (\mathbf{k}_\perp \cdot \mathbf{b} \times \nabla p)^2 - 4\pi \frac{c^2}{\omega^2 B^2} \psi^2 2 \mathbf{k}_\perp \cdot \mathbf{b} \times \nabla p \mathbf{k}_\perp \cdot \mathbf{b} \times \boldsymbol{\kappa}$$

$$+ \sum \int \frac{ds}{B}\, 4\pi \int d^3 v\, \frac{F_0}{T}\, \frac{(\omega - \omega_*)}{(\omega - \overline{\omega}_D)} \left\{ q\overline{\phi} - q\overline{\psi} + \mu \overline{B}_\parallel + \frac{\overline{\omega_D q \psi}}{\omega} \right\} \Bigg] = 0. \tag{56}$$

Substituting

$$\begin{aligned} \Psi &= \phi - \psi \\ \mathcal{B} &= B_\parallel - \psi\, \frac{4\pi c}{\omega B^2}\, \mathbf{k}_\perp \cdot \mathbf{b} \times \nabla p. \end{aligned} \tag{57}$$

We obtain

$$\int \frac{ds}{B} \left[\frac{c^2}{\omega^2}\, k_\perp^2 (\mathbf{b} \cdot \nabla \psi)^2 + \mathcal{B}_\parallel^2 - 4\pi \sum \frac{N_0 q^2}{T}\, \Psi^2 - 4\pi\, \frac{c^2}{\omega B^2}\, 2 \mathbf{k}_\perp \cdot \mathbf{b} \times \nabla p \mathbf{k}_\perp \cdot \mathbf{b} \times \boldsymbol{\kappa} \psi^2 \right]$$

$$+ \sum \int \frac{ds}{B}\, 4\pi \int d^3 v\, \frac{F_0}{T}\, \frac{(\omega - \omega_*)}{(\omega - \overline{\omega}_D)}\, KK = 0. \tag{58}$$

Equation (58) is identical to Eq. (52) with the $\frac{k_\perp c}{\omega}\, \psi \to \xi^\psi$, $\Psi \to \phi$, $\mathcal{B}_\parallel \to Q_L$ where

$$K = q\overline{\Psi} + \mu \overline{\mathcal{B}}_\parallel + \overline{\mu \psi \frac{4\pi c}{\omega B^2}\, \mathbf{k}_\perp \cdot \mathbf{b} \times \nabla p} + \frac{\overline{\omega_D}}{\omega}\, q\psi = q\overline{\Psi} + \mu \overline{\mathcal{B}}_\parallel + \overline{\frac{c}{B\omega}\, \psi (m v_\parallel^2 + \mu B) \mathbf{k}_\perp \cdot \mathbf{b} \times \boldsymbol{\kappa}}. \tag{59}$$

$$\overline{K} = q\overline{\psi} + \mu \overline{\mathcal{B}}_\parallel + \overline{\frac{c}{\omega B}\, \psi (m v_\parallel^2 + \mu B) \mathbf{k}_\perp \cdot \mathbf{b} \times \boldsymbol{\kappa}}$$

3.2 MHD-like stability limit

In the limit of $\phi = 0$ and $|\omega| > |\omega^*|, |\overline{\omega}_D|$, we have the following MHD-like variational expression for ω^2:

$$\omega^2 = \frac{\displaystyle\int \frac{ds}{B}\left\{\frac{1}{4\pi}\frac{\partial \xi^\psi}{\partial s}\frac{\partial \xi^\psi}{\partial s} + \frac{1}{4\pi}Q_L Q_L - 2\frac{\kappa}{B}\frac{\partial p}{\partial \psi}\xi^\psi\xi^\psi - \sum\int d^3v\,\frac{\partial F_0}{\partial \mathcal{E}}\,\overline{K}\overline{K}\right\}}{\displaystyle\int \frac{ds}{B}\frac{mN}{B^2}\xi^\psi\xi^\psi}. \tag{60}$$

The kinetic term is bounded from below by

$$-\sum\left\langle\frac{\partial F_0}{\partial \mathcal{E}}\,\overline{K}\overline{K}\right\rangle = \sum\int dy\,d\psi\frac{15NT}{8B_0}\int_0^1 d\lambda\,\frac{\left\{\displaystyle\int \frac{ds}{(1-\lambda B/B_0)^{1/2}}\left(\lambda Q_L/B_0 + \boldsymbol{\xi}\cdot\boldsymbol{\kappa}(2-\lambda B/B_0)\right)\right\}^2}{\displaystyle\int\frac{ds}{(1-\lambda B/B_0)^{1/2}}}$$

$$\geq \sum\int dy\,d\psi\frac{15NT}{8B_0}\frac{\left\{\displaystyle\int_0^1 d\lambda\int \frac{ds}{(1-\lambda B/B_0)^{1/2}}\left(\lambda Q_L/B_0 + \boldsymbol{\xi}\cdot\boldsymbol{\kappa}(2-\lambda B/B_0)\right)\right\}^2}{\displaystyle\int_0^1 d\lambda\int\frac{ds}{(1-\lambda B/B_0)^{1/2}}}$$

$$= \sum\int dy\,d\psi\frac{5NT}{3}\frac{\left\{\displaystyle\int\frac{ds}{B}\nabla\cdot\boldsymbol{\xi}\right\}^2}{\displaystyle\int\frac{ds}{B}}. \tag{61}$$

Note that

$$Q_L = -B\left(\nabla\cdot\boldsymbol{\xi} + 2\boldsymbol{\xi}\cdot\boldsymbol{\kappa}\right) \tag{62}$$

The expression for ω^2 is minimized by choosing

$$Q_L = -\frac{40\pi p}{3B}\frac{\displaystyle\int\frac{ds}{B}\boldsymbol{\xi}\cdot\boldsymbol{\kappa}}{\displaystyle\int\frac{ds}{B}\left(1+\frac{20\pi p}{3B^2}\right)} \tag{63}$$

from the variation of Eq. (60).

We then obtain the reduced one-field variation form

$$\omega^2 = \frac{\left\langle\left\{\dfrac{1}{4\pi}\dfrac{\partial \xi^\psi}{\partial s}\dfrac{\partial \xi^\psi}{\partial s} - 2\dfrac{\kappa}{B}\dfrac{\partial p}{\partial \psi}\xi^\psi\xi^\psi + \dfrac{20p}{3}\dfrac{(\boldsymbol{\xi}\cdot\boldsymbol{\kappa})^2}{\left\langle 1+\dfrac{20\pi p}{3B^2}\right\rangle}\right\}\right\rangle}{\displaystyle\int\frac{mN}{B^2}\frac{ds}{B}} \tag{64}$$

from Eqs. (60) and (63).

3.3 Ultra-low frequency energy principle

In the limit of $\phi = 0$ and $|\omega| < |\omega^*|, |\varpi_D|$, the kinetic term can be approximated by

$$-\sum\left\langle \frac{\partial F_0}{\partial \mathcal{E}} \frac{\omega^*}{\varpi_D} \overline{K}\overline{K} \right\rangle = \sum \int dy\, d\psi \frac{3}{4B_0} \frac{\partial NT}{\partial \psi} \int_0^1 d\lambda \frac{\left\{ \int \frac{ds}{(1-\lambda B/B_0)^{1/2}} \left(\lambda Q_L/B_0 + \boldsymbol{\xi}\cdot\boldsymbol{\kappa}(2-\lambda B/B_0)\right) \right\}^2}{\int \frac{ds}{(1-\lambda B/B_0)^{1/2}} \frac{1}{B} \left(-\frac{4\pi\lambda}{B_0}\frac{\partial p}{\partial \psi} + (2-\lambda B/B_0)\boldsymbol{\kappa}\cdot\nabla\psi \right)}$$

which is bounded below by

$$\geq \sum \int dy\, d\psi \frac{3}{4B_0} \frac{\partial NT}{\partial \psi} \frac{\left\{ \int_0^1 d\lambda \int \frac{ds}{(1-\lambda B/B_0)^{1/2}} \left(\lambda Q_L/B_0 + \boldsymbol{\xi}\cdot\boldsymbol{\kappa}(2-\lambda B/B_0)\right) \right\}^2}{\int_0^1 d\lambda \int \frac{ds}{(1-\lambda B/B_0)^{1/2}} \frac{1}{B} \left(-\frac{4\pi\lambda}{B_0}\frac{\partial p}{\partial \psi} + (2-\lambda B/B_0)\boldsymbol{\kappa}\cdot\nabla\psi \right)}$$

$$= \int dy\, d\psi \frac{\partial p}{\partial \psi} \frac{\left\{ \int \frac{ds}{B} \nabla\cdot\boldsymbol{\xi} \right\}^2}{\int \frac{ds}{B}\frac{1}{B^2}\left(-4\pi \frac{\partial p}{\partial \psi} + 2\boldsymbol{\kappa}\cdot\nabla\psi \right)} \tag{65}$$

The terms involving Q_L in the quadratic form are minimized by

$$Q_L = -\frac{4\pi}{B}\frac{\partial p}{\partial \psi} \frac{\int \frac{ds}{B} \boldsymbol{\xi}\cdot\boldsymbol{\kappa}}{\int \frac{ds}{B} \frac{\boldsymbol{\kappa}\cdot\nabla\psi}{B^2}}. \tag{66}$$

The quadratic form is now reduced to the one-field stability form

$$\mathcal{L}(\boldsymbol{\xi}^\psi) = \int d^3r \left[-mN\omega^2\, \xi^\psi \xi^\psi + \frac{1}{4\pi} \frac{\partial \xi^\psi}{\partial s} \frac{\partial \xi^\psi}{\partial s} + -2\frac{\kappa}{B}\, \frac{\partial p}{\partial \psi} \xi^\psi \xi^\psi + 2\frac{\partial p}{\partial \psi} \frac{\langle \boldsymbol{\xi}\cdot\boldsymbol{\kappa}\rangle^2}{\left\langle \frac{\kappa}{B} \right\rangle} \right] \tag{67}$$

where $\langle (..) \rangle = \int \frac{ds}{B}(..)/\int \frac{ds}{B}$. Note the essential difference in the compressional stabilization terms in Eq. (64) for higher frequencies and Eq. (67) for lower frequencies.

4 Comparisons with Earlier Works

The bounding integral for the compressional energy we give in Eq. (67) from the Schwartz inequality was derived by Rosenbluth *et al.* (1983a,b), in applying the low-frequency kinetic energy principle of Van Dam *et al.* (1982) and Antonsen and Lee (1982). The same integral occurs in Hurricane *et al.* (1994, 1995) and in Lee and Wolf (1992).

Even though the exact forms of the compressional energies are different in the Hurricane stochastic model, the Lee and Wolf (1992) MHD calculation and our bounce-averaged compressional energy, the *bounding* function used in all three works is the same. Let us see how this unusual situation arises. We change notation to $\xi^{\psi} \to X(s)$ to follow Lee and Wolf (1992) and Lee (1999).

In the MHD form of $W^{\rm MHD}(A)$ from Lee and Wolf (1992) we have

$$W^{\rm MHD} = \int \frac{ds}{B} \left\{ \left(\frac{\partial X}{\partial s}\right)^2 - \frac{2\mu_0 \kappa_A}{B}\frac{dp}{dA} X^2 + \frac{4\gamma\mu_0 p \left(\int \frac{X(s)\kappa_A}{B}\frac{ds}{B}\right)^2}{\int \frac{ds}{B}\left(1 + \frac{\mu_0 \gamma p}{B^2}\right)} \right\} \tag{68}$$

from their Eq. (15). They calculate flute interchange $W^{\rm MHD}(X=1)$ from Eq. (68). They then show that for any system that is interchange (flute) stable $W^{\rm MHD}_{X=1} > 0$, the ballooning mode $W^{\rm MHD}(A)$ is bounded below by

$$W^{\rm MHD}(A) > \int \frac{ds}{B} \left[\left(\frac{\partial X}{\partial s}\right)^2 - \frac{2\mu_0 \kappa_A}{B}\frac{dp}{dA} X^2 + \frac{2\mu_0 \frac{dp}{dA}\left(\int \frac{ds}{B}\frac{X\kappa_A}{B}\right)^2}{\left(\int \frac{ds}{B}\frac{\kappa_A}{B}\right)} \right]. \tag{69}$$

We see immediately that their bounding function on the right-hand side of Eq. (69) is the same as that in Eq. (67) derived in Sec. III by carrying out the pitch angle and energy integrals exactly after using the Schwartz inequality in Eq. (65) on the low-frequency compressional energy. Lee and Wolf (1992) evaluate the bound in the right-hand side of Eq. (69) for $X(\zeta) = \exp(-\zeta^2/\alpha^2)$ and $X = B_n^2/B^2$ and find the lower bound is positive definite for the local Taylor expansion ($dp/dA = \text{const}$) equilibrium model.

In looking at Hurricane *et al.* (1994, 1995a, 1995b), their final formulas for the compressional energy contribution for the stochastic ion orbit model, we see that $\delta W^{\rm comp}$ (stochastic) is precisely the same as the last integral in Eq. (69) used for the lower bound. Thus, the stochastic model is the most unstable in the deep tail region where in fact the model may be the most relevant since the Büchner-Zelenyi chaos parameter is definitely into the chaotic zone in this region as shown in Horton and Tajima (1991) for tearing modes. We plan to reformulate the earlier chaotic wave matrix theory of Horton and Tajima (1991), and Hernandez *et al.* (1993) for the ballooning-interchange mode to compare with the stochastic model of Hurricane *et al.* (1994, 1995). The Hurricane-Pellat model assumes that the chaotic pitch angle scattering is sufficiently strong to make the perturbed ion distribution function independent of pitch angle. The test particle modeling of Hernandez *et al.* (1993) did not have such a strong assumption. In the Hernandez *et al.* (1993) model the chaos results in the resonance broadening of the standard wave-particle resonance functions due to the decay of the two-time velocity correlation function from the chaos.

Thus, we see that the differences in the compressional kinetic energy $\delta W_{\rm comp}$ vary with the dynamical models of (1) MHD, (2) adiabatic ion motion and (3) chaotic ion motion. The adiabatic compressional energy is the largest positive energy, the MHD and the stochastic models can change relative magnitudes. For the deep tail region where the stochastic model applies, the $\delta W^{\rm stoch}$ is smaller than $\delta W^{\rm MHD}$. This is because $\delta W^{\rm stoch}$ is proportional to the pressure gradient whereas $\delta W^{\rm MHD}$ is proportional to the pressure For the near-Earth region the situation changes, but there the adiabatic kinetic theory is the correct theory. It is interesting however, that the stochastic integral based on a kinetic calculation also exceeds the lower bound on the compressional energy released from the adiabatic theory. Thus, we conclude that only the kinetic formulations of the energy release give reliable thresholds for the growth rate of auroral field line flux tubes local interchanges.

With regard to the connection to the ionosphere, we note that Hameiri (1991) shows how the growth rate is reduced as the ionospheric conductance increases. The stability is not changed in the MHD problem, however.

5 Observable Consequences of Instability

The immediate signatures of the instabilities calculated in Secs. 2-4 are the oscillations of the electromagnetic fields and their polarizations. Thus, observations of $\delta E_y, \delta B_\perp$ and $\delta B_\parallel$ at $\omega \approx \omega_{*pi} = k_y \rho_i (v_i/L_p) \cong k_y \rho_i (2\pi/100\,{\rm s})$ are predicted for the growth phase where the MHD-FAST mode condition has yet to be reached. These kinetic modes will locally flatten the x-gradient of the resonant part of the ion velocity distribution. The particle detectors would look for frequency-modulated energetic ion fluxes. The resonant ion energies $\mathcal{E}_k$ are predicted to be related to the wave frequency ω_k through $\omega_k = \overline{\omega}_{Di} \simeq \omega_{D_i}(\mathcal{E}_k/T_i)$ which weakly depend on k_y. Doxas and Horton have started test particle simulations to model the energy resolved modulated ion flux for comparison with spacecraft data.

As the wave growth rate increases by a further steepening of the Earthward gradient of the ion velocity distribution function to the point where $\gamma_k > |\omega_{Di}| \sim |\omega_{*i}|$ the mode becomes the FAST MHD mode and releases a macroscopic energy comparable to the total energy in the local flux tube. The amount of flux in the flux tube is probably best identified by the auroral arc area of the associated auroral activation and its motion measured by the VIS instrument on POLAR.

The auroral activation physics and the integration of these stability results into the WINDMI substorm model occurs through parallel current-voltage relationships for the auroral flux tubes. Before instability there is a steady field-aligned current $j_\parallel$ and associated with the parallel potential drop $\Delta\phi_\parallel = \phi_i - \phi_{ms} > 0$ with the region 1 sense of upward current from downward electron acceleration in the evening sector. With onset of the flux tube convection velocity $v_x = \gamma\xi_r$ there are neighboring flux tubes separated by π/k_y with opposite signs of $\delta j_\parallel$ and thus $\delta\phi_\parallel$. The tubes with a sign of the potential fluctuation $\delta\phi_\parallel$ such as to increase the electron precipitation produce an immediate auroral brightening. The area of the auroral brightening and its westward motion and northward motion give a visualization of the nonlinear dynamics of the flux tubes within seconds.

First we review one set of observations that clearly point to the kinetic interchange driftwave mechanism. Then we estimate the voltage $\delta\phi_\parallel$ from the size of a typical aurora brightening in Frank *et al.* (1998). We use Tsyganenko (1996) to obtain an estimate of δW and other parameters relevant to the system.

Maynard *et al.* (1996) describe the substorm onset scenario derived from a detailed analysis of six events drawn from 20 substorms. A complete array of particle and field measurements were assembled primarily from the CRRES satellite. The correlated ground magnetometer's data for the AL index and the Pi 2 pulsations were analyzed. The substorm onset time defined by the sharp decrease of the AL index from the rapid growth of the westward electrojet current. Prior to this onset from the AL signal, Maynard *et al.* (1996) report oscillations (periods of 2-3 min) about the mean downward $E_y(t)$ with evidence for the rippling of the inner edge of the plasma sheet. Pi 2 oscillations begin up to 20-26 min before the AL signal of the sharp increase in the westward electrojet. Thus, there would be 10-13 oscillations of a 2 minute wave, for example.

CRRES revolution 540 on 4 March 91 is discussed as a candidate for the interchange-ballooning substorm mechanism. In this event irregular E_y oscillations start at 1915 UT 25 min prior to the maximum of westward electrojet current $-$AL (1941). The onset time is given as 1938 UT where the AL first starts its sharp drop. In the period between 1915 UT to 1938 UT there are 10 or more oscillations in $E_y(t)$ about its mean value.

Six peaks of negative $E_y = \overline{E}_y + \delta E_y$ of a few mV/m are specifically labeled in Fig. 6 of Maynard *et al.* (1996). In that work the oscillations of E_y are inferred to ripple the near edge of plasma sheet. The hypothesis is advanced but the oscillations and the rippling are manifestations of the mechanism proposed by Roux *et al.* (1991) for the interchange substorm mechanism. The theory developed here shows the complexity of accurately describing this collisionless, high-pressure plasma dynamics.

Now we estimate the maximum energy release and the increase in the parallel potential $\delta\phi$ associated with auroral brightening from precipitating electrons from the ballooning interchange flux tube motion.

If the area of the footpoint of tube in the auroral region is roughly $A = (50\,\mathrm{km})^2 = 2.5\times10^9\,\mathrm{m}^2$, then the flux is $d\Psi = 105\,\mathrm{Wb}$. Using the Tsyganenko model we can calculate the flux tube volume $V = \int ds/B \simeq 10\,R_E/50$ nT$\simeq 4\times10^{15}\,\mathrm{m/T}$ so that the total energy in the flux tube with a 10 nPa pressure is $pVd\Psi = 10^{-8}\,\mathrm{J/m^3}\cdot 4\times10^{21}\,\mathrm{m^3} \cong 2\times10^{12}\,\mathrm{J}$. If the interchange motion occurs in $1/\gamma = 100\,\mathrm{s}$ and releases 10% of the total energy, we have a power of 2 GW, which is less than the energies and powers associated with other substorm processes. The importance of the unstable motion is that it produces a potential variation $\delta\phi$ that varies both across the field and along the field line. The cross-field variation is estimated from $\delta E_y = \xi_r B_n$ with $\xi_r \sim R_E$ to obtain $\delta E_y \leq 3\,\mathrm{mV/m}$. The potential fluctuation is then $\delta\phi = \delta E_y/k_y \lesssim 2\rho_i E_y \lesssim 500\,\mathrm{V}$. For $k_y\rho_i = 0.5$ then $\delta E_\parallel \sim \frac{1}{2}\,\delta\phi/L_\parallel \simeq 300\,\mathrm{V}/3\,\mathrm{Re} \sim 15\,\mu\mathrm{V/m}$. This potential drop is sufficient to produce a large $\delta j_\parallel \sim 2\,\mu\mathrm{A/m}^2$ current surge of precipitating electrons into the ionosphere. As the growth becomes nonlinear, the visible ($\delta\phi < 0$) flux tube moves tailward and westward due to the nonlinear flux tube motion. This nonlinear dynamics is complex and must be considered in numerical simulations as stated by Hurricane *et al.* (1997a,b, 1999). Hurricane *et al.* report that the motion can be nonlinearly unstable which produces a super accelerated motion that they call the detonation effect. and reproduce a super accelerated motion. Another approach that we prefer is to use coupled low order system of differential equations (Horton *et al.*, 1999). This system has a rich range of behavior that has not been fully explored. For large amplitude motion the low-order models (LOMS) predict nonlinear oscillations going into chaotic pulsations not unlike those in the Pi 2 signals. The LOMS also predict the creation of sheared dawn-dusk flows (Hu and Horton, 1997). Much work on both the M-I coupling processes and the nonlinear dynamics remains to be done before a full assessment of the theoretical modeling can be made.

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Appendix A: Kinetic ballooning instabilities
Magnetic field structure and guiding center drift velocities

A very useful model for geomagnetic tail system valid quantitatively up to geosynchronous orbit is the linear superposition of the 2D dipole and uniform current sheet (j_y = const) magnetic fields. Since the model is translationally invariant in y the eigenmodes are strictly sinusoidal in the y-direction which greatly simplifies the stability problem. Likewise the exact particle orbits are described by a two-degrees-of-freedom Hamiltonian with effective potential $U_{\rm eff} = \left(P_y - q\,A_y(x,z)\right)^2/2m + q\phi(x,z)$. The guiding center orbits also simplify considerably, yet show the small loss cone angle $\alpha_{\ell c} = 1/(R-1)^{1/2}$ due to the large mirror ratio $R = B_{\rm ion}/B_{gt}$. One can move the ionosphere out to perhaps $3\,R_E$ in the analog system to make the model fields closer to those of the 3D magnetosphere. The parameters of the 2D model are $B_0\,r_0^2$, B_x' and B_n and they would be optimized with respect to a given space region $\Omega = L_xL_yL_z$ for the best representation of the 3D fields. The analytical and numerical advantage of the 2D model are clear. The particle simulations of Prichett *et al.* (1997a,b) are performed in such 2D systems with further compressions of the space-time scales by use of small ion-to-electron mass ratios.

The 2D magnetotail model is derived from $\mathbf{B} = \nabla \times (A_y\,\widehat{\mathbf{y}}) = \nabla A_y \times \widehat{\mathbf{y}}$ with

$$A_y(x,z) = -\frac{B_0 r_0^2 x}{x^2+z^2} - \frac{1}{2}\,B_x' z^2 + B_n x \tag{A1}$$

which gives

$$B_z(x,z) = B_n + \frac{B_0 r_0^2(x^2-z^2)}{(x^2+z^2)^2} \tag{A2}$$

$$B_x(x,z) = B_x' z - \frac{2B_0 r_0^2 xz}{(x^2+z^2)^2} \tag{A3}$$

with

$$B^2 = B_n^2 + B_x'^2 z^2 + \frac{2B_0 r_0^2[B_n(x^2-z^2) - 2B_x' xz^2]}{(x^2+z^2)^2}. \tag{A4}$$

The shear matrix $\partial B_i/\partial x_j$ of the magnetic field is

$$B_{x,x} = \frac{2B_0 r_0^2 z(3x^2 - z^2)}{(x^2+z^2)^3} \tag{A5}$$

$$B_{x,z} = B_x' - \frac{2B_0 r_0^2 x(x^2 - 3z^2)}{(x^2+z^2)^3} \tag{A6}$$

$$B_{z,x} = \frac{-2B_0 r_0^2 x(x^2 - 3z^2)}{(x^2+z^2)^3} \tag{A7}$$

$$B_{z,z} = \frac{2B_0 r_0^2 z(z^2 - 3x^2)}{(x^2+z^2)^3} \tag{A8}$$

From Eqs. (A5)-(A8) we see that $\nabla \cdot \mathbf{B} = 0$ and that $\mu_0 j_y = B_{x,z} - B_{z,x} = B_x'$. In the region $x^2 \gg z^2$ of the earlier deep tail model with constant parameters $(B_x',\ B_n)$ is now modified to have

$$\begin{aligned} B_x' &\rightarrow B_x' - \frac{2B_0 r_0^2}{x^3} \\ B_n &\rightarrow B_n + \frac{B_0 r_0^2}{x^2} \end{aligned} \tag{A9}$$

where we note that $x < 0$ in the nightside region of interest.

Curvature drift velocity

Now we compute the curvature vector $\boldsymbol{\kappa} = (\mathbf{b}\cdot\nabla)\mathbf{b}$ and the curvature and grad-B drift velocities. The curvature vector is

$$\boldsymbol{\kappa} = (\mathbf{b}\cdot\nabla)\mathbf{b} = \kappa_x \hat{\mathbf{x}} + \kappa_z \hat{\mathbf{z}}$$

with

$$\kappa_x = \frac{B_x B_{x,x} + B_z B_{x,z}}{B^2} - \frac{B_x}{2B^4}(\mathbf{B}\cdot\nabla)B^2 \tag{A10}$$

$$\kappa_z = \frac{B_x B_{z,x} + B_z B_{z,z}}{B^2} - \frac{B_z}{2B^4}(\mathbf{B}\cdot\nabla)B^2 \tag{A11}$$

and the curvature drift velocity is proportional to

$$\mathbf{B}\times\boldsymbol{\kappa} = \left[-\frac{B_x^2}{B^2}\,B_{z,x} + \frac{B_z^2}{B^2}\,B_{x,z} + \frac{B_x B_z}{B^2}(B_{x,x} - B_{z,z})\right]$$

or

$$B^2(\mathbf{B}\times\boldsymbol{\kappa})_y = \left(B_n + \frac{B_0 r_0^2(x^2-z^2)}{r^4}\right)^2 \left(B'_x - \frac{2B_0 r_0^2 x(x^2-3z^2)}{r^6}\right)$$

$$- \left(B'_x z - \frac{2B_0 r_0^2 xz}{r^4}\right)^2 \left(\frac{2B_0 r_0^2 x(3z^2-x^2)}{r^6}\right) \tag{A12}$$

$$- \left(B_n + \frac{B_0 r_0^2(x^2-z^2)}{r^4}\right)\left(B'_x z - \frac{2B_0 r_0^2 xz}{r^4}\right)\frac{4B_0 r_0^2 z(3x^2-z^2)}{r^6}$$

where $r = (x^2+z^2)^{1/2}$. The curvature drift is in the y-direction and given by Eq. (A12) substituted into

$$v_{\rm curv}(x,z,v_\parallel) = \frac{mv_\parallel^2}{qB^2}(\mathbf{B}\times\boldsymbol{\kappa})_y. \tag{A13}$$

In the limit $z^2/x^2 \to 0$ the curvature drift reduces to

$$v_{\rm curv} = \frac{mv_\parallel^2}{qB_z^2(x)}\left(B'_x - \frac{2B_0 r_0^2}{x^3}\right). \tag{A14}$$

Gradient-B drift velocity

The gradients of B^2 are given by

$$\frac{\partial B^2}{\partial x} = -\frac{4B_0^2 r_0^4 x}{r^6} + \frac{4B_0 r_0^2}{r^6}\left[B_n x(B_z^2 - x^2) - B'_x z^2(z^2-3x^2)\right]$$

$$\frac{\partial B^2}{\partial z} = 2B_x'^2 z - \frac{4B_0^2 r_0^4 z}{r^6} - \frac{4B_0 r_0^2}{r^6}\left[B_n z(z^2-3x^2) - B'_x z(x^2-3x^2)\right] \tag{A15}$$

The gradient-B drift velocity becomes

$$v_{\nabla B}(x,z,v_\perp^2) = \frac{mv_\perp^2}{2qB^4}\left[\left(B_n + \frac{B_0 r_0^2(x^2-z^2)}{r^4}\right)\right]\Big(-\frac{2B_0^2 r_0^4 x}{r^6} + \frac{2B_0 r_0^2}{r^6}$$

$$\left[B_n x(3z^2-x^2) - B'_x z^2(z^2-3x^2)\right]\Big)$$

$$-z^2\left(B'_x - \frac{2B_0 r_0^2 x}{r^4}\right)\left[B_x'^2 - \frac{2B_0^2 r_0^4}{r^6} - \frac{4B_0 r_0^2}{r^6}\left(B_n(z^2-3x^2) - B'_x(x^2-3z^2)\right)\right]. \tag{A16}$$

In the region $z^2/x^2 \ll 1$ the grad-B drift velocity reduces to

$$v_{\nabla B} = \frac{mv_\perp^2}{2qB}\left[-\frac{2B_0^2 r_0^4}{B^2 x^5} - \frac{2B_0 r_0^2 B_n}{B^2 x^3} - \frac{z^2 B_x'^3}{B^3}\right] \tag{A17}$$

where the first two terms are in the positive $\widehat{y}$ direction and the last term is in the negative $\widehat{\mathbf{y}}$ direction.

Adiabatic parallel motion

The parallel motion of the guiding center is given by

$$m\,\frac{dv_\parallel}{dt} = q\,E_\parallel - \mu\,\frac{\partial B}{\partial s} \tag{A18}$$

which must be integrated numerically for general $E_\parallel = -\partial_s\phi - \partial_t\,A_\parallel$ fields. For the case of $\partial_t A_\parallel = 0$ and $\partial B/\partial t = 0$ the particle kinetic energy is conserved. Using the $\epsilon = \frac{1}{2}\,v^2$ (kinetic energy percent of mass) we get

$$\frac{1}{2}\,v_\parallel^2 + \mu B(s) = \epsilon \tag{A19}$$

where

$$\mu = \frac{v_\perp^2}{2B} = \frac{v^2\sin^2\alpha}{2B} \tag{A20}$$

with α the pitch angle defined by $\tan\alpha = v_\perp/v_\parallel$. The parallel motion is then given by

$$\int_0^s \frac{ds'}{\sqrt{2(\epsilon - \mu B(s'))}} = t \tag{A21}$$

giving $s = s(\epsilon, \mu, t)$.

The magnetic field strength is a minimum at $z = 0$ and increases rapidly toward the Earth. For a given field strength B_m at position S_m). For the 3D magnetosphere the auroral field lines have this loss cone boundary defined by $R_m = B_{\rm ion}/B_{\rm ions} \simeq 10^3$ with $\alpha_{\ell c} \simeq 2^\circ$. In the 2D model we have that $R_m = B_{\rm ion}/B_{\rm ions} \sim 100 - 400$ and $\alpha_{\ell c} \sim 0.1$ to 0.5 radians (5.7° to 2.7°). This makes the simulations of the full length of auroral field lines feasible.

Large pitch angle particle in motion

For $\alpha \geq 30^\circ$ the particles mirror in the region $|z| \gtrsim R_c = B_n/B_x'$ and the motion may be approximated as sinusoidal. We write

$$\frac{1}{2}\,s^2 + \mu\left(B_n^2 + B_x'^2 z^2\right)^{1/2} = \epsilon \tag{A22}$$

where $\dot{s} = \dot{z}(1 + z^2/R_c^2)^{1/2}$. For $z^2 < R_c^2$ we have

$$\frac{1}{2}\,\dot{z}^2 + \mu B_n \left(1 + \frac{1}{2}\,\frac{z^2}{R_c^2}\right) = \epsilon. \tag{A23}$$

With the motion given by

$$\begin{aligned} z(t) &= z_t(\alpha) \sin\left(\omega_b(v,\alpha)t\right) \\ \dot{z} &= \omega_b z_t \cos\left(\omega_b(v,\alpha)t\right) \end{aligned} \tag{A24}$$

where

$$\begin{aligned} \omega_b(v,\alpha) &= \frac{v_\perp}{\sqrt{2}\,R_c} = \frac{v\sin\alpha}{\sqrt{2}\,R_c} \\ z_t(\alpha) &= \frac{v_\parallel}{\omega_b} = \sqrt{2}\,\frac{R_c\,v_\parallel}{v_\perp} = \sqrt{2}\,R_c \cot\alpha. \end{aligned} \tag{A25}$$

For these deeply trapped orbits we can calculate the bounce-averaged curvature and guiding center orbits explicitly through $\overline{cos^2\theta} = \overline{\sin^2\theta} = 1/2$ $\overline{\cos^2\theta} = \overline{\sin^2\theta} = 1/2$ to find that

$$\overline{v}_{\mathrm{curve}} = \frac{m\overline{v}_\parallel^2}{q}\,\frac{B_x' B}{B_z^2} = \frac{m v_\parallel^2 B_x'}{2qB_z^2} \tag{A26}$$

and that the dipole-independent part of the gradient-B drift is

$$\overline{v}_{\nabla B} = -\frac{m v_\perp^2}{2qB}\,\frac{\frac{1}{2}\,z_t^2\,B_x'^3}{B^3} = -\frac{m v_\parallel^2\,B_x'}{2qB^2} \tag{A27}$$

and thus exactly cancels the bounce-averaged curvature drift. Thus the total bounce-averaged drift velocity is in the positive y-direction and due to B_0:

$$\overline{V}_D = \frac{m v_\perp^2}{qB}\left[-\frac{B_0 r_0^2 B_n}{B^2 x^3} - \frac{B_0^2 r_0^4}{B^2 x^5}\right] > 0 \tag{A28}$$

where $B = B_n + B_0 r_0^2/x^2$. Now we consider small pitch angle particles.

Appendix B: Energy functional of Kruskal-Oberman

Here record the Kruskal-Oberman variational formulas given in Horton *et al.* (1999) for convenience. Here we emphasize the comparison of $\delta W^{\rm KO}$ with $\delta W^{\rm MHD}$. The variation potential is

$$\delta W^{\rm K0} = \frac{1}{2}\int d^3r \Bigg[\mathbf{Q}_\perp \cdot \mathbf{Q}_\perp + Q_L Q_L - 2\boldsymbol{\xi}\cdot\boldsymbol{\kappa}\cdot\nabla p_0 - \int d^3v \frac{\partial F_0}{\partial H_0}\left\{\frac{ds/v_\parallel\,(\mu Q_L + \boldsymbol{\xi}\cdot\boldsymbol{\kappa}(2H_0-\mu B))}{\int ds/v_\parallel}\right\}^2\Bigg] \tag{B1}$$

where

$$F_0 = F_0(H_0,\psi) \tag{B2}$$

$$\boldsymbol{\xi} = \frac{\mathbf{b}\times\widetilde{\mathbf{A}}_L}{B} \tag{B3}$$

$$\mathbf{Q}_L = B\left\{\mathbf{b}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\mathbf{b} - \mathbf{b}\;\mathbf{b}\cdot(\mathbf{b}\cdot\nabla)\boldsymbol{\xi}\right\} \tag{B4}$$

$$Q_L = -B\{\nabla\cdot\boldsymbol{\xi} + 2\boldsymbol{\xi}\cdot\boldsymbol{\kappa}\} \tag{B5}$$

$$\boldsymbol{\kappa} = (\mathbf{b}\cdot\nabla)\mathbf{b}. \tag{B6}$$

It can be shown that

$$\delta W^K > \delta W^{\rm MHD} = \frac{1}{2}\int d^3r\left[\mathbf{Q}_\perp\cdot\mathbf{Q}_\perp + Q_LQ_L - 2\boldsymbol{\xi}\cdot\boldsymbol{\kappa}\boldsymbol{\xi}\cdot\nabla p_0 + \gamma p_0\left\langle\nabla\cdot\boldsymbol{\xi}_\perp\right\rangle^2\right] \tag{B7}$$

where this expression for $\delta W^{\rm MHD}$ is the MHD energy functional with $\nabla\cdot\boldsymbol{\xi} = \langle\nabla\cdot\boldsymbol{\xi}_\perp\rangle \equiv \int ds/B\;\nabla\cdot\boldsymbol{\xi}_\perp/\int ds'/B$ in order to minimize the plasma compressional energy.

Consider equilibrium independent of y-coordinate magnetic field $\mathbf{B}$ is

$$\mathbf{B} = \nabla\psi\wedge\widehat{y} \tag{B8}$$

Let

$$\widetilde{\mathbf{A}} = \widetilde{\mathbf{A}}_\perp = \widetilde{A}_\psi\nabla\psi + \nabla_\perp\widetilde{\phi}/B. \tag{B9}$$

Let the y-dependence of the perturbations be exponential $\widetilde{\phi} \to \widetilde{\phi} e^{iky}$, and let $\kappa/k \ll 1$. Then

$$\boldsymbol{\xi} \cdot \boldsymbol{\kappa} \to \frac{\boldsymbol{\kappa} \cdot \nabla \psi}{B^2} \xi^{\psi} \tag{B10}$$

$$\nabla \cdot \boldsymbol{\xi}_{\perp} \to -ik\widetilde{A}_{\psi} - \boldsymbol{\xi} \cdot \boldsymbol{\kappa} + \cdots \tag{B11}$$

$$\boldsymbol{\xi} \cdot \nabla p_0 \to \xi^{\psi} \frac{\partial p_0}{\partial \psi} \tag{B12}$$

$$Q_{\perp} \to \frac{\partial \xi^{\psi}}{\partial s} \mathbf{y} \times \widehat{\mathbf{b}} + \mathcal{O}\left(\frac{\kappa}{k}\right) \tag{B13}$$

$$Q_L \to -B\boldsymbol{\xi} \cdot \boldsymbol{\kappa} + ikB\widetilde{A}_{\psi} \tag{B14}$$

$$\xi^{\psi} \to -ik\widetilde{\phi}. \tag{B15}$$

Minimizing δW^{MHD} with respect to $\widetilde{A}_{\psi}$ gives

$$ik\widetilde{A}_{\psi} = \boldsymbol{\xi} \cdot \boldsymbol{\kappa} - \frac{2\gamma p_0}{B^2} \frac{\langle \boldsymbol{\xi} \cdot \boldsymbol{\kappa} \rangle}{\left\langle 1 + \frac{\gamma p_0}{B^2} \right\rangle} \tag{B16}$$

which substituted into Eq. (A35) yields

$$\delta W^{\mathrm{MHD}} \to \int d^3 r \left[\frac{\partial \xi^{\psi}}{\partial s} \frac{\partial \xi^{\psi}}{\partial s} - \frac{2\boldsymbol{\kappa} \cdot \nabla \psi}{B^2} \frac{\partial p_0}{\partial \psi} \xi^{\psi} \xi^{\psi} + \mu \gamma p_0 \frac{\left\langle \frac{\boldsymbol{\kappa} \cdot \nabla \psi}{B^2} \xi^{\psi} \right\rangle \left\langle \frac{\boldsymbol{\kappa} \cdot \nabla \psi}{B^2} \xi^{\psi} \right\rangle}{\left\langle 1 + \frac{\gamma p_0}{B^2} \right\rangle} \right] \tag{B17}$$

as in Lee and Wolf (1992) and Lee (1999).

If $\delta W^{\mathrm{MHD}} > 0$, then $\delta W^K > 0$ and the plasma equilibrium is stable! However, if $\delta W^{\mathrm{MHD}} < 0$, then δW^K is not necessarily negative. Hence, plasma dynamics for which δW^K is the appropriate energy functional is not necessarily unstable if $\delta W^{\mathrm{MHD}} < 0$.

We explore the limit of high plasma beta, $p \to \infty$, and we compare the two energy functionals, δW^{KO} and δW^{MHD}, in this limit.

As $\beta \to \infty$, the dominant term in δW^{KO} is minimized to lowest order in $1/\beta$ by requiring

$$\int \frac{ds}{v_{\parallel}} \left(\mu Q_L + \boldsymbol{\xi} \cdot \boldsymbol{\kappa}(H_0 - \mu B)\right) \longrightarrow \int_0^{s_0} \frac{ds}{(1-\lambda b)^{1/2}} \left(\lambda \frac{Q_L^{(0)}}{B_{\min}} + \boldsymbol{\xi} \cdot \boldsymbol{\kappa}(2 - \lambda b) \right) = 0$$

where $b \equiv B/B_{\min}$.

The solution for $\dfrac{Q_L^{(0)}}{B_{\min}}$ is:

$$\frac{Q_L^{(0)}}{B_{\min}} = -\frac{\partial}{\partial s}\frac{B}{B_{\min}}\int_0^s ds' \boldsymbol{\xi}\cdot\boldsymbol{\kappa} = -\frac{\partial}{\partial s}\frac{B}{B_{\min}}\int_0^s ds' \frac{\boldsymbol{\kappa}\cdot\boldsymbol{\nabla\psi}}{B^2}\xi^\psi. \tag{B18}$$

Neglecting terms which are higher order in $1/\beta$, we have:

$$\delta W^K = \int \frac{d\psi dy ds}{B}\left[\frac{\partial\xi^\psi}{\partial s}\frac{\partial\xi^{\psi^*}}{\partial s} - \frac{2\boldsymbol{\kappa}\cdot\nabla\psi}{B^2}\frac{\partial p_0}{\partial\psi}\xi^\psi\xi^{\psi^*} + \left|\frac{\partial}{\partial s}B\int_0^s ds'\frac{(\boldsymbol{\kappa}\cdot\nabla\psi)}{B^2}\xi^\psi\right|^2\right]. \tag{B19}$$

Variation wrt $\delta B_\parallel$

We take Q_L and $\widetilde{\phi}$ as the field variables to minimize δW^{K0} at fixed ξ^ψ.

Varying with respect to $Q_L^\dagger$ and we obtain integral equation

$$\frac{Q_L(s)}{4\pi} = \sum\int d^3v\frac{\partial F_0}{\partial H_0}\frac{(\omega-\omega_*)}{(\omega-\overline{\omega}_b)}\mu\overline{K} \tag{B20}$$

for Q_L.

We explore the limit of high plasma beta, $\beta\to\infty$, and we consider the case where $\omega\sim\omega^*\gg\overline{\omega}_b$. The right-hand side of Eq. (A48) is proportional to β. Let

$$Q_L = Q_L^{(0)} + \epsilon Q_L^{(1)} + \cdots \tag{B21}$$

where

$$\epsilon = \frac{1}{\beta}\ll 1.$$

Solution of integral equation $\delta B_{\|}(s)$

To lowest order in ϵ, we require that the right-hand side of Eq. (A48) vanish. This condition gives

$$\int \frac{ds}{v_{\|}} \left(\mu Q_L^{(0)} + \boldsymbol{\xi}\cdot\boldsymbol{\kappa}(2H_0 - \mu B)\right) \tag{B22}$$

$$\longrightarrow \int_0^{s_0} \frac{ds}{(1-\lambda b)^{1/2}} \left(\frac{\lambda Q_L^{(0)}}{B_{\min}} + \boldsymbol{\xi}\cdot\boldsymbol{\kappa}(2-\lambda b)\right) = 0 \tag{B23}$$

where $b = B/B_{\min}$.

The solution of Eq. (A51) for $Q_L^{(0)}$ is:

$$Q_L^{(0)}(s) = -\frac{\partial}{\partial s} B \int_0^s ds' \boldsymbol{\xi}\cdot\boldsymbol{\kappa} = -\frac{\partial}{\partial s} B \int_0^s ds' \frac{\boldsymbol{\kappa}\cdot\nabla\psi}{B^2}\xi^{\psi}. \tag{B24}$$

To next order in ϵ, we have from Eq. (A48) that:

$$\frac{Q_L^{(0)}}{4\pi} = \sum \int d^3v \frac{\partial F_0}{\partial H_0} \frac{(\omega-\omega_*)}{\omega} \mu^2 \frac{\int ds/v_{\|} Q_L^{(1)}}{\int ds/v_{\|}}. \tag{B25}$$

Finally, we get the complex, but explicit one-field variation energy quadratic form:

$$\int \frac{d\psi dy dx}{B} \left[-\frac{nw\omega^2}{B^2}\xi^{\psi}\xi^{\psi+} + \frac{1}{4\pi}\frac{\partial \xi^{\psi}}{\partial s}\frac{\partial \xi^{\psi+}}{\partial s} - \frac{2\boldsymbol{\kappa}\cdot\nabla\psi}{B^2}\frac{\partial p_0}{\partial \psi}\xi^{\psi}\xi^{\psi+} \right.$$
$$\left. + \frac{1}{4\pi} Q_l^{(0)} Q_L^{(0)+} + \frac{1}{u^{1/2}} \int_0^u \frac{d\lambda}{(u-\lambda)^{1/2}} \lambda^2 \frac{\left\langle \widehat{Q}_L^{(1)} \right\rangle \left\langle \widehat{Q}_L^{(1)+} \right\rangle}{\alpha(\omega,\lambda)} \right] \tag{B26}$$

where

$$Q_L^{(0)} = \frac{\partial B}{\partial s} \int_0^s ds' \frac{\boldsymbol{\kappa}\cdot\nabla\psi}{B^2}\xi^{\psi}$$
$$\alpha(\omega,\lambda) = \sum\sum_{v_{\|}} \left(\frac{2\pi}{m^2}\right)\left(\frac{m}{2}\right)^{1/2} \frac{1}{B_{\min}^2} \int_0^{\infty} dH_0 H_0^{5/2} \frac{\partial F_0}{\partial H_0} \frac{(\omega-\omega^*)}{(\omega-\overline{\omega}_D)} \tag{B27}$$
$$\left\langle \widehat{Q}_L^{(1)} \right\rangle = \frac{1}{\pi\lambda^2}\frac{\partial}{\partial\lambda}\int_0^{\lambda} du \frac{u^{1/2} Q_L^{(0)}}{(\lambda-\mu)^{1/2}} \frac{1}{4\pi}.$$

References

[1] Antonsen, T.M., and Y.C. Lee, Electrostatic modification of variational principles for anisotropic plasmas, *Phys. Fluids, 25*, 132, 1982.

[2] Beklemishev, A. D., Nonlinear saturation of ideal interchange modes in a sheared magnetic field, *Phys Fluids B 3*, 1425, 1991.

[3] Bernstein, I.B., E.A. Frieman, M.D. Kruskal, and R.M. Kulsrud, An energy principle for hydromagnetic stability problems, *Proc. R. Soc. London, Ser. A, 244*, 17, 1958.

[4] Bhattacharjee, A., Z.W. Ma, and X. Wang, Ballooning instability of a thin current sheet in the high–Lundquist–number magnetotail, *Geophys. Res. Lett., 25*, 861, 1998.

[5] Cheng, C.Z., and A.T.Y Lui, Kinetic ballooning instability for substorm onset and current disruption observed by AMPTE/CCE, *Geophys. Res. Lett., 25*, 4091, 1998.

[6] Delcourt, D.C., and G. Belmont, Particle dynamics in the near–Earth magnetotail and macroscopic consequences, in *New Perspectives on the Earth's Magnetotail*, edited by A. Nishida, D.N. Baker, and S.W.H. Cowley, pp. 193–210, AGU, Washington, D.C., 1998.

[7] Ding, C., Analytical and numerical modeling of the electromagnetic structure of geospace, Ph.D. dissertation, Dept. of Space Phys./Astron., Rice Univ., Houston, Tex., April 1995.

[8] Frank, L.A., J.B. Sigwarth, and W.R. Paterson, High-resolution global images of Earth's auroras during substorms, in *Substorms-4*, edited by S. Kokubun and Y. Kamide, pp. 3-8, Kluwer Acad., Norwell, Mass., 1998.

[9] Hameiri, E., and M.G. Kivelson, Magnetospheric waves and the atmosphere-ionosphere layer, *J. Geophys. Res., 96(A12)*, 21,125-21,134, 1991.

[10] Hernandez, J., W. Horton, and T. Tajima, Low-frequency mobility response functions for the central plasma sheet with application to tearing modes, *J. Geophys. Res. 98*, 5893, 1993.

[11] Hesse, M., and J. Birn, Three-dimensional magnetotail equilibria by numerical relaxation technique, *J. Geophys. Res., 98* (A3), 3973–3982, 1993.

[12] Hong, B.-G., W. Horton, and D.-I. Choi, Pressure gradient-driven modes in finite β toroidal plasmas, *Plasma Phys. Controlled Fusion, 31*, 1291, 1989.

[13] Horton, W., J. Smith, B. Weigel, B. Goode, I. Doxas, and J. Cary, The solar-wind driven magnetosphere-ionosphere as a complex dynamical system, *Phys. Plasmas 6(11)*, 4178-4184, 1999.

[14] Horton, W., H.V. Wong, and J.W. Van Dam, Substorm Trigger Condition, *J. Geophys. Res. 104*, 22745 (1999).

[15] Horton, W., and I. Doxas, A low dimensional dynamical model for the solar wind driven geotail-ionosphere system, *J. Geophys. Res. 103A*, 4561, 1998.

[16] Horton, W., H. Vernon Wong, and R. Weigel, Interchange trigger for substorms in a nonlinear dynamics model, *Phys. Space Plasmas, 15*, 169, 1998.

[17] Horton, W., and I. Doxas, A low-dimensional energy conserving state space model for substorm dynamics, *J. Geophys. Res., 101*, 27,223–27,237, 1996.

[18] Horton, W. and T. Tajima, *J. Geophys. Res., 96*, 15811, 1991.

[19] Horton, W., J. E. Sedlak, Duk-In Choi, and B.-G. Hong, *Phys. Fluids, 28*, 3050, 1985.

[20] Horton, W., D.F. Escande, and G. Schmidt, *Bull. Am. Phys. Soc., 28*, 1209, 1983.

[21] Horton, W., D.-I. Choi and B. G. Hong, *Phys. Fluid, 26*, 1461, 1983.

[22] Horton, W., Drift Modes in Axisymmetric Tandem Mirrors, *Phys. Fluids 24*, 1270-1278, 1981.

[23] Hu, G., and W. Horton, Minimal model for transport barrier dynamics based on ion-temperature-gradient turbulence, *Phys. Plasmas, 4(9)*, 3262-3272, 1997.

[24] Hurricane, O.A., R. Pellat, and F.V. Coroniti, The kinetic response of a stochastic plasma to low frequency perturbations, *Geophys. Res. Lett., 21*, 4253, 1994.

[25] Hurricane, O.A., R. Pellat, and F.V. Coroniti, The stability of a stochastic plasma with respect to low–frequency perturbations, *Phys. Plasmas, 2*, 289, 1995a.

[26] Hurricane, O.A., R. Pellat, and F.V. Coroniti, A new approach to low–frequency "MHD–like" waves in magnetospheric plasmas, *J. Geophys. Res., 100*, 19,421–19,428 1995b.

[27] Hurricane, O.A., R. Pellat, and F.V. Coroniti, Instability of the Lembége–Pellat equilibrium under ideal magnetohydrodynamics, *Phys. Plasmas, 3*, 2472, 1996.

[28] Hurricane, O.A., MHD ballooning stability of a sheared plasma sheet, *J. Geophys. Res., 102*, 19,903, 1997.

[29] Hurricane, O.A., B.H. Fong, and S.C. Cowley, Nonlinear magnetohydrodynamic Detonation: Part I, *Phys. Plasmas, 4*, 3565, 1997.

[30] Hurricane, O.A., B.H. Fong, S.C. Cowley, F.V. Coroniti, C.F. Kennel, and R. Pellat, Substorm Detonation, *J. Geophys. Res., 104(A5)*, 10,221, 1999.

[31] Kruskal, M.D., and C.R. Oberman, On the stability of plasma in static equilibrium, *Phys. Fluids, 1*, 275, 1958.

[32] Lee, D.–Y., Effect of plasma compression on plasma sheet stability, *Geophys. Res. Lett. 26*, 2705, 1999.

[33] Lee, D.–Y., Ballooning instability in the tail plasma sheet, *Geophys. Res. Lett., 25*, 4095, 1998.

[34] Lee, D.–Y., and R.A. Wolf, Is the Earth's magnetotail balloon unstable? *J. Geophys. Res., 97*, 19,251–19,257, 1992.

[35] Lui, A.T.Y., and J.S. Murphree, A substorm model with onset location tied to an auroral arc, *Geophys. Res. Lett., 25*, 1269–1272, 1998.

[36] Lui, A.T.Y., P.H. Yoon, and C.–L. Chang, Quasilinear analysis of ion Weibel instability in the Earth's neutral sheet, *J. Geophys. Res., 98*, 153, 1993.

[37] Lyons, L.R., A new theory for magnetospheric substorms, *J. Geophys. Res.*, **100**, 19,069, 1995.

[38] Maynard, N.C., W.J. Burke, E.M. Basinska, G.M. Erickson, W.J. Hughes, H.J. Singer, A.G. Yahnin, D.A. Hardy, and F.S. Mozer, Dynamics of the inner magnetosphere near times of substorm onsets, *J. Geophys. Res., 101*, 7705-7736, 1996.

[39] Pritchett, P.L., and F.V. Coroniti, Interchange and kink modes in the near–Earth plasma sheet and their associated plasma flows, *Geophys. Res. Lett., 24*, 2925–2928, 1997.

[40] Pritchett, P.L., F.V. Coroniti, and R. Pellat, Connection driven reconnection and the stability of the near–Earth plasma sheet, *Geophys. Res. Lett., 24*, 873–876, 1997.

[41] Rosenbluth, M.N., and C.L. Longmire, Stability of plasmas confined by magnetic fields, *Ann. Phys., 1*, 120, 1957, 1983.

[42] Rosenbluth, M.N., S.T. Tsai, J.W. Van Dam, and M.G. Engquist, Energetic particle stabilization of ballooning modes in tokamaks, *Phys. Rev. Lett., 51*, 1967, 1983.

[43] Roux, A., S. Perraut, P. Robert, A. Morane, A. Pedersen, A. Korth, G. Kremser, B. Aparicio, D. Rodgers, and R. Pellinen, Plasma sheet instability related to the westward traveling surge, *J. Geophys. Res., 96*, 17,697–17,714, 1991.

[44] Sagdeev, R. Z., V. D. Shapiro, and V. I. Shevchenko, Convective Cells in Plasmas, *Fiz. Plazmy, 4*, 551, 1978.

[45] Smith, J.P., J.-L. Thiffeault, W. Horton, Dynamical range of the WINDMI model: An exploration of possible magnetospheric plasma states, *J. Geophys. Res., 105*, 12983, 2000.

[46] Sundaram, A. K., and Fairfield, D.H., Stability of resistive MHD tearing and ballooning modes in the tail current sheet, *J. Geophys. Res., 102(AA9)*, 19,913-19,926, 1997.

[47] Toffoletto, F.R., R.W. Spiro, R.A. Wolf, M. Hesse, and J. Birn, Self-consistent modeling of inner magnetospheric convection, in *Third International Conference on Substorms (ICS-3)*, edited by E.J. Rolfe and B. Kaldeich, pp. 223–230, Eur. Space Agency Pub. Div., Noordwijk, Netherlands, 1996.

[48] Tsyganenko, N.A., and D.P. Stern, Modeling of the global magnetic field of the large-scale Birkeland current systems, *J. Geophys. Res., 101*, 27187-27198, 1996.

[49] Van Dam, J.W., M.N. Rosenbluth, and Y.C. Lee, A generalized kinetic energy principle, *Phys. Fluids, 25*, 1349, 1982.

[50] Yoon, P.H., and A.T.Y. Lui, Nonlinear analysis of generalized cross–field current instability, *Phys. Fluids B, 5*(3), 836, 1993.

FIGURE CAPTIONS

FIG. 1. Sketch showing the relationship between the $\delta W^{\rm MHD}$ computed from the mathematical minimization of the ideal MHD model over all trial functions and the minimization δW (FAST) which takes into account the physical condition for the applicability of MHD. The stabilization of the MHD for $\beta > \beta_2$ is due to compression of the plasma in the flux tubes which requires kinetic theory for a precise determination.

FIG. 2. A comparison of the simple MHD equilibrium constructed from a 2D dipole and a constant $j_y = dp/dA_y$ current sheet (dashed green curve) with one solar wind setting for the standard empirical model (Tsyganenko 96). Panel (a) shows the $B_z(x)$ profiles. Panel (b) shows the pressure profile obtained from axial force balance $dp/dx = j_y B_z$. Panel (c) shows the current density comparison. Panel (d) shows the plasma pressure to the magnetic pressure $\beta(x) = 2\mu_0 p(x)/B_z^2(x,0,0)$ along the nightside geotail axis.

FIG. 3. The logarithmic pressure gradient $R_E/L_p = p^{-1}(dp/dx)$ and the dimensionless radius of curvature $\kappa(x) = \widehat{\mathbf{e}}_x \cdot (\widehat{\mathbf{b}} \cdot \nabla)\mathbf{b}$ along the geotail axis.

FIG. 4. The first approximation to $\gamma^2 \cong -\delta W^{\mathrm{K0}} / \int (\rho_m/B^2)(\xi^\psi)^2 ds/B$ where the formula (12) in Horton *et al.* (1999) used to estimate the line bending stabilization at β_1 ($x \simeq -7$ in this example) and the compressional stabilization at β_2 ($x \simeq -14$ in this example). Typically, the transitional region where $\beta \approx 1$ to 5 is the first region to go unstable to large scale ($k_y\rho_i \ll 1$) MHD-like modes. The maximum growth rate is a fraction of $\gamma_{\max} = v_i/(L_pR_c)^{1/2}$ where $\kappa(x) = 1/R_c(x)$. Without the compressional stabilization the entire geotail would be interchange-ballooning unstable.

FIG. 5. The local kinetic theory growth rate computed from the full 3×3 determinant in Eq. (26) as a function of β. Here $k_y\rho_i = 0.3$, $T_e/T_i = 1$, $L_p/R_c = 1$ and the figure shows that the critical values β_1 and β_2 are still well defined. However, due to the ion kinetic resonances $\omega_k \simeq \omega_{Di}(\mathcal{E})$ there is a residual kinetic instability for all $\beta > \beta_2$. The unstable mode has an MHD-like polarization (with $\psi/\phi \simeq 0.9$) and is westward propagating. In the strongly unstable region the speed is $\omega/k_y \simeq \frac{1}{2}\, v_{di} = \overline{v_i(\rho_i/2L_p)}$ and in the weakly unstable high beta region the speed is $\omega/k_y \simeq v_{di}(x)$. These waves are in the 2 mHz-20 mHz (Pi 2 and Pc5) range and propagate westward with speeds of 5 to 10 km/s.