Radially localized helicon modes in nonuniform plasma

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Abstract

Radial density gradient in a cylindrical plasma column forms a potential well for nonaxisymmetric helicon modes ($m \neq 0$). This paper presents an analytic description of such modes in the limit of small parallel wavenumbers. The corresponding mode equation indicates the possibility of efficient resonant absorption of rf power in helicon discharges at unusually low frequencies.

52.35.Hr, 52.50.Dg, 52.75.Rx

Helicon waves (or whistlers) are widely known for their important role in magnetized plasmas, ranging from space plasma to rf discharges used in plasma processing. A remarkably high efficiency of helicon plasma sources [1, 2] has motivated further theoretical studies of these waves in recent years. Other strong motivations for studying helicon waves include magnetized plasma opening switches, intense-beam plasma interactions, and ionospheric research. One might therefore expect the theory of helicon waves, especially their linear theory, to have been developed in great detail, which is largely true. Still, there appears to be a room for new observations in this well established area. One of them will be discussed in this paper.

We will show that the radial nonuniformity of the axisymmetric plasma column has a surprisingly strong effect on the structure of the helicon modes with nonzero values of the azimuthal mode number m. As a result, the corresponding mode frequency turns out to be significantly lower than what one would expect from the dispersion relation for a uniform plasma. This effect is most pronounced in the limit of small parallel wavenumbers k_z,

$$k_z \ll \min\left(\frac{1}{a};\frac{\omega_{pe}}{c}\right), \tag{1}$$

where a is the plasma radius and ω_{pe} is the electron plasma frequency. The new eigenmode is a helicon mode coupled to space charge effects from electron $\mathbf{E}\times\mathbf{B}$ drift in nonuniform plasma. It has distinct features of a surface wave.

It should be pointed out that this unusual eigenmode cannot be inferred from the early studies of helicons in a homogeneous plasma with a sharp density boundary [3, 4]; nor does this mode appear in axisymmetric problems of electron magnetohydrodynamics (EMHD) [5, 6]. Ironically, the sharp boundary assumption bans this mode: we find that the mode is sensitive to the structure of the density profile when the width of the boundary is smaller than c/ω_{pe}. Also, this mode is necessarily nonaxisymmetric, unlike solutions discussed in Refs. [5, 6].

An importance of the radial density gradient in the helicon discharge was recognized in Ref. [7] and examined in more detail in Ref. [8], based on numerical solutions of the wave equation. However, numerical solutions alone do not demonstrate all the subtleties of the problem. We believe that the analytic approach described below makes these subtleties more visible.

Our starting point is the linear wave equation

$$\nabla(\nabla\mathbf{E}) - \nabla^2\mathbf{E} = \frac{\omega^2}{c^2}\mathbf{D} \tag{2}$$

where the electric displacement $\mathbf{D}$ is related to the field $\mathbf{E}$ by the cold plasma dielectric tensor $\varepsilon_{\alpha\beta}$, namely

$$D_\alpha = \varepsilon_{\alpha\beta}E_\beta. \tag{3}$$

We assume cylindrical symmetry of the plasma with the uniform equilibrium magnetic field $\mathbf{B}_0$ directed along the axis of symmetry (z). The unperturbed plasma density is uniform in the azimuthal (ϕ) and axial (z) directions. The density only depends on radius r.

The three components of Eq. (3) in the cylindrical coordinates (r, ϕ, z) have the form

$$D_r = \varepsilon E_r + igE_\phi,$$

$$D_\phi = -igE_r + \varepsilon E_\phi,$$

$$D_z = \eta E_z$$

with

$$\varepsilon = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \omega_{c\alpha}^2},$$

$$g = -\sum_\alpha \frac{\omega_{c\alpha}\omega_{p\alpha}^2}{\omega\left(\omega^2 - \omega_{c\alpha}^2\right)},$$

$$\eta = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}.$$

Here the subscript α labels particle species (electrons and ions), $\omega_{c\alpha} = q_\alpha B_0/m_\alpha c$ is the gyrofrequency, $\omega_{p\alpha} = \sqrt{4\pi n_\alpha q_\alpha^2/m_\alpha}$ is the plasma frequency.

In what follows, the plasma is assumed to be sufficiently dense so that

$$\omega_{ce} \ll \omega_{pe}. \tag{4}$$

This condition ensures that the displacement current is negligibly small compared to the plasma current. Next, we assume that the wave frequency satisfies the condition

$$\sqrt{\omega_{ci}\omega_{ce}} \ll \omega \ll \omega_{ce}, \tag{5}$$

which allows us to neglect the ion current compared to the electron current. Then the quantities ε, g, and η reduce to

$$\varepsilon = \frac{\omega_{pe}^2}{\omega_{ce}^2}, \qquad g = \frac{\omega_{pe}^2}{\omega\omega_{ce}}, \qquad \eta = -\frac{\omega_{pe}^2}{\omega^2}, \tag{6}$$

with

$$\varepsilon \ll \quad g \ll \quad \eta. \tag{7}$$

As indicated above, we also assume that k_z is sufficiently small to satisfy Eq. (1). This restriction allows us to treat the essential part of the problem analytically.

We now use Fourier expansion of the electric field in z and ϕ and select a single Fourier harmonic that depends on z, ϕ and time as $\mathrm{e}^{ik_z z + im\phi - i\omega t}$. This transforms Eq. (2) into the following set of ordinary differential equations for the electric field components

$$E_r\left(\frac{m^2}{r^2} + k_z^2\right) + \frac{im}{r}\left(\frac{\partial E_\phi}{\partial r} + \frac{E_\phi}{r}\right) + ik_z\frac{\partial E_z}{\partial r} = \frac{\omega^2}{c^2}(\varepsilon E_r + igE_\phi), \tag{8}$$

$$im\frac{\partial}{\partial r}\left(\frac{E_r}{r}\right) - \frac{\partial}{\partial r}\left(\frac{\partial E_\phi}{\partial r} + \frac{E_\phi}{r}\right) + k_z^2 E_\phi - \frac{mk_z}{r}E_z = \frac{\omega^2}{c^2}(\varepsilon E_\phi - igE_r), \tag{9}$$

$$\frac{ik_z}{r}\frac{\partial}{\partial r}(rE_r) - \frac{k_z m}{r}E_\phi - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{m^2}{r^2}E_z = \frac{\omega^2}{c^2}\eta E_z. \tag{10}$$

In this paper we will only discuss the nonaxisymmetric modes ($m \neq 0$) since the axisymmetric mode ($m = 0$) does not exhibit any unexpected features.

To further simplify the problem for $m \neq 0$ we note that assumptions (1) and (5) allow us to formally treat k_z, and ω as small parameters in Eqs. (8) and (9). The final result will show that ω scales as k_z^2 for $m \neq 0$. To lowest order, we put $k_z = 0$ and $\omega = 0$ in Eqs. (8) and (9). In this limit, these equations become degenerate. They both read:

$$\frac{im}{r}E_r = \frac{\partial E_\phi}{\partial r} + \frac{1}{r}E_\phi. \tag{11}$$

A convenient way to resolve the degeneracy is to replace Eq. (9) by the condition $\mathrm{div}\mathbf{D} = 0$, which is exact and does not contain the lowest order terms. This condition has the form

$$\frac{1}{r}\frac{\partial}{\partial r}[r(\varepsilon E_r + igE_\phi)] + \frac{im}{r}[\varepsilon E_\phi - igE_r] + ik_z\eta E_z = 0. \tag{12}$$

Equations (10), (11), and (12) form a closed set that can easily be reduced to just two equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial E}{\partial r}\right]-\frac{m^2}{r^2}E=-\frac{\omega^2}{c^2}\eta E_z, \tag{13}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\varepsilon r\frac{\partial}{\partial r}(E_z-E)\right]-\frac{m}{r}\left[\frac{\partial g}{\partial r}+\frac{\varepsilon m}{r}\right](E_z-E)-k_z^2\eta E_z=0, \tag{14}$$

where the new unknown function

$$E=E_z-\frac{k_z r}{m}E_\phi$$

measures the radial component of the perturbed magnetic field as

$$H_r=\frac{cm}{\omega r}E. \tag{15}$$

We now take into account that ε is much smaller than g for the modes we are interested in. The smallness of ε allows us to decouple Eqs. (13) and (14). Note that the ε-terms in Eq. (14) are negligible unless the radial scalelength of the function E_z-E is much smaller than the radial scalelength of g (we assume m to be not a large number). Hence, for sufficiently smooth modes, we can replace Eq. (14) by an algebraic relationship,

$$E=E_z\left[1+k_z^2 r\eta\left(m\frac{\partial g}{\partial r}\right)^{-1}\right], \tag{16}$$

which leads to

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial E}{\partial r}\right]-\frac{m^2}{r^2}E=-\frac{m}{k_z^2 r}\frac{\omega^2}{c^2}\frac{E\partial g/\partial r}{1+(m\partial g/\partial r)/k_z^2 r\eta}. \tag{17}$$

For the other branch, Eq. (13) indicates that E has to be much smaller than E_z because of very short radial scales associated with the second derivative ε term in Eq. (14). We can then neglect E in Eq. (14) to find

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\varepsilon r\frac{\partial}{\partial r}E_z\right]-\frac{m}{r}\left[\frac{\partial g}{\partial r}+\frac{\varepsilon m}{r}\right]E_z-k_z^2\eta E_z=0. \tag{18}$$

This equation describes nearly electrostatic perturbations (the Gould-Trivelpiece modes) somewhat modified by plasma density gradient.

The effect of the density gradient is much stronger for the mode described by Eq. (17). It is straightforward to solve Eq. (17) numerically for any given density profile. With some natural simplifications, this equation can also be solved analytically. To find analytical solutions, we consider two limiting cases: $\omega_{pe}a/c \gg 1$ and $\omega_{pe}a/c \ll 1$.

The first condition ($\omega_{pe}a/c \gg 1$) actually means that the parallel electric field is negligibly small. Formally, this allows us to neglect the $\partial g/\partial r$ term in the denominator on the right-hand side of Eq. (17). We can then construct a surface-wave solution by choosing a step-like radial profile of the plasma density:

$$n = \begin{cases} n_- & \text{if } r < r_0 \\ n_+ & \text{if } r > r_0, \end{cases}$$

so that

$$\frac{\partial g}{\partial r} = (g(n_+) - g(n_-))\delta(r - r_0). \qquad (19)$$

Note, however, that the width of the narrow transition layer at $r = r_0$ still has to be large compared to the skin depth c/ω_{pe}. Outside the layer, the solution has the form

$$E = E_0 \begin{cases} (r/r_0)^{|m|} & \text{if } r < r_0 \\ (r/r_0)^{-|m|} & \text{if } r > r_0, \end{cases} \qquad (20)$$

with E_0 a constant. We have taken into account that the solution must be continuous at $r = r_0$. Integration of Eq. (17) through the transition layer with expression (19) for $\partial g/\partial r$ gives the following dispersion relation:

$$\omega = 2\frac{m}{|m|}\frac{\omega_{ce}k_z^2c^2}{\omega_{pe}^2(n_+) - \omega_{pe}^2(n_-)}. \qquad (21)$$

The second limiting case ($\omega_{pe}a/c \ll 1$) also gives an eigenmode associated with the density gradient. In this case, the mode is centered at the point $r = r_*$ where the function

$$f(r) = -\frac{1}{rn}\frac{dn}{dr}$$

has a maximum, and the mode frequency turns out to be very close to

$$\omega_* = -\omega_{ce}\frac{k_z^2}{mf(r_*)}, \qquad (22)$$

the value that makes the denominator in Eq. (17) vanish at $r = r_*$. Therefore, we take

$$\omega = \omega_*(1 - \triangle^2), \tag{23}$$

where the value of the small parameter $\triangle$ is to be determined by the boundary condition for the outside solution similar to (20)

$$E = E_* \begin{cases} (r/r_*)^{|m|} & \text{if } r < r_* \\ (r/r_*)^{-|m|} & \text{if } r > r_*, \end{cases} \tag{24}$$

with E_* a constant. The right-hand side of equation (17) is now sharply peaked around the point $r = r_*$, so that Eq. (17) can be simplified to

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial E}{\partial r}\right] - \frac{m^2}{r^2}E = -\frac{\omega_{pe}^2(r_*)}{c^2}E\left(\triangle^2 + \left|\frac{1}{2f}\frac{d^2 f}{dr^2}\right|(r - r_*)^2\right)^{-1}, \tag{25}$$

where f and $d^2 f/dr^2$ should be evaluated at $r = r_*$. Except for the narrow vicinity of the point $r = r_*$, the right–hand side of this equation is negligibly small, which readily gives the dispersion relation via integrating Eq. (25) through the transition layer at $r = r_*$. The result is

$$\triangle^2 = \frac{\pi^2}{2m^2}\frac{\omega_{pe}^4(r_*)r_*^2}{c^4}\left|\frac{f}{d^2 f/dr^2}\right|_{r=r_*} \ll 1. \tag{26}$$

This result can be easily generalized to the case of small but finite resistivity, which formally requires to change the expression for η from that given by Eq. (6) to $\eta = -\omega_{pe}^2/\omega(\omega + i\nu)$, where ν is a small collision frequency. The derivation of the corresponding dispersion relation repeats the one used to obtain Eq. (26), but Δ^2 now becomes a complex quantity given by

$$\Delta^2 = \frac{\pi^2}{2m^2}\frac{\omega_{pe}^4(r_*)r_*^2}{c^4}\left|\frac{f}{d^2 f/dr^2}\right|_{r=r_*} + i\nu(r_*)/\omega_*. \tag{27}$$

The derivation of Eq. (27) implies that $\nu \ll \omega_*$, a condition that breaks for very steep density profiles. This bans the mode in the sharp boundary problem [3, 4].

It is significant that, in both limiting cases ($\omega_{pe}\, a/c \gg 1$ and $\omega_{pe}\, a/c \ll 1$), the mode involves a perturbed "surface" current that is localized near the peak of the eigenfunction $E(r)$. It is due to this current that the radial derivative of $E(r)$ as well as the azimuthal component of the perturbed magnetic field are discontinuous in the outside solutions (20) and (24). The existence of this surface current distinguishes the above modes from those discussed in Refs. [3, 4].

Although the modes described by Eqs. (17), (21), and (23) are radially localized modes with distinct surface-wave features, their coupling to the antenna current in helicon plasma sources can be sufficiently strong, especially for $|m| = 1$, a typical mode number for such devices. Note that these modes are asymmetric with respect to the change of sign of m or $\mathbf{B}_0$ separately. However, the mode frequency does not change when the signs of m and $\mathbf{B}_0$ change simultaneously. The asymmetry, which results from the Hall effect in nonuniform plasma, may cause significant changes in the operation of helicon sources with the change of the magnetic field sign. A related phenomenon is the co-counter asymmetry in fast wave heating and current drive (see Ref. [9]).

It should be possible to clearly identify the described modes in a well diagnosed experiment similar to the one reported in Refs. [10,11], where helicon modes were observed in a homogeneous plasma. This will of course require to have a plasma with controllable radial density gradient.

It follows from the structure of Eq. (17) that an externally driven solution for $E(r)$ is regular at the points where the denominator on the right-hand side of this equation vanishes. A straightforward analysis shows that the function $E(r)$ remains finite at these points. However, with finite value of E, Eq. (16) exhibits a singularity in the parallel electric field, which in turn suggests a resonant-type absorption of the rf power near the singularity. It is plausible that such an absorption is responsible for unusually high efficiency of the helicon sources, especially at frequencies below the typical helicon frequency for a uniform plasma. It should be emphasized that this absorption mechanism is qualitatively different from the one discussed in Refs. [8, 12], where the enhanced absorption was attributed to linear coupling between the helicon and the Gould-Trivelpiece modes. In our case, the absorption results from the surface-type helicon mode alone, which is actually decoupled from the Gould-Trivelpiece mode described by Eq. (18). The particularly low frequency of the localized modes translates into their low phase velocity, which should facilitate Landau damping of these modes.

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