A topological knot in a dissipative fifth-order system

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Abstract

In order to show that some quasiperiodic orbits of a fifth–order system are embedded in a three–dimensional subspace, we numerically investigate main projections onto a three–dimensional subspace from the five–dimensional space. We find that the quasiperiodic orbits are topologically equivalent to a (p, q)–torus knot, which has q strands traveling p times meridionally about a two–torus in a three–manifold. In terms of a braid word for the torus knot, we finally obtain a (2,7)–torus knot in the fifth–order system through complicated bifurcations under parameter variation. This suggests that topological invariants embedded in a three–manifold can be extracted from realistic dissipative higher–dimensional dynamical systems.

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After a new polynomial for knots and links was discovered by Jones [1], numerous other polynomials have been found in knot theory [2]. Jones' discovery of the polynomial as well as Lorenz' discovery of the strange attractor [3] has caused great excitement among many biochemists [4] and physicists [5,6]. Birman and Williams [7,8] applied braid and knot theory to dynamical systems such as the celebrated Lorenz model [3] and the forced Duffing equation [9], and presented the concept of a template which is a certain branched two–manifold carrying a semiflow. Specifically, they investigated some periodic orbits of certain flows on a three–manifold and reconstructed *isotopic* flows to the original orbits from a template for the codimension two. Williams [10] also studied in detail Lorenz knots for the Lorenz attractor in terms of a template. The essentials of the underlying dynamics of higher–dimensional systems [11] may be embedded in the lowest dimension, which unfolds a chaotic attractor so that none of the periodic orbits of its chaotic attractor overlap, this being called the embedding dimension d_E (positive integer). However, it is difficult to find such a low–dimension subspace containing a chaotic attractor for a general infinite–dimension dynamical system, which is usually described by a set of partial differential equations, for example, the generalized Ginzburg–Landau equation [12,13], the set of magnetohydrodynamic equations [14], and so forth. Fortunately, Nozaki and Bekki [15] succeeded in reducing an infinite–dimensional system (a forced dissipative soliton) to a finite–order system of ordinary differential equations and found a subspace containing a chaotic attractor; that is, the embedding dimension of the forced dissipative soliton is four ($d_E = 4$). It should be noted that the embedding dimension of dissipative systems is significantly different from that of Hamiltonian systems [16].

The Lorenz model of the third–order system cannot have any quasiperiodic solutions since the divergence of the flow is everywhere negative and constant. It is *not* natural that quasiperiodic orbits of tori are also created from the Lorenz template [10] for the Lorenz model. Therefore, in order to investigate any topological properties related to the quasiperiodic orbits embedded in a three–dimension subspace, it is crucial to introduce a higher–dimensional system than the Lorenz model of the three–dimension space. If we show numerically that some quasiperiodic orbits of a higher–dimensional system are embedded in a lower three–dimension subspace, the template concept of Birman and Williams [8] may be powerful enough to be applied to realistic higher–dimensional dynamical systems [17]. The aim of the present Letter is twofold: (1) to exhibit a key example that some periodic orbits are embedded in a three–dimensional subspace of a higher–dimensional dynamical system, and (2) to extract some topological invariants embedded in a three–dimensional subspace from a chaotic attractor through complicated bifurcations under parameter variation in a fifth–order system of magnetoconvection.

A fifth–order system for magnetoconvection [18,19] is designed to describe nonlinear coupling between Rayleigh–Bénard convection and an external magnetic field. This type of system was first presented by Veronis [20] in studying a rotating fluid. The fifth–order autonomous system of magnetoconvection is given as follows [21]:

$$\dot{a}(t) = \sigma \left[-a(t) + rb(t) - qd(t) \left(1 + \frac{w(3-w)}{\zeta^2(4-w)} e(t) \right) \right], \tag{1}$$

$$\dot{b}(t) = -b(t) + a(t) - a(t)c(t), \tag{2}$$

$$\dot{c}(t) = w(-c(t) + a(t)b(t)), \tag{3}$$

$$\dot{d}(t) = -\zeta(d(t) - a(t)) - \frac{w}{\zeta(4-w)} a(t)e(t), \tag{4}$$

$$\dot{e}(t) = -\zeta(4-w)(e(t) - a(t)d(t)), \tag{5}$$

where a dot denotes differentiation with respect to the characteristic time t, and $a(t)$ represents the first order velocity perturbation, while $b(t)$, $c(t)$ and $d(t)$, $e(t)$ are measures of the first and second order perturbations to the temperature and to the magnetic flux function, respectively. In the fifth–order system, there are five fundamental parameters: ζ, σ, r, q, and w, where ζ is the magnetic Prandtl number(the ratio of the magnetic to the thermal diffusivity), σ is the Prandtl number, r is a normalized Rayleigh number, q is a normalized Chandrasekhar number, and w is a geometrical parameter($0 < w < 4$). In the case of $q = 0$, the fifth–order system Eqs. (1)–(5) can be transformed to the Lorenz system [3]. The divergence of the flow in phase space can be calculated from Eqs. (1)–(5):

$$\frac{\partial}{\partial a}\dot{a} + \frac{\partial}{\partial b}\dot{b} + \frac{\partial}{\partial c}\dot{c} + \frac{\partial}{\partial d}\dot{d} + \frac{\partial}{\partial e}\dot{e} = -[1 + \sigma + w + \zeta(5-w)], \tag{6}$$

which is always negative since $0 < w < 4$. These trajectories of the fifth–order system are attracted to a set of measure zero in the phase space; a fixed point, a limit cycle (one–sphere S^1), a two–torus ($T^2 = S^1 \times S^1$), a three–torus ($T^3 = S^1 \times S^1 \times S^1$), and a strange attractor [21–23]. The system of Eqs. (1)–(5) possesses an important symmetry, in that it is invariant under the transformation

$$(a, b, c, d, e) \rightarrow (-a, -b, c, -d, e). \tag{7}$$

In a fifth–order system of five–degrees of freedom, there are five Lyapunov exponents $\lambda_5 \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1$ [24]. Their sum is always negative due to Eq. (6).

We have used the fourth–order Runge–Kutta scheme with a time step of either $\Delta t = 0.1$ or 0.01 and carried out numerical integrations with appropriately chosen initial conditions: $a(0) = \pm 0.1$, $b(0) = c(0) = d(0) = 0.0$ [21]. If three Lyapunov exponents are equal to zero ($\lambda_1 = \lambda_2 = \lambda_3 = 0.000$) within a numerical accuracy of 10^{-4}, we call an attractor a three–torus. The magnetic Prandtl number ζ is chosen as the control parameter, and the other four parameters are fixed: $\sigma = 1.0$, $w = 0.1081$, $q = 5.0$, and $r = 14.47$ [21]. It is convenient to separate the parameter region into three parts: Region (I) $0.0968 < \zeta < 0.1020$, Region (II) $0.1020 \lesssim \zeta \lesssim 0.1060$, and Region (III) $0.1060 < \zeta \lesssim 0.1108$. Our fifth–order system for Region (I) shows that the winding numbers of a two–torus create an incomplete devil's staircase versus ζ and lead to the last phase–locked winding number $W_\infty = 5/7$ as ζ approaches 0.1020 [25]. When ζ is increased beyond Region (I), that is, for Region (II), we can observe complicated bifurcations: T^3 ($\lambda_1 = \lambda_2 = \lambda_3 = 0$) through the third Hopf bifurcation $\rightarrow$ Chaos ($\lambda_1 > 0$) $\rightarrow$ Period–7 $\rightarrow T^3 \rightarrow T^2 \rightarrow T^3 \rightarrow$ Hyper–Chaos ($\lambda_1, \lambda_2 > 0$) $\rightarrow \cdots$. The fifth–order system for Region (III) behaves like a Lorenz attractor and transits to a symmetric limit cycle via saddle–node bifurcation [23] as ζ approaches 0.1108. We are motivated to provide a deeper understanding of these complicated bifurcations for Region (II). Attractors for Regions (I) and (II) are asymmetric. This asymmetry depends on the choice of initial conditions: if an attractor appears in the positive region of b when $a(0) = +0.1$, then its mirror image appears in the negative region of b when $a(0) = -0.1$ on account of the symmetry of Eq. (7). Hereafter, for convenience, all attractors are shown in the positive region of b, and we will not distinguish between a closed periodic orbit and its mirror image in a three–dimensional subspace. The projection of a typical chaotic attractor onto the (b, c) plane from the five–dimensional space is shown in Fig. 1 ($\zeta = 0.1031$). There are three fundamental frequencies in a three–torus; for example, $f_1 = 0.1009$, $f_2 = 0.1413$ and $f_3 = 0.0020$ for $\zeta = 0.1029$. This three–torus is related to the last two–torus [25] since $W_\infty \sim f_1/f_2$ and $f_3 = f_1 f_2/7$. This means that the period–7 of the two–torus is *resonant* with the period of f_3; we will call it a resonant three–torus. Such a resonant three–torus is shown in Fig. 2(a), which is projected onto the (b, c) plane from the five–dimensional space for $\zeta = 0.1029$.

First, let us consider the main projections of the resonant three–torus onto a three–dimensional subspace from the five–dimensional space. For simplicity, two of the three coordinates of the three–dimensional subspace are fixed, that is, the (b, c) plane is chosen as the planar representation of the resonant three–torus embedded in a three–dimensional subspace. Then, there are eight possibilities for projections onto a three–dimensional subspace from the five–dimensional space; (a, b, c), $(\dot{a}, b, c)$, $(\dot{b}, b, c)$, $(\dot{c}, b, c)$, (d, b, c), $(\dot{d}, b, c)$, (e, b, c), and $(\dot{e}, b, c)$. Using the time–series data of Eqs. (1)–(5), we can easily determine the relation between the upper and the lower curves at each crossing point of a planar representation of the resonant three–torus in the three–dimensional subspace [26]. There are twelve crossing points in the projection of the resonant three–torus, as shown in Fig. 2(a). The number of crossings $c[K] = 12$ is one of the topological invariants for a knot K if a knot is embedded in a three–dimensional space [27]. The investigation of a set of crossing relations for each projection leads to the fact that each closed orbit for its projection is topologically the same as a planar representation of Fig. 2(b). This allows us to assume $d_E = 3$ for the resonant three–torus. Since a closed orbit of the resonant three–torus in a three–dimensional subspace does not intersect itself anywhere due to the existence–uniqueness theorem for solutions of ordinary differential equations, we can hereafter regard a closed orbit of the resonant three–torus as forming a knot K in a three–dimensional space, where the torsion of the orbit is neglected for simplicity [28]. Therefore, from Fig. 2(b), we obtain a positive braid representation [7] for the resonant three–torus, which is shown in Fig. 3. The arrows denote the orientation of K, which corresponds to the direction of time evolution. From the positive braid representation, we have $c[K] = 12$ and the number of strands $N_s = 7$. Then, the theorem of Birman–Williams [7] gives $g[K] = 3$, where $g[K]$ is the genus of K. This $g[K] = 3$ is also one of the topological invariants for K. A closure of the braid representation is shown in Fig. 4, which gives birth to a knot K. Let us take the generators of the fundamental group to be x on the left and y on the right for the closure of the positive braid. Then, our braid word $w[K]$ is written as

$$w[K] = x^2y^2xy^2. \tag{8}$$

If K is a (p, q)–torus knot, its braid word [29] is given by

$$w(p, q) = x^{n_1}y^2x^{n_2}y^2 \cdots x^{n_p}y^2, \tag{9}$$

where $q - 2p = \sum_{1\leq j\leq p} n_j$ $(n_1 > n_p)$, with p an isotopy invariant and a putative braid number [7] and q the period of knot K. From Eqs. (8) and (9), we find a (2,7)–torus knot since $n_1 = 2$, $n_2 = 1$, $p = 2$, and $q = 7$. Then, the Alexander polynomial [30] for the $(2, 7)$–torus knot [2] is given by

$$\Delta_K(t) = t^{-3} - t^{-2} + t^{-1} - 1 + t - t^2 + t^3. \tag{10}$$

The Alexander polynomial of Eq. (10) yields $g[K] = 3$ since a maximum index is three in the polynomial. This is consistent with the previous result obtained by the theorem of Birman–Williams. Meanwhile, the closure of the braid yields the $(2, 7)$–torus knot via the Reidemeister moves [31]. Finally, we conclude that the knot of the resonant three–torus K is the (2,7)–torus knot, which is shown in Fig. 5. By use of the formula of the skein relation [1], the Jones polynomial for Fig. 5 is given as

$$V_K(t) = -t^{-10} + t^{-9} - t^{-8} + t^{-7} - t^{-6} + t^{-5} + t^{-3}. \tag{11}$$

Next, let us consider the recurrence properties of a chaotic attractor. The time–series data of the fifth–order system are defined by $\mathbf{x}(i) = [\ a(i), b(i), c(i), d(i), e(i)\ ]$ with $1 \leq i \leq$

N. According to Ref. [11], let us introduce the distance between unstable periodic orbits,

$$\delta = \|\mathbf{x}(i) - \mathbf{x}(i+n)\| \leq 1, \tag{12}$$

where δ is the distance normalized by its maximum and n is related to the fundamental period ($n = kn_0$). The unstable orbits can be estimated by choosing the orbit with the best recurrence properties (minimum δ). By means of such a procedure, unstable periodic orbits can be extracted from the chaotic time–series data. The fifth–order system of Eqs. (1)–(5) for $\zeta = 0.1031$ is integrated for 2^{13} periods with $N = 2^{13}$ steps per period, where the minimum fundamental period (n_0) corresponds to $1/f_1$. The time–series data of the chaotic attractor are sampled and stored every 2^7 steps, so that 2^6 points are sampled per period. Periodic orbits are reconstructed from the sampled data with a standard Euclidean metric. The distances δ are plotted as a function of i for fixed $n_0 = 2^6$ and $k = 7$. Such a recurrence plot for a chaotic attractor is shown in Fig. 6. The bottom of each window is a good approximation to the nearby unstable periodic orbit. Choosing $i = 2581$ (minimum δ) from the data of the recurrence plot (Fig. 6), we obtain a reconstructed period–7 orbit extracted from the chaotic time–series data, which is shown in Fig. 7. Surprisingly, this reconstructed period–7 orbit in a three–dimensional subspace is also isotopic to that of the planar representation for the resonant three–torus except the case of embedding onto the $(\dot{e}, b, c)$ subspace.

In conclusion, we have shown that the topological properties of the fifth–order magneto-convection system for Region (II) can be compacted into the (2,7)–torus knot embedded in a three–dimensional subspace. This implies that a higher–dimensional template theory for dissipative dynamical systems can be constructed.

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REFERENCES

[1] V.F.R. Jones, Bull. Amer. Math. Soc. **12**, 103(1985).

[2] A. Kawauchi, *A Survey of Knot Theory* (Birkhäuser, Verlag, 1996) and references cited therein.

[3] E.N. Lorenz, J. Atmos. Sci. **20**, 130(1963).

[4] S. Wasserman, J. Dungan and N. Cozzarelli, *Science* **229**, 171(1985).

[5] E. Witten, Comm. Math. Phys. **121**, 355(1989).

[6] L. Kauffman, *Knots and Physics* (World Scientific, Singapore, 1991).

[7] J. Birman and R.F. Williams, *Topology* **22**, 47(1983).

[8] J. Birman and R.F. Williams, Cont. Math. **20**, 1(1983).

[9] Y. Ueda, J. Stat. Phys. **20**, 181(1979).

[10] R.F. Williams, Ergod. Th. and Dynam. Sys. **4**, 147(1983).

[11] N.B. Tufillaro, H.G. Solari and R. Gilmore, Phys. Rev. A **41**, 5717(1990).

[12] N. Bekki, J. Phys. Soc. Jpn **50**, 659(1981).

[13] N. Bekki and K. Nozaki, Phys. Lett. **110A**, 133(1985).

[14] N. Bekki and Y. Kaneda, Phys. Rev. Lett. **57**, 2176(1986).

[15] K. Nozaki and N. Bekki, Physica **21D**, 381(1985).

[16] F. Takens, *Springer Lecture Notes in Mathematics* **898**, 365(1980).

[17] J. Maselko and H.L. Swinney, J. Chem. Phys. **85**, 6439(1986) and references cited therein.

[18] E. Knobloch and M.R.E. Proctor, J.Fluid Mech. **108**, 291(1981); E. Knobloch *et al.*,

ibid. **113**, 153(1981); L.N. da Costa *et al.*, *ibid.* **109**, 25(1981); E. Knobloch and N.O. Weiss, Physica **9D**, 379(1983).

[19] E. Knobloch *et al.*, Phys. Plasmas **3**, 2475(1996).

[20] G. Veronis, J. Fluid Mech. **4**, 545(1966).

[21] N. Bekki and T. Karakisawa, Phys. Plasmas **2**, 2945(1995).

[22] N. Bekki and T. Karakisawa, Phys. Plasmas **3**, 2477(1996).

[23] T. Karakisawa, N. Bekki and I. Shimada, Phys. Lett. **247A**, 235(1998).

[24] I. Shimada and T. Nagashima, Prog. Theor. Phys. **59**, 1033(1978).

[25] N. Bekki and T. Karakisawa, to be published.

[26] D.P. Lathrop and E.J. Kostelich, Phys. Rev. A **40**, 4028(1989).

[27] K. Murasugi, Trans. Amer. Math. Soc. **326**, 237(1991).

[28] In general, the torsion plays an important role in determining topological properties; T. Uezu and Y. Aizawa, Pro. Theor. Phys. **68**, 1907(1982); H.G. Solari and R. Gilmore, Phys. Rev. A **37**, 3096(1988).

[29] P. Holmes and R.F. Williams, Archive for Rational Mech. and Anal. **90**(2), 115(1985).

[30] J.W. Alexander, Proc. Nat. Acad. Sci. USA **9**, 93(1923).

[31] K. Reidemeister, *Knotentheorie* (Springer–Verlag, Berlin, 1932).

Figure Captions

1. Chaotic attractor projected onto the (b, c) plane from five–dimensional space.

2. (a) Closed orbit of resonant three–torus projected onto the (b, c) plane from five–dimensional space.

   (b) Planar representation for the closed orbit of a resonant three–torus projected onto the (b, c) plane from the (a, b, c) subspace, where orbit is oriented counter–clockwise.

3. Braid representation for a resonant three–torus embedded in a three–dimensional subspace.

4. Closure of the braid for Fig. 3, which gives birth to a knot.

5. This knot is isotopic to a $(2, 7)$–torus knot as a positive braid on two strands with period–7 via the Reidemeister moves for the closure of braid.

6. Distance δ between unstable periodic orbits versus i for the chaotic time–series data with $\zeta = 0.1031$. The bottom of each window is a good approximation to the nearby unstable periodic orbit.

7. Projection of the reconstructed period–7 orbit onto the (b, c) plane. This reconstructed period–7 orbit in a three–dimensional subspace is also isotopic to that of the planar representation for the resonant three–torus (See Fig. 2(b)).