Simultaneous Beltrami Conditions in Coupled Vortex Dynamics

Z. Yoshida

Graduate School of Frontier Sciences, The University of Tokyo, Tokyo 113-8656, Japan

S.M. Mahajan

Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712

(February 9, 1999)

Abstract

The two-fluid model of a plasma describes the strong coupling between the magnetic and the fluid aspects of the plasma. The Beltrami condition which demands alignment of vortices and flows becomes a system of simultaneous equations in the magnetic field and the flow velocity. Combining these equations yields the double curl Beltrami equation. General solvability of the equation has been proved using the spectral theory of the curl operator. The set of solutions contains field configurations which can be qualitatively different from the conventional constant-α-Beltrami fields (which are naturally included in the set). The larger new set may help us understand a variety of structures generated in plasmas.

52.30.Bt,95.30.Qd,02.30.Sa,47.65.+a

I. INTRODUCTION

The Beltrami condition, an expression of the alignment of a vorticity with its flow, describes the simplest and perhaps the most fundamental equilibrium state in a vortex dynamics system (Sec. II). The resulting Beltrami fields constitute a null set for the generator of the evolution equation describing the vortex dynamics. It is also believed that the Beltrami fields are accessible and robust in the sense that they emerge as the nonlinear dynamics of vortices tends to self-organize the system through a weakly dissipative process (Appendix I).

The simplest example of a Beltrami condition is provided by a three dimensional solenoidal field (flow) $\boldsymbol{u}$ obeying

$$\begin{cases} \nabla \times \boldsymbol{u} = \lambda \boldsymbol{u} & \text{(in } \Omega\text{)}, \\ \\ \boldsymbol{n} \cdot \boldsymbol{u} = 0 & \text{(on } \partial\Omega\text{)}, \end{cases} \tag{1}$$

where λ is a real (or complex) constant number, Ω ($\subset \mathbf{R}^3$) is a bounded domain with a smooth boundary $\partial\Omega$ and $\boldsymbol{n}$ is the unit normal vector onto $\partial\Omega$. This system of linear equations is regarded as an eigenvalue problem with respect to the curl operator. The spectral theory of the curl operator reveals an interesting relation of this problem with the cohomology theory [1]. We have the following theorem.

(i) If Ω is simply connected, then (1) has a non-zero solution for special λ included in a set of discrete real numbers; these numbers represent the point spectrum of the self-adjoint part of the curl operator.

(ii) If Ω is multiply connected, then (1) has a non-zero solution for every $\lambda \in \mathbf{C}$ [2].

The aim of this paper is to generalize this theory for "coupled" (or higher-order) Beltrami conditions [3] that describe structures far richer than the ones contained in the single curl Beltrami equation (1). In an ideal plasma, the coupling between the magnetic field and the plasma flow yields the "double curl Beltrami equation"

$$
\begin{cases}
\nabla \times (\nabla \times \boldsymbol{u}) + \alpha \nabla \times \boldsymbol{u} + \beta \boldsymbol{u} = 0 & \text{(in } \Omega\text{)}, \\
\\
\boldsymbol{n} \cdot \boldsymbol{u} = 0, \quad \boldsymbol{n} \cdot (\nabla \times \boldsymbol{u}) = 0 & \text{(on } \partial\Omega\text{)},
\end{cases}
\tag{2}
$$

where $\boldsymbol{u}$ is either the magnetic field or the flow velocity of the plasma (Sec. III). Applying the spectral theory of the curl operator, we will show that (2) has a non-zero solution for arbitrary complex numbers α and β, if the domain Ω is multiply connected (Sec. IV). The method of present theory applies for general multi-curl Beltrami equations obtained from simultaneous Beltrami conditions in coupled systems.

II. VORTEX DYNAMICS AND BELTRAMI CONDITION

We start with reviewing the prototype equation for vortex dynamics. Let $\boldsymbol{\omega}$ be a three-dimensional vector field representing a certain vorticity (contravariant vector field) in $\mathbf{R}^3$. We consider an incompressible flow $\boldsymbol{U}$ that transports $\boldsymbol{\omega}$. When the circulation associated with the vorticity is conserved everywhere, this $\boldsymbol{\omega}$ obeys the equation

$$
\frac{\partial}{\partial t}\boldsymbol{\omega} - \nabla \times (\boldsymbol{U} \times \boldsymbol{\omega}) = 0. \tag{3}
$$

In $\mathbf{R}^2$, the vorticity becomes a pseudo-scalar field ω, and the vortex dynamics equation can be cast in the form of a Liouville equation

$$
\frac{\partial}{\partial t}\omega + \{\phi, \omega\} = 0, \tag{4}
$$

where ϕ is the Hamiltonian of an incompressible flow and $\{\ ,\ \}$ is the Poisson bracket, i.e.,

$$
\boldsymbol{U} = \begin{pmatrix} \partial\phi/\partial y \\ -\partial\phi/\partial x \end{pmatrix}, \quad \{\phi, \omega\} = \frac{\partial\phi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\phi}{\partial x}\frac{\partial\omega}{\partial y}. \tag{5}
$$

The Beltrami condition with respect to (3) is

$$
\boldsymbol{U} = \mu\boldsymbol{\omega}, \tag{6}
$$

where μ is a certain scalar function. This condition assures the vanishing of the generator of the vortex dynamics equation (3). For (4), the Beltrami condition is simply

$$\phi = f(\omega), \tag{7}$$

which implies the commutation of the vorticity and the Hamiltonian of the flow.

The simplest example of the vortex dynamics equation is that of the Euler equation of incompressible ideal flows. Let $\boldsymbol{U}$ be an incompressible flow that obeys

$$\frac{\partial}{\partial t}\boldsymbol{U} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{U} = -\nabla p, \tag{8}$$

where p is the pressure. Taking the curl of (8), we obtain the evolution equation for the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{U}$, which reads, in $\mathbf{R}^3$, as (3), and in $\mathbf{R}^2$, as (4). In the Beltrami flow, $\boldsymbol{\omega}$ parallels $\boldsymbol{U}$, i.e.,

$$\nabla \times \boldsymbol{U} = \mu \boldsymbol{U}. \tag{9}$$

We note that (9) is not Galilei-transform invariant. We thus consider a bounded domain and impose a boundary condition (see (1)) to remove the freedom of the Galilei transform. Taking the divergence of (9), we find that the scalar function μ must satisfy

$$\boldsymbol{U} \cdot \nabla \mu = 0, \tag{10}$$

demanding that μ must remain constant along each streamline of the flow $\boldsymbol{U}$. An analysis of the nonlinear system of elliptic-hyperbolic partial differential equations (9)-(10) involves extremely difficult mathematical issues. The characteristic curve of (10) is the streamline of the unknown flow $\boldsymbol{U}$, which can be chaotic (nonintegrable) in general three-dimensional problems. If we assume, however, that μ is a constant number, the analysis reduces into a simple but nontrivial problem, i.e., the eigenvalue problem of the curl operator. In this paper, our analysis is restricted to this mathematically well-defined subclass of Beltrami fields.

We end this section by reviewing another example of vortex dynamics; the magneto-hydrodynamic (MHD) description of a plasma. The two principal equations of the ideal (dissipation-less) conducting-fluid model are

$$\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B} = 0, \tag{11}$$

$$\frac{\partial}{\partial t}\boldsymbol{U} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{U} = \frac{1}{\rho}\left(\boldsymbol{J} \times \boldsymbol{B} - \nabla p\right), \tag{12}$$

where $\boldsymbol{U}$, $\boldsymbol{J}$, $\boldsymbol{E}$ and $\boldsymbol{B}$ are, respectively, the flow velocity, the current density, the electric field and the magnetic field measured in a certain fixed coordinates, and ρ is the fluid mass density that is assumed to be constant. We may write

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla \phi, \tag{13}$$

$$\boldsymbol{J} = \frac{1}{\mu_0}\nabla \times \boldsymbol{B} \tag{14}$$

in terms of a vector potential $\boldsymbol{A}$ (such that $\nabla \times \boldsymbol{A} = \boldsymbol{B}$) and a scalar potential ϕ. Using Faraday's law

$$\partial \boldsymbol{B}/\partial t = -\nabla \times \boldsymbol{E},$$

and taking the curl of (11) and (12), we obtain

$$\frac{\partial}{\partial t}\boldsymbol{B} - \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) = 0, \tag{15}$$

$$\frac{\partial}{\partial t}\boldsymbol{\omega} - \nabla \times \left(\frac{\boldsymbol{J} \times \boldsymbol{B}}{\rho} + \boldsymbol{U} \times \boldsymbol{\omega}\right) = 0, \tag{16}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{U}$. The Beltrami conditions for this system of vortex dynamics equations are

$$\boldsymbol{J} = \mu_1 \boldsymbol{B} = \mu_2 \boldsymbol{U} = \mu_3 \boldsymbol{\omega}. \tag{17}$$

Using (14) in the first equality of (17), we get

$$\nabla \times \boldsymbol{B} = \mu \boldsymbol{B}, \tag{18}$$

which implies that $\boldsymbol{B}$ parallels its own vorticity (cf. (9)). This configuration, for which the magnetic stress $\boldsymbol{J} \times \boldsymbol{B}$ vanishes, is aptly called "force-free".

In order to characterize the stellar magnetic fields, solutions to (18) were intensively studied in 1950s [4–6]. For $\mu \neq 0$, the magnetic field $\boldsymbol{B}$ has a finite curl, and hence, the field lines are twisted. The current (proportional to $\nabla \times \boldsymbol{B}$), flowing parallel to the twisted field lines, creates what may be termed as "paramagnetic" structures. Such twisted magnetic field lines appear commonly in many different plasma systems such as the magnetic ropes created in solar and geomagnetic systems [7], and galactic jets [8]. Some laboratory experiments have also shown that the "relaxed state" generated through turbulence is well described as solutions of the force-free equation [9,10]

In the next section, we will show that a more adequate formulation of the plasma dynamics allows a much wider class of special equilibrium solutions. The set of new solutions contains field configurations which can be qualitatively different from the force-free magnetic fields.

III. DOUBLE CURL BELTRAMI FIELD

The two-fluid model for the macroscopic dynamics of a plasma differentiates between the electron and ion velocities. Denoting the electron (ion) flow velocity by $\boldsymbol{V}_e(\boldsymbol{V}_i)$, the macroscopic evolution equations become

$$\frac{\partial}{\partial t}\boldsymbol{V}_e + (\boldsymbol{V}_e \cdot \nabla)\boldsymbol{V}_e = \frac{-e}{m}\left(\boldsymbol{E} + \boldsymbol{V}_e \times \boldsymbol{B}\right) - \frac{1}{mn}\nabla p_e, \tag{19}$$

$$\frac{\partial}{\partial t}\boldsymbol{V}_i + (\boldsymbol{V}_i \cdot \nabla)\boldsymbol{V}_i = \frac{e}{M}\left(\boldsymbol{E} + \boldsymbol{V}_i \times \boldsymbol{B}\right) - \frac{1}{Mn}\nabla p_i, \tag{20}$$

where $\boldsymbol{E}$ is the electric field, p_e and p_i are, respectively, the electron and the ion pressures, e is the elementary charge, n is the number density of both electrons and ions (we consider a quasineutral plasma with singly charged ions), m and M are, respectively, the electron and the ion masses. In the electron equation, the inertial terms (left-hand side of (19)) can be safely neglected, because of their small mass ($m \ll M$). Therefore, (19) reduces to

$$\boldsymbol{E} + \boldsymbol{V}_e \times \boldsymbol{B} + \frac{1}{en}\nabla p_e = 0. \tag{21}$$

When electron mass is neglected, $\boldsymbol{V}_i = \boldsymbol{V}$, the fluid velocity. We introduce the following set of dimensionless variables,

$$\begin{cases} \boldsymbol{x} = \lambda_i \hat{\boldsymbol{x}}, \quad \boldsymbol{B} = B_0 \hat{\boldsymbol{B}}, \\ \\ t = (\lambda_i / V_A)\hat{t}, \quad p = (B_0^2/\mu_0)\hat{p}, \quad \boldsymbol{V} = V_A \hat{\boldsymbol{V}}, \\ \\ \boldsymbol{A} = (\lambda_i B_0)\hat{\boldsymbol{A}}, \quad \phi = (V_A \lambda_i B_0)\hat{\phi}, \end{cases} \tag{22}$$

where the ion skin-depth

$$\lambda_i = \frac{c}{\omega_{pi}} = \frac{V_A}{\omega_{ci}} = \sqrt{\frac{M}{\mu_0 n e^2}}.$$

is a characteristic length scale of the system, and the Alfvén speed is given by $V_A = B_0/\sqrt{\mu_0 M n}$ (we assume $n =$ constant, for simplicity) with B_0 as an appropriate measure of the magnetic field.

Writing $\hat{\boldsymbol{E}} = -\partial \hat{\boldsymbol{A}}/\partial \hat{t} - \hat{\nabla}\hat{\phi}$, the dimensionless version of (21) and (20) now read as

$$\frac{\partial}{\partial \hat{t}} \hat{\boldsymbol{A}} = \left(\hat{\boldsymbol{V}} - \hat{\nabla} \times \hat{\boldsymbol{B}}\right) \times \hat{\boldsymbol{B}} - \hat{\nabla}\left(\hat{\phi} + \hat{p}_e\right), \tag{23}$$

$$\frac{\partial}{\partial \hat{t}} (\hat{\boldsymbol{V}} + \hat{A}) = \hat{\boldsymbol{V}} \times \left(\hat{\boldsymbol{B}} + \hat{\nabla} \times \hat{\boldsymbol{V}}\right) - \hat{\nabla}\left(\hat{V}^2/2 + \hat{p}_i + \hat{\phi}\right). \tag{24}$$

In what follows, we shall drop the *hat* for a simpler notation. Taking the curl of (23) and (24), we can cast them in a revealing symmetric form

$$\frac{\partial}{\partial t} \boldsymbol{\omega}_j - \nabla \times (\boldsymbol{U}_j \times \boldsymbol{\omega}_j) = 0 \quad (j = 1, 2) \tag{25}$$

in terms of a pair of generalized vorticities

$$\boldsymbol{\omega}_1 = \boldsymbol{B}, \quad \boldsymbol{\omega}_2 = \boldsymbol{B} + \nabla \times \boldsymbol{V},$$

and the effective flows

$$\boldsymbol{U}_1 = \boldsymbol{V} - \nabla \times \boldsymbol{B}, \quad \boldsymbol{U}_2 = \boldsymbol{V}.$$

The simplest equilibrium solution to (25) is given by the "Beltrami conditions"

$$\boldsymbol{U}_j = \mu_j \boldsymbol{\omega}_j \quad (j = 1, 2), \tag{26}$$

which implies the alignment of the vorticities and the corresponding flows. Writing $a = 1/\mu_1$ and $b = 1/\mu_2$, and assuming that a and b are constants, the Beltrami conditions (26) read as a system of simultaneous linear equations in $\boldsymbol{B}$ and $\boldsymbol{V}$

$$\boldsymbol{B} = a(\boldsymbol{V} - \nabla \times \boldsymbol{B}), \tag{27}$$

$$\boldsymbol{B} + \nabla \times \boldsymbol{V} = b\boldsymbol{V}. \tag{28}$$

These equations have a simple and significant connotation; the electron flow $(\boldsymbol{V} - \nabla \times \boldsymbol{B})$ parallels the magnetic field $\boldsymbol{B}$, while the ion flow $\boldsymbol{V}$ follows the "generalized magnetic field" $(\boldsymbol{B} + \nabla \times \boldsymbol{V})$. This generalized magnetic field contains the Coriolis' force induced by the ion inertia effect on a circulating flow.

Combining (27) and (28) yields a second order partial differential equation

$$\nabla \times (\nabla \times \boldsymbol{B}) + \alpha \nabla \times \boldsymbol{B} + \beta \boldsymbol{B} = 0, \tag{29}$$

where

$$\alpha = \frac{1}{a} - b, \quad \beta = 1 - \frac{b}{a}.$$

The double curl Beltrami equation (29) encompasses a wide class of steady-state equations of mathematical physics. The conventional force-free-field equation (18), which describes paramagnetic fields, is included in this system as a special case: $\alpha = 0$ and $\beta < 0$. On the other hand, when $\alpha = 0$ and $\beta > 0$, (29) resembles London's equation of super conductivity with its well-known fully diamagnetic solutions. We note that, in this version of the London equation, the characteristic shielding length for the magnetic field is the ion skin depth c/ω_{pi}, instead of the usual electron skin depth c/ω_{pe}, because it is the ion-dynamics that brings about the coupling of the magnetic field with the collective motion of the medium.

In the next section, we will study the mathematical structure of the double curl Beltrami equation with arbitrary complex α and β [11].

IV. BELTRAMI FIELDS AND HARMONIC FIELDS

The single Beltrami condition (1) is known to have a non-zero solution for arbitrary complex number λ, if the domain Ω is multiply connected [1]. The harmonic field which represents the cohomology class of the differential forms in Ω plays an essential role to generate the Beltrami field. Similar relation holds in the double curl Beltrami equations (2). Here, we study the relation between the topology of the domain Ω and the degree of freedom in the solution of the double curl Beltrami fields.

It is convenient to denote the curl derivative $\nabla\times$ by "curl" to use it as an operator. Let us rewrite the differential equation of (2) in the form

$$(\text{curl} - \lambda_+)(\text{curl} - \lambda_-)\boldsymbol{u} = 0, \tag{30}$$

where

$$\lambda_\pm = \frac{1}{2}\left[-\alpha \pm (\alpha^2 - 4\beta)^{1/2}\right]. \tag{31}$$

Because the two operators $(\text{curl} - \lambda_\pm)$ commute, the general solution to (30) is given by a linear combination of two Beltrami fields. Let $\boldsymbol{G}_\pm$ be the Beltrami field such that

$$\begin{cases} (\text{curl} - \lambda_\pm)\boldsymbol{G}_\pm = 0 & (\text{in } \Omega), \\ \boldsymbol{n}\cdot\boldsymbol{G}_\pm = 0 & (\text{on } \partial\Omega). \end{cases}$$

Then, for arbitrary constants $c_\pm$, the sum

$$\boldsymbol{u} = c_+\boldsymbol{G}_+ + c_-\boldsymbol{G}_- \tag{32}$$

solves (30). Since $\boldsymbol{n}\cdot(\nabla\times\boldsymbol{G}_\pm) = \lambda_\pm\boldsymbol{n}\cdot\boldsymbol{G}_\pm = 0$ on $\partial\Omega$, $\boldsymbol{u}$ satisfies the boundary conditions given in (2). Therefore, the existence of a nontrivial solution to the double curl Beltrami equations (2) will be predicated on the existence of the appropriate pair of single Beltrami fields (cf. Appendix II). Let us briefly review the mathematical theory of single Beltrami fields [1].

Suppose that Ω ($\subset \boldsymbol{R}^3$) is a bounded domain with a smooth boundary $\partial\Omega = \cup_{i=1}^n \Gamma_i$. We consider cuts of the domain Ω. Let $\Sigma_1, \cdots, \Sigma_\nu$ ($\nu \geq 0$) be the cuts such that $\Sigma_i \cap \Sigma_j = \emptyset$ ($i \neq j$), and such that $\Omega \setminus (\cup_{j=1}^{\nu} \Sigma_j)$ becomes a simply connected domain. The number ν of such cuts is the first Betti number of Ω. When $\nu > 0$, we define the flux through each cut by

$$\Phi_j(\boldsymbol{u}) = \int_{\Sigma_j} \boldsymbol{n} \cdot \boldsymbol{u} \, ds \qquad (j = 1, \cdots, \nu),$$

where $\boldsymbol{n}$ is the unit normal vector on Σ_j with an appropriate orientation. By Gauss' formula, $\Phi_j(\boldsymbol{u})$ is independent of the place of the cut Σ_j, if $\nabla \cdot \boldsymbol{u} = 0$ in Ω and $\boldsymbol{n} \cdot \boldsymbol{u} = 0$ on $\partial\Omega$.

Let $L^2(\Omega)$ the Lebesgue space of square-integrable (complex) vector fields in Ω, which is endowed with the standard innerproduct

$$(\boldsymbol{a}, \boldsymbol{b}) = \int_\Omega \boldsymbol{a} \cdot \overline{\boldsymbol{b}} \, dx.$$

We define the following subspaces of $L^2(\Omega)$:

$$L^2_\Sigma(\Omega) = \{\boldsymbol{w};\ \nabla \cdot \boldsymbol{w} = 0 \text{ in } \Omega,\ \boldsymbol{n} \cdot \boldsymbol{w} = 0 \text{ on } \partial\Omega,\ \Phi_j(\boldsymbol{w}) = 0\ (j = 1, \cdots, \nu)\},$$

$$L^2_H(\Omega) = \{\boldsymbol{h};\ \nabla \cdot \boldsymbol{h} = 0, \nabla \times \boldsymbol{h} = 0 \text{ in } \Omega,\ \boldsymbol{n} \cdot \boldsymbol{h} = 0 \text{ on } \partial\Omega\},$$

$$L^2_G(\Omega) = \{\nabla\phi;\ \Delta\phi = 0 \text{ in } \Omega\},$$

$$L^2_F(\Omega) = \{\nabla\phi;\ \phi = c_i\ (\in \mathbf{C}) \text{ on } \Gamma_i\ (i = 1, \cdots, n)\}$$

in terms of which we have an orthogonal decomposition [12]

$$L^2(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_H(\Omega) \oplus L^2_G(\Omega) \oplus L^2_F(\Omega).$$

The space of the solenoidal vector fields with vanishing normal components on $\partial\Omega$ is

$$L^2_\sigma(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_H(\Omega).$$

The subspace $L^2_H(\Omega)$ corresponds to the cohomology class, whose member is a harmonic vector field and $\dim L^2_H(\Omega) = \nu$ (the first Betti number of Ω). When Ω is simply connected, then $\nu = 0$ and $L^2_H(\Omega) = \emptyset$. We have the following expression

$$L^2_\Sigma(\Omega) = \{\nabla \times \boldsymbol{w};\ \boldsymbol{w} \in H^1(\Omega),\ \nabla \cdot \boldsymbol{w} = 0 \text{ in } \Omega,\ \boldsymbol{n} \times \boldsymbol{w} = 0 \text{ on } \partial\Omega\},$$

where $H^1(\Omega)$ is the Sobolev space of first order. This says that a member of $L^2_\Sigma(\Omega)$ can be expressed as the curl of a vector potential with the boundary condition $\boldsymbol{n} \times \boldsymbol{w} = 0$.

The spectral theory of the curl operator provides the basic understanding of the mathematical structure of the Beltrami equations. We repeat Theorem 1 of Yoshida-Giga [1].

Theorem 1. *Suppose that Ω is a smoothly bounded domain in $\mathbf{R}^3$. We define a curl operator $\mathcal{S}$ in the Hilbert space $L^2_\Sigma(\Omega)$ by*

$$\mathcal{S}\boldsymbol{u} = \nabla \times \boldsymbol{u},$$

$$D(\mathcal{S}) = \{\boldsymbol{u} \in L^2_\Sigma(\Omega) \ ; \ \nabla \times \boldsymbol{u} \in L^2_\Sigma(\Omega)\}.$$

The $\mathcal{S}$ is a self-adjoint operator. The spectrum of $\mathcal{S}$ consists of only point spectrum $\sigma_p(\mathcal{S})$ which is a discrete set of real numbers.

This theorem says that the Beltrami equation (1) together with the zero-flux condition (see the definition of the space $L^2_\Sigma(\Omega)$) has non-zero solution only for special discrete real numbers $\lambda \in \sigma_p(\mathcal{S})$. If Ω is simply connected ($\nu = 0$), the topological flux $\Phi_j(\ )$ does not exist, so that $L^2_\Sigma(\Omega) = L^2_\sigma(\Omega)$. If Ω is multiply connected ($\nu \geq 1$), however, we can remove the zero-flux condition assumed in Theorem 1, and consider a wider set of functions to find solutions of (1). This is done by considering the curl operator defined in the space $L^2_\sigma(\Omega)$. Let us trace the method of Yoshida-Giga [1].

Lemma 1. *For every $\boldsymbol{f} \in L^2_\sigma(\Omega)$, the equation*

$$\nabla \times \boldsymbol{u} = \boldsymbol{f} \quad (\text{in } \Omega) \tag{33}$$

has a unique solution in $L^2_\Sigma(\Omega)$.

Proof. Let $\tilde{\boldsymbol{f}}$ be the 0-extension of $\boldsymbol{f}$ over $\mathbf{R}^3$, i.e.

$$\tilde{\boldsymbol{f}}(x) = \begin{cases} \boldsymbol{f}(x) & x \in \Omega, \\ 0 & x \notin \Omega. \end{cases}$$

Since $\boldsymbol{f} \in L^2_\sigma(\Omega)$, we have $\nabla \cdot \tilde{\boldsymbol{f}} = 0$ in $\mathbf{R}^3$. We denote by $(-\Delta)^{-1}$ the vector Newtonian potential. We define

$$\boldsymbol{w}_0 = \nabla \times [(-\Delta)^{-1}\tilde{\boldsymbol{f}}] \quad \text{in } \Omega.$$

We denote by $\mathcal{P}_\Sigma$ the orthogonal projection in $L^2(\Omega)$ onto $L^2_\Sigma(\Omega)$, and define $\boldsymbol{u}_0 = \mathcal{P}_\Sigma \boldsymbol{w}_0$. Since $L^2_\Sigma(\Omega)$ is orthogonal to Ker(curl), we observe

$$\nabla \times \boldsymbol{u}_0 = \nabla \times \boldsymbol{w}_0 = \nabla \times \{\nabla \times [(-\Delta)^{-1}\tilde{\boldsymbol{f}}]\}.$$

Since $\nabla \cdot [(-\Delta)^{-1}\tilde{\boldsymbol{f}}] = 0$, we obtain

$$\nabla \times \{\nabla \times [(-\Delta)^{-1}\tilde{\boldsymbol{f}}]\} = -\Delta[(-\Delta)^{-1}\tilde{\boldsymbol{f}}] = \tilde{\boldsymbol{f}}.$$

We thus find that $\boldsymbol{u}_0$ ($\in L^2_\Sigma(\Omega)$) is the solution of (33). Since $L^2_\Sigma(\Omega)$ is orthogonal to Ker(curl), this $\boldsymbol{u}_0$ is the unique solution. $\square$

This lemma shows that every solenoidal vector field (member of $L^2_\sigma(\Omega)$) has a unique vector potential in the space $L^2_\Sigma(\Omega)$. We apply this result to determine the vector potential of the harmonic field (member of $L^2_H(\Omega)$). Let ν (≥ 1) be the dimension of $L^2_H(\Omega)$ (first Betti number of Ω), and $\boldsymbol{h}_j$ ($j = 1, \cdots, \nu$) be the orthogonal basis of $L^2_H(\Omega)$ such that

$$\Phi_i(\boldsymbol{h}_j) = \int_{\Sigma_i} \boldsymbol{n} \cdot \boldsymbol{h}_j \, ds = \delta_{i,j}. \tag{34}$$

By solving (33) for $\boldsymbol{f} = \boldsymbol{h}_j$, we obtain the corresponding vector potential which we denote by $\boldsymbol{g}_j$, i.e.,

$$\nabla \times \boldsymbol{g}_j = \boldsymbol{h}_j \quad (\text{in } \Omega), \quad \boldsymbol{g}_j \in L^2_\Sigma(\Omega) \quad (j = 1, \cdots, \nu).$$

Let us consider an arbitrary harmonic field and its vector potential, and write them as

$$\boldsymbol{h} = \sum_{j=1}^{\nu} \xi_j \boldsymbol{h}_j, \quad \boldsymbol{g} = \sum_{j=1}^{\nu} \xi_j \boldsymbol{g}_j. \tag{35}$$

For every $\lambda \notin \sigma_p(\mathcal{S})$, the resolvent operator $(\mathcal{S} - \lambda)^{-1}$ defines a unique continuous map on $L^2_\Sigma(\Omega)$. We consider

$$\boldsymbol{v} = \lambda \boldsymbol{g} + \lambda^2 (\mathcal{S} - \lambda)^{-1} \boldsymbol{g}.$$

This $\boldsymbol{v}$ is the unique solution (in $L^2_\Sigma(\Omega)$) of

$$(\text{curl} - \lambda)\boldsymbol{v} = \lambda \boldsymbol{h} \quad (\text{in } \Omega). \tag{36}$$

Now we find that $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{h}$ solves

$$\begin{cases} (\mathrm{curl}\ \ -\lambda)\boldsymbol{u} = 0 & (\text{in } \Omega), \\ \\ \boldsymbol{n}\cdot\boldsymbol{u} = 0 & (\text{on } \partial\Omega), \end{cases}$$

Since $\boldsymbol{h}$ $(\in L^2_H(\Omega))$ and $\boldsymbol{v}$ $(\in L^2_\Sigma(\Omega))$ are orthogonal, $\boldsymbol{u} \not\equiv 0$.

We have shown that the single curl Beltrami equation (1) has a non-zero solution for every complex number λ, if the domain Ω is multiply connected. For $\lambda \notin \sigma_p(\mathcal{S})$, the solution is uniquely determined by the harmonic field $\boldsymbol{h}$. Although (1) appears as a homogeneous equation, the harmonic field (member of the kernel of curl) plays a role of a hidden inhomogeneous term; see (36). On the other hand, for $\lambda \in \sigma_p(\mathcal{S})$, the solution is given by the eigenfunction of the self-adjoint curl operator $\mathcal{S}$. Therefore, the solution is a zero-flux field, and $\boldsymbol{h}$ must be set to zero. The solution is not unique in the sense that any constant multiple of the eigenfunction is a solution.

Because of (32), it is now straightforward to generalize the theory for the double curl (and multi curl) Beltrami equations.

Theorem 2. *For a multiply connected smoothly bounded domain Ω, and for all complex numbers λ_1 and λ_2, the equation*

$$(\mathrm{curl} - \lambda_1)(\mathrm{curl} - \lambda_2)\boldsymbol{u} = 0 \tag{37}$$

has a non-zero solution.

Let us examine the relations among the solutions, the harmonic fields and the fluxes. If $\lambda_1, \lambda_2 \notin \sigma_p(\mathcal{S})$, then the solution is given by

$$\boldsymbol{u} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2,$$

$$\boldsymbol{u}_j = \boldsymbol{h} + \lambda_j\boldsymbol{g} + \lambda_j^2(\mathcal{S} - \lambda_j)^{-1}\boldsymbol{g} \quad (j = 1, 2),$$

where $\boldsymbol{h} \in L^2_H(\Omega)$, $\nabla\times\boldsymbol{g} = \boldsymbol{h}$ and $\boldsymbol{g} \in L^2_\Sigma(\Omega)$. Let us decompose $\boldsymbol{h}$ in terms of the normalized bases as (35). The coefficients $c_1, c_2, \xi_1, \cdots, \xi_\nu$ are related to the fluxes of $\boldsymbol{u}$ and $\nabla \times \boldsymbol{u}$ by

$$\begin{cases} (c_1 + c_2)\xi_j = \Phi_j(\boldsymbol{u}), \\ \\ (c_1\lambda_1 + c_2\lambda_2)\xi_j = \Phi_j(\nabla \times \boldsymbol{u}) \end{cases} \qquad (j = 1, \cdots, \nu),$$

where $\Phi_j(\;)$ is the flux through the cut Σ_j. When $\nu = 1$ (as in the case of a simple toroid), we can give the fluxes of both $\boldsymbol{u}$ and $\nabla \times \boldsymbol{u}$ independently to determine ξ_1 and c_1 with setting $c_2 = 1 - c_1$ (cf. Appendix II). For $\nu > 1$, the fluxes of $\nabla \times \boldsymbol{u}$ are not totally independent.

If $\lambda_1 \in \sigma_p(\mathcal{S})$ and $\lambda_2 \notin \sigma_p(\mathcal{S})$, we take $\boldsymbol{u}_1$ to be the eigenfunction corresponding to λ_1. Then, $\boldsymbol{u}_1$ is a zero-flux function, and hence, c_1 is an arbitrary constant. The other component $\boldsymbol{u}_1$ carries fluxes. Taking $c_2 = 1$, we can determine

$$\xi_j = \Phi_j(\boldsymbol{u}) \quad (j = 1, \cdots, \nu).$$

If $\lambda_1, \lambda_2 \in \sigma_p(\mathcal{S})$, then both $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are the corresponding eigenfunctions. Solution exists only for $\xi_j = 0$ $(j = 1, \cdots, \nu)$.

V. SUMMARY

The study of the solvability of the double curl equation is warranted both by physical as well as mathematical considerations. A more adequate modeling of plasma dynamics, containing a coupling of the magnetic and fluid aspects of a plasma, necessarily leads to a departure from the conventional single Beltrami equilibria (1) which are restricted to only force-free equilibria. This departure, then, leads to an immensely larger class of physically interesting equilibria which can be constructed by a superposition of several different Beltrami fields. In the example dealt with in this paper (where the coupling is introduced by the Hall term), a superposition of two Beltrami fields suffices. Notice that in the nonlinear vortex dynamics models such as (3) with coupled $\boldsymbol{\omega}$ and $\boldsymbol{U}$, a linear combination of Beltrami fields is no longer a Beltrami field. Hence, a finite pressure and coupled flows can exist in conjunction with the magnetic field, and the structures that are far richer than those of single Beltrami fields come within the scope of the theory [3].

The mathematical content of the paper may be summarized as follows: We have elucidated the general relation between the (double curl) Beltrami fields and the harmonic fields which, being members of Ker(curl), play the role of a hidden inhomogeneous term in the Beltrami equations. The existence of harmonic fields invokes the multiply-connectedness of the domain. For every $\lambda \in \mathbf{C} \setminus \sigma_p(\mathcal{S})$ (point spectrum of the self-adjoint curl operator), a harmonic field generates non-zero unique Beltrami field corresponding to λ. When $\lambda \in \sigma_p(\mathcal{S})$, the corresponding eigenfunction gives the Beltrami field. The linear combination of two Beltrami fields yields the double curl Beltrami field.

ACKNOWLEDGMENTS

This work was supported by Grant-in-Aid for Scientific Research from the Japanese Ministry of Education, Science and Culture No. 09308011. A part of this study was done at the workshop "Various Approaches to Complex Systems" held at the International Institute for Advanced Studies in Kyoto.

APPENDIX I
(VARIATIONAL PRINCIPLE AND RELAXATION THEORY)

The Beltrami condition can be derived by a variational principle invoking the "helicity". Woltjer [13] derived the force-free equation (18) by minimizing the magnetic energy with the constraint that the magnetic helicity is conserved. Here, the magnetic helicity is, for a magnetic field $\boldsymbol{B}$ and its vector potential $\boldsymbol{A}$,

$$H = \frac{1}{2} \int_\Omega \boldsymbol{A} \cdot \boldsymbol{B} \, dx.$$

Minimization of the magnetic field energy

$$E = \frac{1}{2} \int_\Omega |\boldsymbol{B}|^2 \, dx = \frac{1}{2} \int_\Omega |\nabla \times \boldsymbol{A}|^2 \, dx$$

with keeping H constant is represented by the variational principle

$$\delta(E - \lambda H) = 0, \tag{38}$$

where λ is the Lagrange multiplier. Assuming a boundary condition

$$\boldsymbol{n} \times \boldsymbol{A} = 0 \quad (\text{on } \partial\Omega), \tag{39}$$

(note that $\boldsymbol{n} \cdot \boldsymbol{B} = 0$ (on $\partial\Omega$) follows from (39)), the formal Euler-Lagrange equation with respect to (38) yields (18); see [14] for a more rigorous treatment of the variational principle.

Using Maxwell's equations, we obtain

$$\frac{\partial}{\partial t}(\boldsymbol{A} \cdot \boldsymbol{B}) = -2\boldsymbol{E} \cdot \boldsymbol{B} - \nabla \cdot (\phi \boldsymbol{B} + \boldsymbol{E} \times \boldsymbol{A}).$$

In an ideal plasma, $\boldsymbol{E} \cdot \boldsymbol{B} = 0$ (see (11)). When Ω is surrounded by a perfectly conducting wall, we find that H is a constant of motion. Taylor [15] introduced the far-reaching concept of relaxation; he conjectured that a small amount of resistivity would tend to relax all constraints restricting an ideal plasma leaving only the "rugged" constraint on the global helicity H. When the magnetic energy achieves its minimum under the constraint on H, the "relaxed state" is characterized by the variational principle (38), and hence, the magnetic field satisfies the force-free equation (18). Many authors have examined the selective dissipation of the magnetic field energy E with respect to the helicity H (see Hasegawa [16] and papers cited there). Montgomery *et al.* [17] studied the statistical mechanical properties of the relaxed state using the Beltrami functions to expand fields (see also [18]).

The conservation of helicity applies for general vortex dynamics. Let $\boldsymbol{\omega}$ be a vorticity that satisfies (3) and boundary condition

$$\boldsymbol{n} \times (\boldsymbol{U} \times \boldsymbol{\omega}) = 0 \quad (\text{on } \partial\Omega).$$

The general "helicity" is defined as

$$\mathcal{H} = \frac{1}{2} \int_\Omega (\text{curl}^{-1} \boldsymbol{\omega}) \cdot \boldsymbol{\omega} \; dx,$$

where curl^{-1} is the inverse operator of the curl that is represented by the Biot-Savart integral (see Lemma 1 for a more suitable treatment). By this definition, we easily verify the conservation of $\mathcal{H}$. For our two-fluid MHD model, we have two helicities

$$H_1 = \frac{1}{2}\int_\Omega \boldsymbol{A}\cdot\boldsymbol{B}\ dx,$$
$$H_2 = \frac{1}{2}\int_\Omega (\boldsymbol{A}+\boldsymbol{V})\cdot(\boldsymbol{B}+\nabla\times\boldsymbol{V})\ dx.$$

Minimization of the total energy

$$E = \frac{1}{2}\int_\Omega \left(|\boldsymbol{B}|^2 + |\boldsymbol{V}|^2\right)\ dx$$

with the constraints on H_1 and H_2 will directly lead us to (27) and (28) (cf. [19]).

APPENDIX II

(EXAMPLES OF SOLUTIONS)

Some explicit forms of the Beltrami fields may help understanding of the structures of the solutions.

When we consider a cubic volume that has sides of length a and assume the periodic boundary condition, we have the so-called ABC flow. Let A, B and C be real (complex) constants and $\lambda = 2\pi n/a$ ($n \in \mathbf{N}$). In the cartesian coordinates, we define

$$\boldsymbol{u} = \begin{pmatrix} A\sin\lambda z + C\cos\lambda y \\ B\sin\lambda x + A\cos\lambda z \\ C\sin\lambda y + B\cos\lambda x \end{pmatrix}. \tag{40}$$

We easily verify that (40) gives an eigenfunction of the curl belonging to an eigenvalue λ. The linear combination of two ABC flows give the double curl Beltrami flow.

Solutions with the zero-normal boundary conditions are known for a cylindrical domain. In the (r,θ,z) cylindrical coordinates, the Chandrasekhar-Kendall function [6] is defined as

$$\boldsymbol{u} = \lambda(\nabla\psi\times\nabla z) + \nabla\times(\nabla\psi\times\nabla z) \tag{41}$$

with

$$\lambda = \pm(\mu^2 + k^2)^{1/2}, \tag{42}$$

$$\psi = J_m(\mu r) e^{i(m\varphi - kz)}, \quad k = 2\pi n/L,\ m, n \in \mathbf{N}, \tag{43}$$

where J_m is the ordinary Bessel function and L is the length of the periodic cylinder. We find that $\boldsymbol{u}$ is an eigenfunction of the curl corresponding to the eigenvalue λ ($\in \mathbf{R}$). The eigenvalue is determined by the boundary condition that the normal component of $\boldsymbol{u}$ vanishes on the surface of the cylindrical domain. This condition becomes trivial when $k = m = 0$. For these axisymmetric modes, we impose the "zero-flux condition"

$$\Phi(\boldsymbol{u}) = \int_\Sigma \boldsymbol{n} \cdot \boldsymbol{u} ds = 0, \tag{44}$$

where Σ is a cut of the cylinder (cf. Theorem 1).

When we do not impose the zero-flux condition, however, the eigenvalue μ can be an arbitrary real (and even complex) number for the $k = m = 0$ mode [2]. Therefore, we have non-zero Beltrami fields for arbitrary λ. For such a solution that has a finite flux $\Phi(\boldsymbol{u})$, the flux can be regarded as the variable of state. The double curl Beltrami field is a combination of two Beltrami fields, and hence, the degree of freedom is two and two fluxes $\Phi(\boldsymbol{u})$ and $\Phi(\nabla \times \boldsymbol{u})$ can be assigned.

In a two dimensional system, we can apply the Clebsch representation of solenoidal vector fields (cf. (4) and (5)). For example, let us assume that the fields are homogeneous in the direction of z in the cartesian coordinates x-y-z. We write $\boldsymbol{B}$ in a contravariant-covariant combination form

$$\boldsymbol{B} = \nabla\psi \times \boldsymbol{e} + \phi\, \boldsymbol{e}, \tag{45}$$

where $\boldsymbol{e} = \nabla z$. The ψ and ϕ are scalar functions of x and y. We have

$$\nabla \times \boldsymbol{B} = \nabla\phi \times \boldsymbol{e} + (-\Delta\psi)\boldsymbol{e},$$

$$\nabla \times (\nabla \times \boldsymbol{B}) = \nabla(-\Delta\psi) \times \boldsymbol{e} + (-\Delta\phi)\boldsymbol{e}.$$

Using these expressions in the double curl Beltrami equation (29), we obtain a system of coupled Helmholtz equations

$$\begin{cases} -\Delta\psi + \alpha\phi + \beta\psi = C, \\ \\ -\Delta\phi - \alpha\Delta\psi + \beta\phi = 0, \end{cases} \tag{46}$$

where C is a constant. Biasing the potential ψ with $-C/\beta$, we can eliminate this constant. The system (46) can be casted into a symmetric form

$$\Delta \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ -\alpha\beta & \beta - \alpha^2 \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix}. \tag{47}$$

Similar algebra applies for the case of axisymmetric (toroidal) systems, where we must take $\boldsymbol{e} = \nabla\theta$ in (45) and assume that ψ and ϕ are functions of r and z in the r-θ-z cylindrical coordinates. Then, the Laplacian Δ is replaces by the Grad-Shafranov operator

$$L = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}.$$

The coupled Grad-Shafranov equation of the type (46) was derived previously for the analysis of toroidal equilibrium in a plasma-beam system, where the inertia force of the beam particles brings about coupling of the magnetic field and the beam flow [20]

REFERENCES

[1] Z. Yoshida and Y. Giga, Math. Z. **204**, 235 (1990).

[2] Without the zero-flux condition, the curl operator is not self-adjoint [1], and hence, the potency of the set of the eigenvalues (point spectrum) can be uncountable without violating the separability of the Hilbert space $L^2(\Omega)$.

[3] S.M. Mahajan and Z. Yoshida, Phys. Rev. Lett. **81**, 4863 (1998).

[4] R. Lüst and A. Schlüter, Z. F. Ap. **34**, 263 (1954).

[5] S. Chandrasekhar, Proc. Nat. Acad. Sci. **42**, 1 (1956).

[6] S. Chandrasekhar and P.C. Kandall, Astrophys. J. **126**, 457 (1957).

[7] L. Acton *et al.*, Science **258**, 618 (1992).

[8] A. Königl and A.R. Choudhuri, Astrophys. J. **289**, 173 (1985).

[9] J.B. Taylor, Rev. Mod. Phys. **58**, 741 (1986).

[10] Z. Yoshida *et al.*, J. Plasma Phys. **59**, 103 (1998).

[11] We remark that the solution $\boldsymbol{u}$ is a real function, if α and β are real. Indeed, if $\boldsymbol{u}$ is a solution for real α and β, then $\overline{\boldsymbol{u}}$ is also a solution (take the complex conjugate of the equations). Since the solution is unique (see Sec. IV), $\boldsymbol{u} = \overline{\boldsymbol{u}}$, and hence $\boldsymbol{u}$ is a real function.

[12] R. Temam, *Navier-Stokes Equations*, North-Holland, Amsterdam (1984).

[13] L. Woltjer, Proc. Natl. Acad. Sci. U.S.A. **44**, 489 (1958).

[14] P. Laurence and M. Avellaneda, J. Math. Phys. **32**, 1240 (1991).

[15] J. B. Taylor, Phys. Rev. Lett. **33**, 1139 (1974).

[16] A. Hasegawa, Adv. Phys. **34**, 1 (1985).

[17] D. Montgomery, L. Turner, and G. Varala, Phys. Fluids **21**, 757 (1978).

[18] N. Ito and Z. Yoshida, Phys. Rev. E **53**, 5200 (1996).

[19] L. C. Steinhauer and A. Ishida, Phys. Rev. Lett. **79**, 3423 (1997).

[20] Z. Yoshida, Phys. Fluids B **1**, 1702 (1989).