Scale Separation in Two-Fluid Plasmas and its Implications for Dynamo Theory

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Abstract

By relating the velocity and the magnetic fields, the Hall term in the two-fluid model of a plasma leads to a singular perturbation that couples physical quantities varying at vastly different length scales. In a Beltrami model of the steady states, then, the dynamo mechanism emerges naturally. The scale separation also suggests a dissipative mechanism for heating the solar coronal structures embedded in relatively smooth magnetic fields.

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The conventional macroscopic description of a plasma viewed as a single conducting-fluid naturally fails in capturing the physics which stems from the two-fluid nature of the electron-ion plasmas. The absence (or a very inadequate treatment) of the mutual interaction (possibly strong) between the velocity and the magnetic fields is a rather serious omission of this approach. The purpose of this letter is to show that a proper two-fluid treatment leads to the delineation of qualitatively new phenomena in magnetofluids resulting from the self-consistent relationships which must exist between the velocity and the magnetic fields. It is shown, for example, that the Hall term (the principal term that distinguishes the single-fluid model from the two-fluid model), usually assumed to be negligibly small, causes a "singular perturbation" introducing a new "short" equilibrium scale. Consequently a small "short" length scale perturbation in an otherwise smooth magnetic (velocity) field is associated with a large perturbation in the velocity (magnetic) field because the fields are self-consistently related by a singular perturbation term. It becomes possible, thus, to have equilibria in which two related physical quantities can vary on vastly different length scales.

In order to trace the origin of the singular perturbation, we start with a re-examination of the "single-fluid" plasma dynamics described by the flow velocity $\boldsymbol{v}$ and the electric current $\boldsymbol{j}$. For a quasineutral plasma with singly charged ions, these two variables are related to the electron flow velocity $\boldsymbol{V}_e$ and the ion flow velocity $\boldsymbol{V}_i$ by

$$\boldsymbol{v} = \frac{M\boldsymbol{V}_i + m\boldsymbol{V}_e}{M+m} \approx \boldsymbol{V}_i, \quad \boldsymbol{j} = en(\boldsymbol{V}_i - \boldsymbol{V}_e),$$

where m (M ($\gg m$))is the electron (ion) mass, n is the number density and e is the elementary charge. The two principal equations of the ideal (dissipation-less) single-fluid model are

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0, \tag{1}$$

$$\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \frac{1}{Mn}\left(\boldsymbol{j} \times \boldsymbol{B} - \nabla p\right), \tag{2}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are, respectively, the electric and the magnetic fields. We may write

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\phi, \quad \boldsymbol{j} = \frac{1}{\mu_0}\nabla \times \boldsymbol{B}$$

in terms of a vector potential $\boldsymbol{A}$ ($\nabla \times \boldsymbol{A} = \boldsymbol{B}$) and a scalar potential ϕ. The system (1)-(2) does not contain any scaling coefficients (such as the Reynolds number or the Lundquist number), and hence, $\boldsymbol{v}$, $\boldsymbol{j}$ and the electromagnetic fields have no characteristic length scales. Indeed, we can rewrite (1) and (2) in a scale-invariant dimension-less form. Choosing an arbitrary L_0 and B_0, we normalize $\boldsymbol{x}$ and $\boldsymbol{B}$ as

$$\boldsymbol{x} = L_0 \hat{\boldsymbol{x}}, \quad \boldsymbol{B} = B_0 \hat{\boldsymbol{B}}. \tag{3}$$

With the Alfvén speed written as $V_A = B_0/\sqrt{\mu_0 M n}$ (n is some representative value), and the normalizations

$$\begin{cases} t = (L_0/V_A)\hat{t}, \quad p = (B_0^2/\mu_0)\hat{p}, \quad \boldsymbol{v} = V_A \hat{\boldsymbol{v}}, \\ \boldsymbol{A} = (L_0 B_0)\hat{\boldsymbol{A}}, \quad \phi = (V_A L_0 B_0)\hat{\phi}, \end{cases} \tag{4}$$

equations (1) and (2) transform to

$$\frac{\partial}{\partial \hat{t}} \hat{\boldsymbol{A}} = \hat{\boldsymbol{v}} \times \hat{\boldsymbol{B}} - \hat{\nabla}\hat{\phi}, \tag{5}$$

$$\frac{\partial}{\partial \hat{t}} \hat{\boldsymbol{v}} = -(\hat{\boldsymbol{v}} \cdot \hat{\nabla})\hat{\boldsymbol{v}} + (\hat{\nabla} \times \hat{\boldsymbol{B}}) \times \hat{\boldsymbol{B}} - \hat{\nabla}\hat{p}. \tag{6}$$

The one-fluid system, thus, is scale invariant.

This is in marked contrast to the two-fluid model which has a characteristic length scale representing ion inertia effects. The two-fluid formulation retains both the electron and the ion velocities. Neglecting the small electron inertia, the electron equation of motion (under the Lorentz force and the electron pressure $-\nabla p_e$) is

$$\boldsymbol{E} + \boldsymbol{V}_e \times \boldsymbol{B} + \frac{1}{en}\nabla p_e = 0. \tag{7}$$

Comparing (7) with (1), we notice that $\boldsymbol{v}$ is replaced by $\boldsymbol{V}_e = \boldsymbol{V}_i - \boldsymbol{j}/(en) \approx \boldsymbol{v} - \boldsymbol{j}/(en)$. We thus have two new terms $-\boldsymbol{j}/(en)$ and $\nabla p_e/(en)$; the former being the Hall term. The equation of motion of ions is a little less affected because $\boldsymbol{V}_i \approx \boldsymbol{v}$. Denoting $\boldsymbol{V}_i = \boldsymbol{V}$, the ions obey

$$\frac{\partial}{\partial t}\boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = \frac{e}{M}\left(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}\right) - \frac{1}{Mn}\nabla p_i. \tag{8}$$

Applying the normalization used in (3)-(4) for (7) and (8), we obtain

$$\frac{\partial}{\partial \hat{t}}\hat{\boldsymbol{A}} = \left(\hat{\boldsymbol{V}} - \varepsilon\hat{\nabla}\times\hat{\boldsymbol{B}}\right)\times\hat{\boldsymbol{B}} - \hat{\nabla}\left(\hat{\phi} + \varepsilon\hat{p}_e\right), \tag{9}$$

$$\frac{\partial}{\partial \hat{t}}(\varepsilon\hat{\boldsymbol{V}} + \hat{A}) = \hat{\boldsymbol{V}}\times\left(\hat{\boldsymbol{B}} + \varepsilon\hat{\nabla}\times\hat{\boldsymbol{V}}\right) - \hat{\nabla}\left(\varepsilon V^2/2 + \varepsilon\hat{p}_i + \hat{\phi}\right). \tag{10}$$

where the scaling coefficient $\varepsilon = \lambda_i/L_0$ is a measure of the ion skin depth

$$\lambda_i = \frac{c}{\omega_{pi}} = \frac{V_A}{\omega_{ci}} = \sqrt{\frac{M}{\mu_0 n e^2}}.$$

When ε is a small parameter, the terms of order unity in (9), as well as those in (10), imply (1), while (after eliminating order unity terms) remaining terms of order ε yield (2). All terms in the set, however, can balance on the length scale of the ion skin depth. For $L_0 = \lambda_i$, $\varepsilon = 1$, and the ion gyration time normalizes the time $\hat{t}$.

The Hall term $(\hat{\nabla}\times\hat{\boldsymbol{B}})\times\hat{\boldsymbol{B}}$ of (9), which may be regarded as a singular perturbation to the conventional induction law (5), plays an essential role in connecting the two different length scales, i.e., the characteristic length scale λ_i, and a possible longer scale that describes the global structure of the system. In what follows, we shall consider the most fundamental stationary structures of the electromagnetic fields and the associated electron and ion flows that may be common to various slowly evolving (in the time scale much longer than the ion gyration time) plasmas, and derive a fundamental relation between the short and the long length scale components.

Let us take $L_0 = \lambda_i$, and simplify the notation by dropping the $\hat{}$ on the normalized variables. Taking the curl of (9) and (10), we can cast them in a revealing symmetric form

$$\frac{\partial}{\partial t}\boldsymbol{\Omega}_j - \nabla\times(\boldsymbol{U}_j\times\boldsymbol{\Omega}_j) = 0 \quad (j = 1, 2) \tag{11}$$

in terms of a pair of generalized vorticities

$$\boldsymbol{\Omega}_1 = \boldsymbol{B}, \quad \boldsymbol{\Omega}_2 = \boldsymbol{B} + \nabla\times\boldsymbol{V},$$

and the effective flows

$$\boldsymbol{U}_1 = \boldsymbol{V} - \nabla \times \boldsymbol{B}, \quad \boldsymbol{U}_2 = \boldsymbol{V}.$$

The simplest equilibrium solution to (11) is given by the "Beltrami conditions"

$$\boldsymbol{U}_j = \mu_j \boldsymbol{\Omega}_j \quad (j = 1, 2), \tag{12}$$

implying the alignment of the vorticities with the corresponding flows. Writing $a = 1/\mu_1$ and $b = 1/\mu_2$, and assuming that a and b are constants, the Beltrami conditions (12) translate to the simultaneous linear equations

$$\boldsymbol{B} = a(\boldsymbol{V} - \nabla \times \boldsymbol{B}), \tag{13}$$

$$\boldsymbol{B} + \nabla \times \boldsymbol{V} = b\boldsymbol{V}, \tag{14}$$

which have a simple and significant connotation; the electron flow $(\boldsymbol{V} - \nabla \times \boldsymbol{B})$ parallels the magnetic field $\boldsymbol{B}$, while the ion flow $\boldsymbol{V}$ follows the "generalized magnetic field" $(\boldsymbol{B} + \nabla \times \boldsymbol{V})$. The generalized magnetic field contains the Coriolis' force induced by the ion inertia effect on a circulating flow.

Combining (13) and (14) yields a second order partial differential equation

$$\nabla \times (\nabla \times \boldsymbol{B}) + \alpha \nabla \times \boldsymbol{B} + \beta \boldsymbol{B} = 0, \tag{15}$$

where $\alpha = (1/a) - b$ and $\beta = 1 - b/a$. If "curl" denotes the curl derivative $\nabla\times$, Eq.(15) may be written as

$$(\mathrm{curl} - \Lambda_+)(\mathrm{curl} - \Lambda_-)\boldsymbol{B} = 0, \tag{16}$$

where

$$\Lambda_\pm = \frac{1}{2}\left[-\alpha \pm (\alpha^2 - 4\beta)^{1/2}\right]. \tag{17}$$

Since the operators $(\mathrm{curl} - \Lambda_\pm)$ commute, the general solution to the "double curl Beltrami equation" (16) is given by the linear combination of the two Beltrami fields, i.e., for $\boldsymbol{G}_\pm$ such that $(\mathrm{curl} - \Lambda_\pm)\boldsymbol{G}_\pm = 0$, and for arbitrary constants $C_\pm$, the sum

$$\boldsymbol{B} = C_+\boldsymbol{G}_+ + C_-\boldsymbol{G}_- \tag{18}$$

solves (16). The corresponding ion flow is given by

$$\boldsymbol{V} = \left(\frac{1}{a} + \Lambda_+\right) C_+\boldsymbol{G}_+ + \left(\frac{1}{a} + \Lambda_-\right) C_-\boldsymbol{G}_-. \tag{19}$$

We emphasize that these combinations of the two different Beltrami fields are the consequences of the flow-magnetic field coupling generated by the singular perturbation associated with the Hall effect (the source of the highest derivative term in (15)). In fact, if we omit the corresponding term $\nabla \times \boldsymbol{B}$ in the right-hand side of (13), we obtain the conventional single Beltrami condition [2] on $\boldsymbol{B}$, as well as on $\boldsymbol{V}$ that parallels $\boldsymbol{B}$.

We note that the parameter Λ_+ (Λ_-), being the eigenvalue of the curl operator, characterizes the reciprocal of the length scale on which $\boldsymbol{G}_+$ ($\boldsymbol{G}_-$) changes significantly. As the "Beltrami parameters" a and b vary, $\Lambda_\pm$ can assume quite different magnitudes. Figure 1 shows the separation of these two length scales in the parameter space spanned by a and b. When $a = b$, $\beta = 0$, so that one of $\Lambda_\pm$ is zero and the other is α. Therefore, in the region $a \approx b$, these two length scales separate significantly.

A fascinating consequence of such a scale separation is that the "regularity" of $\boldsymbol{B}$ and that of the coupled $\boldsymbol{V}$ may have an appreciable difference. It is possible that the magnetic field is dominated by the long scale field ($|\Lambda_j| \ll 1$), while the flow may have a large component varying on the short scale ($|\Lambda_j| \gg 1$). The opposite relation may also pertain.

Let us examine two different regimes of parameters. First, we consider the case when both $|a|$ and $|b|$ are relatively large and $a \approx b$. Then the Beltrami conditions (13) and (14) can hold if $|\boldsymbol{V}| \ll |\boldsymbol{B}|$, i.e., the flow is "extremely sub-Alfvénic". For this case, we may approximate

$$\Lambda_+ \approx \frac{1}{b} - \frac{1}{a}, \quad \Lambda_- \approx b - \frac{1}{b},$$

leading to $|\Lambda_+| \ll 1$ and $|\Lambda_-| \gg 1$, and

$$\boldsymbol{V} \approx \frac{1}{b} C_+\boldsymbol{G}_+ + bC_-\boldsymbol{G}_-. \tag{20}$$

In comparison with the magnetic field (18), we find that, in the flow velocity, the short-scale (Λ_-) component is amplified by a factor b, while the long-scale (Λ_+) component is reduced by a factor $1/b$. Therefore, when $|C_-|/|C_+| < O(1/|b|)$, a primarily smooth magnetic field $\boldsymbol{B} \approx C_+\boldsymbol{G}_+$ couples with a jittery flow $\boldsymbol{V} \approx bC_-\boldsymbol{G}_-$ ($|bC_-| < O(|C_+|)$); see Fig. 2. This simple theoretical result is of fundamental significance and it exposes the mechanism operating in the systems where irregular flows are observed in conjunction with smooth magnetic fields. One example is the generic turbulent dynamo which is a process for generating ordered magnetic fields through complex flows. The short scale length (Λ_-) is to be viewed as the correlation length of the turbulent velocity field.

Another important implication concerning short-scale flows is that the viscous dissipation can be appreciably large in such a system. As far as one assumes that the magnetic field and the flow have the same length scales, a plasma with a smooth magnetic field cannot cause appreciable energy dissipation (both resistive and viscous damping) and resultant plasma heating. However, the above result shows that smooth magnetic fields (for example, those generated by turbulent dynamo) can be associated with big and jittery flows which yield significant ion heating when they are damped by a finite viscosity. Although the present theory does not contain viscosity, it suggests an important mechanism of inducing the coupling of the magnetic fields and flows with far different length scales. A possible consequence could be the heating of the solar corona by the viscous dissipation of the jittery part of the flow kinetic energy even for relatively smooth solar magnetic fields.

Next, we suppose that both $|a|$ and $|b|$ are small and $a \approx b$. This is the case when the flow dominates the dynamics and the magnetic field is small. Here we can approximate

$$\Lambda_+ \approx b - a, \quad \Lambda_- \approx -\frac{1}{a} + a - b.$$

We find again that $|\Lambda_+| \ll 1$ and $|\Lambda_-| \gg 1$. The flow is given by

$$\boldsymbol{V} \approx \frac{1}{a}C_+\boldsymbol{G}_+ + (a-b)C_-\boldsymbol{G}_-. \tag{21}$$

In contrast to the above result (20), the factor $1/a$ in the first term (long scale component) of (21) is a big parameter. We thus find that, for $|C_+|/|C_-| > O(|a|)$, a primarily smooth flow

$\boldsymbol{V} \approx (C_+/a)\boldsymbol{G}_+$ is coupled with a short-scale magnetic field $\boldsymbol{B} \approx C_-\boldsymbol{G}_-$ ($|C_+/a| > O(|C_-|)$); see Fig. fig:smoothV. These solutions are relevant to the fast dynamo [5], that is, a kinematic process of generating magnetic field through the induction effect. It is possible that the short-scale magnetic fields have a long-term temporal order, since the magnetic field couples with the long-scale flow which dominates the system.

A treatment of the magnetofluid in which the velocity and the magnetic fields are treated at par (we do this by keeping the Hall term) yields structures which are much more complex, interesting and richer than the ones contained in the conventional conducting-fluid model. The new formalism, in addition to introducing a new characteristic length scale λ_i (ion skin depth) to the unperturbed scale-invariant system, couples the magnetic field with the flow allowing the possibility of transforming one into the other. Mathematically the equilibrium system appears as a coupling of two Beltrami conditions with two spatial scale lengths which, through their self-consistent coupling, allow the simultaneous existence of the two fields varying on vastly different scales. This disparate variation of the velocity and the magnetic field is precisely the required recipe which lies at the heart of the dynamo mechanism: the turbulent dynamo in which the short scale velocity field produces a relatively smooth magnetic field, and the kinematic (fast) dynamo in which the length scales for the two fields are reversed. We have just shown that the seeds of both these possibilities are there even in the simplest manifestation of the magnetofluid equilibria.

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FIGURES

FIG. 1. Separation of the two length scales of double Beltrami fields. Figure (a) shows the contour of $\log_{10}(|\Lambda_-|/|\Lambda_+|)$ as the function of the two Beltrami parameters a and b. In the region 'C', $\Lambda_\pm$ are the complex conjugates, and hence, $|\Lambda_-| = |\Lambda_+|$. Figure (b) shows the section of Fig. (a) at $b = 0.1, 1$, and 10.

FIG. 2. Smooth magnetic field coupled with jittery flow ($a = 15, b = 12$). Analytical solution is obtained in a cylindrical geometry [1]. r, θ and z are the cylindrical coordinates. In the graph of the flow, we plot $10 \times V$.

FIG. 3. Jittery magnetic field coupled with smooth flow ($a = 0.1, b = 0.2$). In the graph of the magnetic field, we plot $10 \times B$.