Incomplete devil's staircase in a model of magnetoconvection

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Abstract

A fifth-order autonomous system of magnetoconvection is numerically investigated in the context of a transition from two-torus to three-torus through the third Hopf bifurcation. By use of a return-map constructed from the Poincaré section for the orbits of two-torus, it is first shown that a phase-locking series of winding numbers creates an incomplete devil's staircase as the magnetic Prandtl number is varied in a certain parameter region. A relation between hierarchies of a set of winding numbers is leading to some scaling-laws and to a final stable two-torus; an accumulated rational winding number, which suggests strongly the existence of some topological invariants through the complicated bifurcations under the parameter variation.

Keywords: Devil's staircase, Two-torus, Phase-locking, Winding number
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I. INTRODUCTION

Natural phenomenon is a nonlinear open system and exhibits amazing complex behaviors. Any viscous or thermal dissipation is always occurring in any real hydrodynamic system instead of ideal Hamiltonian systems [1]. Ever since Lorenz [2] found *strange attractor* by his dissipative truncated model designed to represent the Rayleigh-Bénard convection (the forced dissipative hydrodynamic system) [3-5], various routes to chaos have been investigated in dissipative systems. In order to generate turbulence, Landau [6] proposed an infinite number of Hopf bifurcations. After Lorenz had shown that only three ordinary nonlinear differential equations can lead to a complex and chaotic behavior, Landau's scenario to turbulence was critically scrutinized. Contrary to this Landau's scenario, Ruelle and Takens [7] have shown that certain perturbations are capable of destroying a three-torus (T^3) after three successive Hopf bifurcations and transforming T^3 into a strange attractor. One of the most common and experimentally well studied routes to chaos is the two-frequency mode-locking route [8-13]. Nonlinear interaction of pairs of frequencies is of *profound* theoretical interest due to the generality of its synchronizing phenomenon, which is related to the Rayleigh-Bénard convection, the rotating fluid, the Belousov-Zhabotinsky reaction [14], and the biological systems [15-17]. A fifth-order system of the two-dimensional magnetoconvection [18-20] is designed to describe nonlinear coupling between the Rayleigh-Bénard convection and an external magnetic field. This type of system was first presented by Veronis [21] in studying a rotating fluid. The fifth-order system and the Lorenz model are analogous in conception. Namely, this fifth-order system of magnetoconvection is a straightforward extension of the Lorenz model for the Boussinesq convection interacting with the magnetic field. Nonlinear interaction between plasma convection and magnetic field may explain some magnetohydrodynamic (MHD) features in the solar convection zone and the magnetically confined high-temperature plasma [22]. The fifth-order system has exhibited the typical bifurca-

tions [23,24]; the period-doubling bifurcation [19], the intermittent transition to chaos, and the codimension-two bifurcation [25] by Takens and Bogdanov. It should be noted that the Lorenz model cannot have a quasiperiodic solution since the divergence of the flow is everywhere negative and constant. Excepting the circle-map [8-13] and the double-scroll circuit [26], very little is known about the properties of the transition from two-torus to chaos or three-torus in a realistic autonomous system of ordinary differential equations [27-30]. Although transition from two-torus to chaos has been intensively studied only by use of the circle map, as well as by use of the other discrete maps [12], we are still far from a complete understanding of its transition to chaos through the collapse of three-torus in an autonomous dissipative system. Therefore, it is of great importance to investigate a global structure of rational winding numbers of two-torus. In the present Letter, a global structure of rational winding numbers of two-torus is numerically investigated. In the fifth-order system under the parameter variation, we observe that the rational winding numbers construct a set of hierarchy of the Farey series [31]: an incomplete devil's staircase of the winding numbers versus the magnetic Prandtl number. We find a relation between the generations of the hierarchies of a set of winding numbers when parameters are changed. We also show some scaling-laws near the accumulation points of the winding numbers in a set of the Farey series and the last stable phase-locked two-torus, which suggests strongly the existence of some topological invariants through the complicated bifurcations under the parameter variation.

II. A FIFTH-ORDER MODEL

According to Ref. [23], the fifth-order autonomous system of magnetoconvection is given as follows;

$$\dot{a} = \sigma \left[-a + rb - qd \left(1 + \frac{w(3-w)}{\zeta^2(4-w)} e \right) \right], \tag{1}$$

$$\dot{b} = -b + a - ac, \tag{2}$$

$$\dot{c} = w(-c + ab), \tag{3}$$

$$\dot{d} = -\zeta(d - a) - \frac{w}{\zeta(4 - w)} ae, \tag{4}$$

$$\dot{e} = -\zeta(4 - w)(e - ad), \tag{5}$$

where dot denotes differentiation with respect to the characteristic time t, $a(t)$ represents the first order velocity perturbation, while $b(t)$, $c(t)$ and $d(t)$, $e(t)$ are measures of the first and second order perturbations to the temperature and to the magnetic flux function, respectively. In the fifth-order system, there are five fundamental parameters; ζ, σ, r, q, and w, where ζ is the magnetic Prandtl number(the ratio of the magnetic to the thermal diffusivity), σ is the Prandtl number, r is a normalized Rayleigh number related to the free energy source (destabilizing effect), q is a normalized Chandrasekhar number related to the magnitude of the imposed magnetic field (stabilizing effect), and w is a geometrical parameter related to the aspect-ratio ($0 < w < 4$). In the case of $q = 0$, the fifth-order system (1)–(5) can be transformed to the Lorenz system [2]. The divergence of the flow in phase space can be calculated from Eqs. (1)–(5),

$$\frac{\partial}{\partial a}\dot{a} + \frac{\partial}{\partial b}\dot{b} + \frac{\partial}{\partial c}\dot{c} + \frac{\partial}{\partial d}\dot{d} + \frac{\partial}{\partial e}\dot{e} = -[1 + \sigma + w + \zeta(5 - w)], \tag{6}$$

is always negative since $0 < w < 4$. So trajectories are attracted to a set of measure zero in the phase space; they may be attracted to a fixed point(corresponding to a steady convection), a limit cycle and a strange attractor [2].

III. NUMERICAL RESULTS

It is difficult to find analytical solutions for the fifth-order system of nonlinear ordinary differential equations (1)–(5) in the five-dimensional parameter space. Therefore a numerical integration enables us to get numerical solutions in certain parameter regions. Here we use the fourth-order Runge-Kutta scheme with appropriately chosen time difference between

steps $\Delta t = 0.1$ or 0.01, and carry out numerical integrations with appropriately chosen initial condition: $a(0) = \pm 0.1$, $b(0) = c(0) = d(0) = 0.0$ [23]. Numerical results are not so sensitive to the choice of a set of the initial conditions in the framework of a transition from two-torus to chaos. If two Lyapunov exponents [32] equal zero within the numerical accuracy of 10^{-4}, we call here an attractor a two-torus. The choice of parameter region in the five-dimensional parameter space is crucial to the numerical integrations. In order to investigate a global structure of rational winding numbers of two-torus, except for the magnetic Prandtl number ζ, four other parameters are fixed: $\sigma = 1.0$, $w = 0.1081$, $q = 5.0$, and $r = 14.47$ [23]. The magnetic Prandtl number is chosen as the control parameter, leading to a qualitative or a quantitative change of the system (1)–(5).

First of all, let us demonstrate that the fifth-order system for $\zeta \gtrsim \zeta_T (= 0.09683)$ can be organized to a two-torus via the second Hopf bifurcation from the limit cycle. If a map on the Poincaré section of the successive maxima of $a(t)$ is transformed from a fixed point into a closed curve when ζ is changed, it is concluded that the limit cycle transits to a two-torus through the second Hopf bifurcation. We call its map the Poincaré map ($a_{n+1} = f(a_n)$). The Poincaré sections of Fig. 1 clearly show that the trajectories of the fifth-order system lie on the surface of a two-torus since they draw a closed curve. Figure 1(b) allows us to recognize that it is organized to a two-torus through the second Hopf bifurcation from the limit cycle of Fig. 1(a), since $\zeta = 0.097 > \zeta_T$, and Fig. 1(c)–(f) show the change of two-torus for $0.0987 \lesssim \zeta \lesssim 0.10195$. A return-map can be constructed from the data of the Poincaré section for a two-torus, which is defined by the function $\theta_{n+1} = F(\theta_n)$ (mod 1), where θ_n is the angle of the nth point measured with respect to a polar coordinate system whose origin is inside the closed loop formed by the Poincaré section, and θ_n is also normalized by 2π. Such a constructed return-map is similar to the circle-map [8-13]. For a case of irrational winding number, as is shown in Fig. 2(a), a return-map which governs the map of the invariant closed curve can be constructed from the Poincaré section in Fig. 1(c). In the case of a phase-locked

two-torus, by use of the return-map, we can define the winding number by

$$W_n = \frac{P_n}{Q_n}, \tag{7}$$

where P_n is the number of the times when the return-map $F(\theta_n)$ exceeds one in the Q_n periodic orbit. Namely, P_n and Q_n are relatively prime integers with $P_n \leq Q_n$, the trajectory is closed after Q_n orbits and the motion of system is periodic. The winding number represents the ratio of the two frequencies of the two-torus. The rational winding number of Fig. 2(b) can be constructed from the Poincaré section in Fig. 1(d): $W_2 = 19/27$. If the winding number is irrational for any ζ, it can be also defined by

$$W = \lim_{k\to\infty} \frac{F^k(\theta) - \theta}{k}. \tag{8}$$

An irrational winding number means the average revolution per iteration in a return-map for any ζ.

Next, let us show a set of phase-locking series of winding numbers as ζ is monotonically increased. For $\zeta_T \lesssim \zeta \lesssim 0.09782$, we obtain a main series of winding numbers as follows,

$$W_n^{(1)} = \left\{\frac{30}{43}, \frac{37}{53}, \frac{44}{63}, \frac{51}{73}, \ldots\right\} = \left\{\frac{7n+23}{10n+33}\right\}, \tag{9}$$

where the character (1) of winding number represents the first generation between the hierarchies of the Farey series [31] and the natural number n denotes the order in a set of the Farey series. The branches created by the median rule ($\oplus$) of the Farey arithmetic are excluded from the main Farey series $W_n^{(1)}$. For example, a branch $W(= 95/136)$ for $\zeta \sim 0.09735$ is created by $W_3^{(1)}(= 44/63)$ for $0.09727 \lesssim \zeta \lesssim 0.09730$ and $W_4^{(1)}(= 51/73)$ for $0.09739 \lesssim \zeta \lesssim 0.09743$; $W = W_3^{(1)} \oplus W_4^{(1)}$. This means that the winding number is a highly nonlinear, monotonically increasing, continuous but not differentiable function of ζ, and that the choice of the generation and order of the Farey series is generally arbitrary. Hereafter we regard ζ as defined by $\zeta_n^{(j)}$, where (j) denotes the (j)th generation which is corresponding to

$W_n^{(j)}$. Similarly, for $0.09782 \lesssim \zeta \lesssim 0.102$, we also obtain

$$
\begin{aligned}
W_n^{(2)} &= \left\{\frac{7}{10}, \frac{19}{27}, \frac{31}{44}, \cdots\right\} = \left\{\frac{12n-5}{17n-7}\right\}, \\
W_n^{(3)} &= \left\{\frac{12}{17}, \frac{41}{58}, \frac{70}{99}, \cdots\right\} = \left\{\frac{29n-17}{41n-24}\right\}, \\
W_n^{(4)} &= \left\{\frac{29}{41}, \frac{75}{106}, \frac{121}{171}, \cdots\right\} = \left\{\frac{46n-17}{65n-24}\right\}, \\
W_n^{(5)} &= \left\{\frac{46}{65}, \frac{109}{154}, \frac{172}{243}, \cdots\right\} = \left\{\frac{63n-17}{89n-24}\right\}, \\
W_n^{(6)} &= \left\{\frac{63}{89}, \frac{80}{113}, \frac{97}{137}, \cdots\right\} = \left\{\frac{17n+46}{24n+65}\right\}, \\
W_n^{(7)} &= \left\{\frac{17}{24}, \frac{73}{103}, \frac{129}{182}, \cdots\right\} = \left\{\frac{56n-39}{79n-55}\right\}, \\
W_n^{(8)} &= \left\{\frac{56}{79}, \frac{95}{134}, \frac{134}{189}, \cdots\right\} = \left\{\frac{39n+17}{55n+24}\right\}, \\
W_n^{(9)} &= \left\{\frac{39}{55}, \frac{61}{86}, \frac{83}{117}, \cdots\right\} = \left\{\frac{22n+17}{31n+24}\right\}, \\
&\cdots, \cdots, \cdots.
\end{aligned}
\tag{10}
$$

It should be noted that a series of the winding numbers $W_n^{(j)}$ for the fifth-order system is *not* the Fibonacci's sequence, which has been extensively studied by the circle-map [8-13]. From Eqs. (9) and (10), we find a relation between the (j)th generation and the $(j+1)$th generation of winding numbers;

$$W_1^{(j+1)} = \lim_{n\to\infty} W_n^{(j)}. \tag{11}$$

Equation (11) suggests that the hierarchy of the winding numbers has a Cantor structure [31]. In a transition from two-torus to three-torus, a winding number for the last stable phase-locked two-torus (W_∞) can be defined in terms of the limitation of the rational winding number $W_n^{(j)}$;

$$W_\infty = \lim_{j\to\infty} \lim_{n\to\infty} W_n^{(j)}, \tag{12}$$

where $W_\infty \sim 5/7$ for $\zeta \geq 0.1026$ and the return-map is not invertible any longer. This rational winding number of the last phase-locked two-torus is significantly different from

that of the KAM-torus of the Hamiltonian systems [8-13]. Equations (9)–(11) generate an incomplete devil's staircase [33-35] of the phase-locked winding numbers ($30/43 \leq W_n^{(j)} \leq W_\infty$) since the winding number is a monotonically increasing continuous but not differentiable function of ζ. We also find a scaling property of the difference between $\zeta_\infty^{(j)}$ and $\zeta_n^{(j)}$ for the fixed (j), which is corresponding to $W_n^{(j)}$. As is shown in Fig. 4, if we introduce $\triangle_n^{(j)} \equiv (\min(\zeta_\infty^{(j)}) - \zeta_n^{(j)}) \times 10^4$, we obtain the scaling property for each generation (j): $\triangle_n^{(1)} \propto n^{-1}$, $\triangle_n^{(2)} \propto n^{-2}$, and $\triangle_n^{(9)} \propto n^{-3}$, where $\min(\zeta_\infty^{(1)}) = 0.098$, $\min(\zeta_\infty^{(2)}) = 0.1005$, and $\min(\zeta_\infty^{(9)}) = 0.10187$, respectively. A scaling property of $\triangle_n^{(1)} \propto n^{-1}$ and $\triangle_n^{(2)} \propto n^{-2}$ can be explained in connection with the theory of intermittency by Pomeau and Manneville [36], which was first presented by Kaneko [10]. However, an explanation for $\triangle_n^{(9)} \propto n^{-3}$ is not made yet.

IV. CONCLUDING REMARKS

A transition from two-torus to chaos has been qualitatively explained by the stretching and folding process of the wrinkles of torus, since it is difficult to give a quantitative explanation for the collapse of two-torus. However, it has been shown that the return-map of two-torus captures all the essential physics of the dissipative fifth-order system in the context of a transition from two-torus to three-torus. In other words, we have revealed that the evolution of two-torus up to the final stable two-torus, W_∞, is completely determined by a set of the phase-locked winding number $W_n^{(j)}$ with the generation (j) and the order n in the set of the Farey series as the magnetic Prandtl number is increased. We have presented the first evidence of generating an incomplete devil's staircase and a new type of scaling laws ($\triangle_n^{(j)} \propto n^{-\alpha}$ with $1 \lesssim \alpha \lesssim 3$).

The existence of a final stable phase-locked two-torus may be robustly tied to the occurrence of a three-torus through the third Hopf bifurcation and strongly related to that of some topological invariants in the fifth-order system of magnetoconvection. Details will be

published later.

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FIGURE CAPTIONS

FIG. 1. The Poincaré sections of the successive maxima of $a(t)$ for $\zeta = 0.096(a)$, $\zeta = 0.097(b)$, $\zeta = 0.0995(c)$, $\zeta = 0.0997(d)$, $\zeta = 0.101(e)$, and $\zeta = 0.1015(f)$, respectively.

FIG. 2. (a) A return-map constructed from the Poincaré section in Fig. 1(c), where the origin of the coordinate system is as indicated in Fig. 1(c) for $\zeta = 0.0995$. (b) A return-map constructed from the Poincaré section in Fig. 1(d) for $\zeta = 0.0997$. The phase-locked winding number is defined by $W_2^{(2)} = 19/27$ in this case.

FIG. 3. An incomplete devil's staircase of the winding numbers($30/43 \leq W_n^{(j)} \leq 5/7$) versus the magnetic Prandtl number (ζ) in the fifth-order autonomous system.

FIG. 4. Scaling-laws versus the order n of the magnetic Prandtl number $\zeta_n^{(j)}$ for the fixed generation j. By use of a definition $\triangle_n^{(j)} \equiv (\min(\zeta_\infty^{(j)}) - \zeta_n^{(j)}) \times 10^4$, we can observe scaling laws: $\triangle_n^{(1)} \propto n^{-1}$, $\triangle_n^{(2)} \propto n^{-2}$, and $\triangle_n^{(9)} \propto n^{-3}$, respectively.