Alfvén–Wave Particle Interaction in Finite-Dimensional Self-Consistent Field Model

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Abstract

A low–dimensional Hamiltonian model is derived for the acceleration of ions in finite amplitude Alfvén waves in a finite pressure plasma sheet. The reduced low–dimensional wave–particle Hamiltonian is useful for describing the reaction of the accelerated ions on the wave amplitudes and phases through the self–consistent fields within the envelope approximation. As an example, we show for a single Alfvén wave in the central plasma sheet of the Earth's geotail, modeled by the linear pinch geometry called the Harris sheet, the time variation of the wave amplitude during the acceleration of fast protons.

1 Introduction

The acceleration of particles by Alfvén waves is important because the phenomenon is ubiquitous in space, astrophysical, and laboratory plasmas. In astrophysical plasmas the acceleration of ions due to continuous Kolmogorov–like Alfvén wave spectra is a source of energetic ions. See Arons *et al.* [1] for a wide range of acceleration processes associated with magnetohydrodynamic waves.

The solar wind contains a rich spectrum of both shear and compressional Alfvén waves which accelerate ions. In magnetospheres, the dense magnetotail plasma trapped by the current sheet produced by the solar wind acting on planets with strong dipolar magnetic fields contains a spectrum of Alfvén waves that is thought to play a role in energizing ions.

The usual method of infinite uniform plasma theory is to make the quasilinear approximation and use a Fokker–Planck wave–kinetic equation to describe the particle scattering by the wave spectrum [2, 3]. This method ensures the conservation of momentum and energy of the system. In many cases of practical interest, however, the waves are stronger and the correlations between the waves and particles are relatively coherent so that the quasilinear theory is not applicable. Even in cases where the system is tailored to satisfy the assumption of quasilinear theory, important wave–particle correlations arise in the long time limit [4].

Here we develop a new, low–dimensional model, based on the field Lagrangian for the entire wave–particle system that describes the interaction. In developing the theory we employ an averaging procedure [5] to reduce the field Lagrangian to a finite–dimensional one. The resulting Hamiltonian for the closed system of M waves and N particles conserves the total energy and momentum. Simple functions of the amplitude and phase of each wave are canonically conjugate variables in this formalism.

We develop the theory for the case of the plasma sheet where the pinch effect traps the plasma in a particularly simple configuration called the Harris sheet [6]. The complex orbits of charged particles in the absence of the waves are well–known in terms of elliptic functions. The phase space is divided by a separatrix with periodic orbits on one side and non–crossing orbits on the other side. Alfvén waves mix the orbits producing chaotic motion [7]. Here we do not dwell on the chaos in the orbits, but present results for the integration of a small ensemble of ions integrated over many wave periods. We show explicitly the conservation of energy and momentum for the example.

Consider a high plasma pressure sheet pinch in which the current $j_y(z)$ is localized to the scale $|z| \leq L$ and the plasma pressure is trapped by the reversed magnetic field $B_x(z)$ created by $j_y(z)$. The primary example of this sheet pinch is the Harris sheet. We show how to derive a reduced low–dimensional Hamiltonian description of the Alfvén wave–particle interactions by using the standard techniques of the envelope approximation applied to the total Lagrangian for the system of field and particles.

The Alfvén waves are represented by the eikonal approximation with their slowly–varying amplitudes a_ℓ and phases θ_ℓ. The time and space (x only) frequencies of the waves are denoted by ω_ℓ and k_ℓ respectively. The field Hamiltonian transforms to an N–particle Hamiltonian with M new canonically conjugate pairs of wave variables $(I_\ell, \psi_\ell)_{\ell=1}^{M}$, where $I_\ell(a_\ell)$ is the wave action and $\psi_\ell := \omega_\ell\, t + \theta_\ell$. The total energy and x–momentum $P_x = \sum_{\ell=1}^{M} k_\ell\, I_\ell + \sum_{i=1}^{N} p_{xi}$ of the M waves and N particles is conserved in this framework. Thus the system describes the acceleration of the particles with the corresponding reaction on the wave energy and momentum. In the terminology of laser–plasma interaction physics the beam loading effect is fully accounted for in the $M + 3\,N$ degree–of–freedom Hamiltonian.

In space physics the acceleration of ions by Alfvén waves is important in many contexts. The particular situation of interest in this work is the generation of Alfvén waves by currents connecting the plasma sheet to the nightside ionosphere. Here the waves mediate a coupling between the near–Earth magnetotail and the ionosphere. The interaction with fast (keV energy) hydrogen and oxygen ions is important for understanding the energy spectra observed in the region of the magnetosphere. Here we restrict ourselves to the theoretical formulation of such problems.

2 Two–Dimensional Particle–Field Problem

The self–consistent field equations can be derived from the variation of the electromagnetic field action S which is the space–time integral of the Lagrangian density $\mathcal{L}$. The Lagrangian density for the particle–field model is

$$\mathcal{L}(\mathbf{A}, \partial\mathbf{A}, \mathbf{x}_i, \dot{\mathbf{x}}_i) := \frac{\epsilon_0}{2}\left(\frac{\partial \mathbf{A}}{\partial t}\right)^2 - \frac{1}{2\mu_0}(\nabla \times \mathbf{A})^2 + p(\mathbf{A})$$
$$+ \sum_{i=1}^{N} e\,\dot{\mathbf{x}}_i \cdot \mathbf{A}(\mathbf{x}, t)\,\delta\left(\mathbf{x} - \mathbf{x}_i(t)\right) + \sum_{i=1}^{N} \frac{1}{2} m\left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)\,\delta\left(\mathbf{x} - \mathbf{x}_i(t)\right), \quad (1)$$

where $\mathbf{A}$ is the vector potential, N is the number of particles, and $\mathbf{x}_i(t)$ is the trajectory of the ith particle. When the current $\mathbf{j}$ is specified independently, the potential $p(\mathbf{A})$ is replaced by $\mathbf{j} \cdot \mathbf{A}$ [8]. The action is

$$S[\mathbf{A}, \mathbf{x}_i] := \int dt \int d^3x\,\mathcal{L}\left(\mathbf{A}, \partial\mathbf{A}, \mathbf{x}_i, \dot{\mathbf{x}}_i\right), \quad (2)$$

and it is assumed that there are no external sources present so that the electric field can be given by

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \quad (3)$$

It may be verified that the equations of motion obtained upon setting $\delta S/\delta\mathbf{A}$ and $\delta S/\delta\mathbf{x}_i$ to zero are consistent with Maxwell's equations for the particle–field problem, where the current $\mathbf{j}$ is identified with $\partial p/\partial\mathbf{A}$.

For the problem at hand, only the y component of the vector potential is non–zero: $\mathbf{A} = (0, A(x, z, t), 0)$. The corresponding magnetic field is $(-\partial A/\partial z, 0, \partial A/\partial x)$. Representation of the potential as a sum of an equilibrium piece $A_0(z)$ and a piece that depends on (x, z, t),

$$A(x, z, t) = A_0(z) + A_1(x, z, t), \tag{4}$$

yields an expansion for the Lagrangian density,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2, \tag{5}$$

where

$$\mathcal{L}_0 := p(A_0) - \frac{1}{2\mu_0}\left(\frac{dA_0}{dz}\right)^2, \tag{6}$$

$$\mathcal{L}_1 := j_0\, A_1 - \frac{1}{\mu_0}\frac{dA_0}{dz}\frac{dA_1}{dz}, \tag{7}$$

$$\begin{aligned}\mathcal{L}_2 := {} & \frac{1}{2}\left(\epsilon_0 + \frac{m_0\, n_0}{B_0^2}\right)\left(\frac{dA_1}{dt}\right)^2 - \frac{1}{2\mu_0}\left[\left(\frac{\partial A_1}{\partial x}\right)^2 + \left(\frac{\partial A_1}{\partial z}\right)^2\right] + \frac{1}{2}\frac{dj_0}{dA_0}A_1^2 \\ & + \sum_{i=1}^{N}\left\{e\,\dot{y}_i\left[A_0(z) + A_1(x, z, t)\right] + \frac{1}{2}m\left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)\right\}\delta\left(\mathbf{x} - \mathbf{x}_i(t)\right).\end{aligned} \tag{8}$$

In the above equations we have denoted the equilibrium current dp/dA_0 by j_0, and neglected terms of order A_1^3 and higher. Note that the contribution to the action from $\mathcal{L}_1$ vanishes when A_0 is an equilibrium corresponding to $\mathcal{L}_0$. In $\mathcal{L}_2$, we have also included a term that gives rise to the polarization current $-(m_0\, n_0/B_0^2)\,\partial^2 A_1/\partial t^2$, where m_0, n_0, B_0 represent the background ion mass, density, and magnetic field, resepectively.

The thermal plasma response in Eq. (8) arises from the adiabatic response given by the fluid response of plasma, namely

$$j_1 = \frac{mn}{B^2}\frac{\partial E^{(1)}}{\partial t} = -\frac{mn}{B^2}\frac{\partial^2 A_1}{\partial t^2} \tag{9}$$

$$j_2 = \frac{dj_0}{dA_0}\, A_1. \tag{10}$$

Now to reduce the problem to a finite number of degrees of freedom we represent A_1 by

M waves with slowly varying amplitudes a_ℓ and phases θ_ℓ:

$$A_1(x,z,t) = \sum_{\ell=1}^{M} a_\ell(t) f_\ell(z) \cos\left(k_\ell x - \omega_\ell t - \theta_\ell\right) \tag{11}$$

where $\dot{a}_\ell \ll \omega_\ell\, a_\ell$ and $\dot{\theta}_\ell \ll \omega_\ell$. This is the eikonal approximation [5]. In this presentation of $A_1(\mathbf{x},t)$ the a_ℓ, θ_ℓ become new dynamical variables determined by the variational principle for the action S of the system.

Now

$$\frac{\partial A_1}{\partial t} = \sum_{\ell=1}^{M} \dot{A}_\ell\, f_\ell\, \cos\psi_\ell + A_\ell\, f_\ell\, (\omega_\ell + \dot{\theta}_\ell) \sin\psi_\ell. \tag{12}$$

$$\left(\frac{\partial A}{\partial t}\right)^2 = \dot{A}^2 f^2 \cos^2\psi - 2A\dot{A}f^2(-\omega - \dot{\theta}) \cos\psi \sin\psi + A^2 f^2 (\omega + \dot{\theta})^2 \sin^2\psi \tag{13}$$

Now, we compute

$$\int dxdz \left(\frac{\partial A}{\partial t}\right)^2 = \frac{1}{2} A^2 \left(\omega^2 + 2\omega\dot{\theta} + \dot{\theta}^2\right) \int f^2 dz + \frac{\dot{A}^2}{2} \int f^2 dz \tag{14}$$

$$\int dxdz \left[\left(\frac{\partial A_y}{\partial x}\right)^2 + \left(\frac{\partial A_y}{\partial z}\right)^2\right] = \frac{k^2 A^2}{2} \int f^2\, dz + \frac{A^2}{2} \int \left(\frac{df}{dz}\right)^2 dz$$

$$\int dxdz\, \frac{dj_0}{dA}\, A^2\, f^2 \cos^2\psi = \frac{1}{2}\, A^2 \int f^2\, \frac{\partial j_0}{\partial A}\, dz. \tag{15}$$

Upon averaging over x, and using the eikonal approximation, the action may be written as

$$S = \int L_{\rm env}\, dt, \tag{16}$$

where the envelope Lagrangian is given by

$$L_{\rm env} = \frac{1}{4}\sum_{\ell=1}^{M} a_\ell^2 \Big[(\omega_\ell^2 + 2\omega_\ell \dot\theta_\ell) \int f_\ell^2 \left(\frac{mn}{B^2} + \epsilon_0\right) dz - \frac{k_\ell^2}{\mu_0} \int f_\ell^2 dz$$

$$- \frac{1}{\mu_0} \int \left(\frac{df_\ell}{dz}\right)^2 dz + \int f_\ell^2 \frac{dj_0}{dA_0} dz \Big] L_x L_y$$

$$+ \sum_{i=1}^{N} e\dot y_i \Big[A_0(z_i) + \sum_{\ell=1}^{M} a_\ell(t) f_\ell(z_i) \cos(k_\ell x_i - \omega_\ell t - \theta_\ell) \Big] + \sum_{i=1}^{N} \frac{1}{2} m(\dot x_i^2 + \dot y_i^2 + \dot z_i^2), \quad (17)$$

where L_x and L_y are the x and y spatial extents of the system.

The dominant term from the variation with respect to a_ℓ is required to vanish, which is true when each wave satisfies the dispersion relation:

$$\omega_\ell^2 \int f_\ell^2 \left(\frac{mn}{B^2} + \epsilon_0\right) dz - \frac{k_\ell^2}{\mu_0} \int f_\ell^2 dz - \frac{1}{\mu_0} \int \left(\frac{df_\ell}{dz}\right)^2 dz + \int f_\ell^2 \frac{dj_0}{dA_0} dz = 0. \quad (18)$$

We are thus left with the Lagrangian,

$$L = \sum_{1}^{N} \frac{m}{2} \left(\dot x_i^2 + \dot y_i^2 + \dot z_i^2\right) + \sum_{i=1}^{N} e\, \dot y_i \Big[A_0(z_i) + \sum_{\ell=1}^{M} a_\ell(t)\, f_\ell(z_i)\, \cos(k_\ell x_i - \omega_\ell t - \theta_\ell) \Big]$$

$$+ \frac{1}{2} a_\ell^2\, \omega_\ell\, \dot\theta_\ell\, L_x\, L_y \int f_\ell^2 \left(\frac{mn}{B^2} + \epsilon_0\right) dz. \quad (19)$$

Finally, we note that there is a narrow boundary surrounding the $B_x = 0$ reversal layer inside of which the polarization current formula (9) breaks down. The calculation of the kinetic current in this layer requires numerical evaluation of the complex ion orbits [9]. Here we resolve this difficulty by the physical argument that there is effectively a lower bound on $|B_{\rm min}|$ due to other sources of magnetic fields, such as the interplanetary magnetic field. This allows us to assume that the polarization current is defined through the kinetic boundary layer.

2.1 Canonical momenta

The generalized momenta $p = \partial L/\partial\dot{q}$ are given by

$$p_{yi} := m\dot{y}_i + e\,A_0(z_i) + e\sum_{\ell=1}^{M} a_\ell(t) f(z_i)\cos(k_\ell x_i - \omega_\ell t - \theta_\ell),$$

$$p_{xi} := m\dot{x}_i, \qquad p_{zi} := m\dot{z}_i, \qquad p_{a\ell} := 0,$$

$$p_{\theta\ell} := \frac{1}{2}\,a_\ell^2\,\omega_\ell\,L_x\,L_y C_\ell =: I_\ell, \tag{20}$$

where we have defined the wave action $I_\ell = p_{\theta\ell}$ and the capacitance,

$$C_\ell := \left\langle \frac{m_0\,n_0}{B_0^2} + \epsilon_0 \right\rangle_\ell := \int \left(\frac{m_0\,n_0}{B_0^2} + \epsilon_0 \right) f_\ell^2 dz. \tag{21}$$

produced by the polarization currents.

The Euler–Lagrange equations of motion corresponding to variations in θ_ℓ and a_ℓ are

$$\frac{dp_{\theta\ell}}{dt} = \frac{d}{dt}\left[\frac{1}{2}a_\ell^2\omega_\ell\Big\langle\frac{mn}{B^2} + \epsilon_0\Big\rangle_\ell\right] = \frac{\partial L}{\partial\theta_\ell} = e\,a_\ell\sum_i \dot{y}_i f_\ell(z_i)\sin(k_\ell x_i - \omega_\ell t - \theta_\ell)$$

$$\frac{dp_{a\ell}}{dt} = 0 = \frac{\partial L}{\partial a_\ell} = a_\ell\omega_\ell\dot{\theta}_\ell\Big\langle\frac{mn}{B^2} + \epsilon_0\Big\rangle_\ell + e\sum_i^N \dot{y}_i f_\ell(z_i)\cos(k_\ell x_i - \omega_\ell t - \theta_\ell). \tag{22}$$

2.2 Hamiltonian equations of motion

Introducing $\psi_\ell = \theta_\ell + \omega_\ell t$ as the phase variables, the autonomous Hamiltonian may be written as

$$H(I_\ell, \psi_\ell, p_{xi}, x_i, p_{yi}, p_{zi}, z_i) = \sum_{\ell=1}^{M} I_\ell\omega_\ell + \sum_{i=1}^{N}\left(\frac{p_{xi}^2}{2m} + \frac{p_{zi}^2}{2m}\right)$$
$$+ \sum_{i=1}^{N}\frac{1}{2m}\left[p_{yi} - e\,A_0(z_i) - \sum_{\ell=1}^{M} e\,a_\ell(I_\ell)f_\ell(z_i)\cos(k_\ell x_i - \psi_\ell)\right]^2 \tag{23}$$

where the relation in Eqs. (20) gives $a_\ell(I_\ell)$, while ω_ℓ, k_ℓ, and $A_0(z_i)$ are considered known.

Equation (23) is obtained by the usual Legendre transformation ($H = \dot{q}\,\partial L/\partial\dot{q} - L$) from the Lagrangian to the Hamiltonian. The terms linear in the generalized velocities, $\dot{\psi}_\ell$ and $\dot{I}_\ell$, drop out of the Hamiltonian. At first glance, it may seem necessary to use (Dirac's) constraint theory due to this degeneracy, but it is not required in this case. The phase variables ψ_ℓ and I_ℓ turn out to be canonically conjugate to each other and the momenta corresponding to each of them can be ignored. The equations of motion are thus given by

$$\dot{x}_i = \frac{\partial H}{\partial p_{xi}} = \frac{p_{xi}}{m} \qquad \dot{p}_{xi} = -\frac{\partial H}{\partial x_i} = -\sum_{\ell=1}^{M} e\,\dot{y}_i\,a_\ell(I_\ell)\,f_\ell\,k_\ell\,\sin(k_\ell x_i - \psi_\ell), \tag{24}$$

$$\dot{y}_i = \frac{\partial H}{\partial p_{yi}} = \frac{1}{m}\Big(p_{yi} - e\,A_0(z_i) - e\sum_{\ell=1}^{M} a_\ell(I_\ell)\,f_\ell(z_i)\,\cos(k_\ell x_i - \psi_\ell)\Big), \tag{25}$$

$$\dot{p}_{yi} = -\frac{\partial H}{\partial y_i} = 0, \tag{26}$$

$$\dot{z}_i = \frac{\partial H}{\partial p_{zi}} = \frac{p_{zi}}{m}, \qquad \dot{p}_{zi} = -\frac{\partial H}{\partial z_i} = \dot{y}_i\Big(e\,\frac{dA_0}{dz_i} + e\sum_{\ell=1}^{M} a_\ell(I_\ell)\,\frac{df_\ell}{dz_i}\,\cos(k_\ell x_i - \psi_\ell)\Big), \tag{27}$$

$$\dot{\psi}_\ell = \frac{\partial H}{\partial I_\ell} = \omega_\ell - e\,\frac{da_\ell}{dI_\ell}\sum_{i=1}^{N}\dot{y}_i\,f_\ell(z_i)\,\cos(k_\ell x_i - \psi_\ell)\,, \tag{28}$$

$$\dot{I}_\ell = -\frac{\partial H}{\partial \psi_\ell} = e\,a_\ell(I_\ell)\sum_{i=1}^{N}\dot{y}_i\,f_\ell(z_i)\,\sin(k_\ell x_i - \psi_\ell), \tag{29}$$

where it is understood that whenever $\dot{y}_i$ appears on the right–hand side, it is an abbreviation for the expression given by the right–hand side of Eq. (25). It may be verified that the above equations are equivalent to the Euler–Lagrange equations of motion.

The $M + 3\,N$ degrees–of–freedom system has the integrals of total energy H, total x–momentum $P_x := \sum_{\ell=1}^{M} k_\ell I_\ell + \sum_{i=1}^{N} p_{xi}$, and the N canonical y–momenta p_{yi} for each particle.

A test particle in a prescribed wave corresponds to setting $dI_\ell/dt = 0$ and $d\theta_\ell/dt = 0$. The new system of M waves and N particles has many of the same features of the test particle problem when the number of particles N is small. As the energy in the particles increases, however, phase correlations build up as given by the right–hand side of Eqs. (29) and (30) and limit the total energy available for acceleration. These phase correlations also present a simple description of the acceleration process.

¿From the standard calculation of the Alfvén wave energy W_ℓ we verify that the wave action I_ℓ and x–momentum $P_{x\ell}$ can be written as

$$I_\ell = \frac{W_\ell}{\omega_\ell} \quad \text{and} \quad P_{x\ell} = \frac{k_{x\ell} W_\ell}{\omega_\ell} \tag{30}$$

where W_ℓ is the wave energy. The action I_ℓ can thus be interpreted as the number of quanta in the wave packet. This interpretation is useful for making bounds on the maximum energy particles can gain during the decay of the wave.

3 Example of Self–Consistent Wave–Particle Interactions

Here we relate the present general nonlinear formulation given by the Hamiltonian in Eq. (23) to the weak quasilinear theory limit. We present thre numerical examples of the interactions that occur for small and large numbers of particles with one Alfvén wave. We reserve for later work the case of multiple wave particle simulations.

In general, even with a large number of particles, it is important to assign statistical weights to each particle. The weights are chosen to represent samples of N particles taken from the desired distribution function of the particles. Examples are the initial $\mathrm{sech}^2(z/L)\exp(-mv^2/2T)$ distribution, or the typically observed power law distributions $(1+\varepsilon/\varepsilon_0)^{-\gamma}$ that are observed for high energy ($\varepsilon = mv^2/2 \gg T$) particles in space physics. The energy density in the waves localized over a volume $V = (2\pi/kL_yL_z$ is $I_\ell\,\omega_\ell/V$, while the particle energy density is $\sum_1^N w_i\,\varepsilon_i/V$, where $\varepsilon_i = mv_i^2/2$. Both energy densities are small compared to that of the ambient plasma internal energy $p = nT \approx B_0^2/2\mu_0$.

3.1 Quasilinear Limit

The quasilinear limit appears in the case where there is effectively one weighted particle in the interaction with each wave. In this case the solvability conditions from the conservation laws reduces to the $N = 1$ problem for each interaction

$$\omega\, I_i + w\, \varepsilon(p_i) = \omega\, I_f + w\, \varepsilon(p_f),$$

$$k\, I_i + w\, p_i = k\, I_f + w\, p_f. \tag{31}$$

For a small change $\Delta I = I_f - I_i$ in the wave action leads to the quasilinear resonance condition when the expansion $\varepsilon(p_f) = \varepsilon(p_i) + (p_f - p_i)\partial\varepsilon/\partial p_i$ is used along with $v = \partial\varepsilon/\partial p_i$. The reader may easily verify that the condition for a nontrivial solution to the two equations (24) with the two unknowns being the final particle momentum p_f and wave action I_f, is the linear wave–particle resonance condition $v = \partial\varepsilon/\partial p_i = \omega/k$. Thus, in the small ΔI and Δp limit we recover the quasilinear wave–particle interactions.

A special feature of Alfvén wave quasilinear theory is that the polarization factor vanishes at the Landau resonance, i.e. $(E_y - v_x\delta B_z)\pi\delta(\omega - k_x v_x)$ is zero due to $\omega\delta B_z = k_x\delta E_y$ polarization relation. For the complex, nonlinear orbits in the self–consistent field this relation does not apply. The degree of magnetization of particles varies strongly along the orbits given by elliptic functions in the inner sheet region.

The constraints of energy and momentum conservation allow a range of solutions for the case of N particles. Perhaps the most important features may be shown by the case of two weighted particles.

3.2 Accelerated and Recoiled Particles

For strong waves the general situation is that a few particles gain a large amount of energy from the wave, and a large number of particles recoil to absorb the momentum change. For the $N = 2$ problem we write $\Delta I = I_f - I_i$, which we consider as being negative and rewrite Eq. (41) for two particles,

$$w_1 \Big(\varepsilon(p_1 + \Delta p_1) - \varepsilon(p_1)\Big) + w_2 \Big(\varepsilon(p_2 + \Delta p_2) - \varepsilon(p_2)\Big) = -\omega\, \Delta I$$

$$w_1\, \Delta p_1 + w_2\, \Delta p_2 = -k\, \Delta I \tag{32}$$

where the final momenta are written as $p_f = p_i + \Delta p$. There is a one parameter (Δp_1) family of solutions to the two equations with three unknowns Δp_1, Δp_2, ΔI.

The case of most interest is where $w_2 < w_1$ and particle 2 picks up almost all the lost wave energy. In the extreme limit Δp_2 and Δp_1 are determined by

$$\Delta p_2 \approx \left(\frac{2m\omega|\Delta I|}{w_2}\right)^{1/2}$$

$$\Delta p_1 \approx \frac{k|\Delta I|}{w_1} - \frac{w_2}{w_1}\left(\frac{2m\omega|\Delta I|}{w_2}\right)^{1/2}.$$

This is the strong acceleration limit consistent with the conservation laws. To what degree the system dynamics equations produces these two–particle one–wave interactions is a problem beyond the scope of this work.

3.3 Numerical Examples with Small N and Large N

We first show the results for $N = 10^3$ particles and then consider an example with $N = 10^4$ protons. In Figs. 1 and 2 we show a result for $N = 10^3$ with a single wave to make explicit the capability of the theory to describe the reaction of the particle acceleration on the waves. We take the simplified model $f(z) = f_1 \exp(-k\,|z|)$. We take $k = 2\pi/L_z$ and load the particles uniformly spaced in $k\,x_i = 2\,\pi\, i/N$ for $i = 1, ..., N$. We have carried out the numerical experiment for a range of z and initial energy ε values. Here we choose the single, interesting experiment with $z/L = 0.5$. That is, we take ions in the high energy tail of the thermal distribution.

¿From the Alfvén wave dispersion relation we find that $\omega/\omega_{cio} = k\,c/\omega_{pi} = 2\pi\,c/L_z\,\omega_{pi}$. For the example we take $L_z = 20\pi\,c/\omega_{pi}$ so that $\omega/\omega_{cio} = 1/10$. Working out the equations of motion for this case and using an adaptive step Runge–Kutta integrator, we obtain results shown in Fig. 1 for the x–momentum components and Fig. 2 for the energy components. It is found that the momentum conservation is satisfied to a very high accuracy and the total energy is conserved to an error of order the truncation integration. In future study we will examine these particle trajectories in the $d = 4$ phase space.

In Fig. 3 we show the results for choosing the parameters to correspond to a reversed equilibrium field of 20 nT and Alfvén wave in the central plasma sheet. At $t = 0$ we release $N = 10^4$ protons in Fig. 3 at the position $z/L_z = 0.5$ corresponding to the mid–point of current profile. The initial v_x–velocity of the particles is taken as the resonant value equal to the wave phase velocity. Initial ions gain and lose energy as their phase relative to how the wave varies. The net particle energy summed over all particles shows the growth to an oscillatory, saturated state for $N = 10^3$. For $N = 10^4$ we find a more nearly monotonic growth and approach to a well–defined saturated state is obtained. We leave the determination of the parametric dependence of the final particle energy to future studies. When the simulation in Fig. 3 is repeated with a fixed Alfvén wave ($\dot{a} = \dot{\theta} = 0$) the total particle energy increases by approximately 10% more.

4 Conclusions

We have presented a new description of the Alfvén wave–particle interaction problem that provides a Hamiltonian formulation of the interactions of the total system consisting of a finite number of degrees of freedom for the M waves and N particles. The theory uses the eikonal approximation for the reduction of the field Lagrangian field density containing the background thermal plasma and a set of N discrete particles. Consequently, the problem conceptually parallels that of the electron plasma waves interacting with a weak electron beam considered first by Mynick and Kaufman [10]. Due to the more complex form of the Lagrangian for charged particles in a magnetic field and the structure of the Alfvén wave, the mathematical structure of the two problems differs considerably.

We discuss the relations that reduce the problem to the weak quasilinear limit without taking up the issue of long–time wave–particle correlations that may well modify the quasi-linear problem as in the case of the electron plasma wave problem considered by Doxas and Cary [4].

We show a few examples of how the self–consistent field problem can produce a secular increase in the net particle energy while conserving the total energy and momentum of the wave–particle system.

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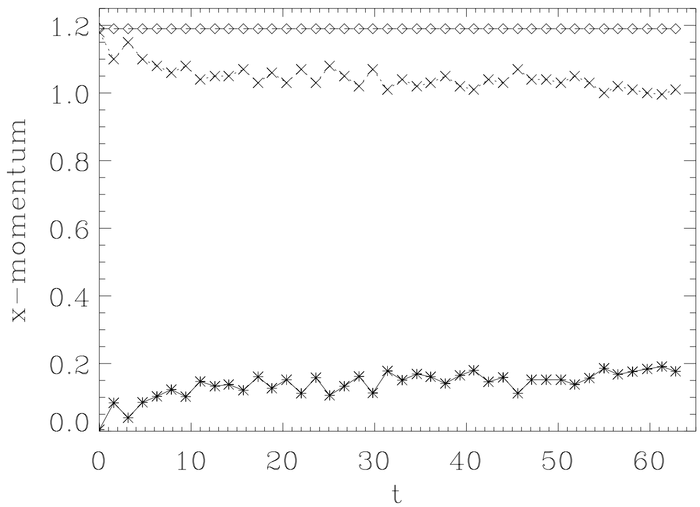

Figure 1: X–momentum transfer from the wave to ions. Solid line shows ion momentum; dotted line shows wave momentum; the sum is preserved.

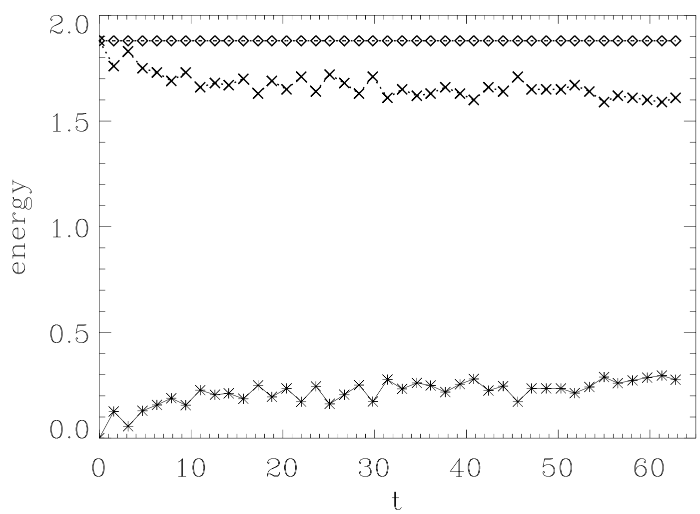

Figure 2: Energy transfer from the wave to ions. Solid line shows ion kinetic energy; dotted line shows wave energy; the sum is preserved.

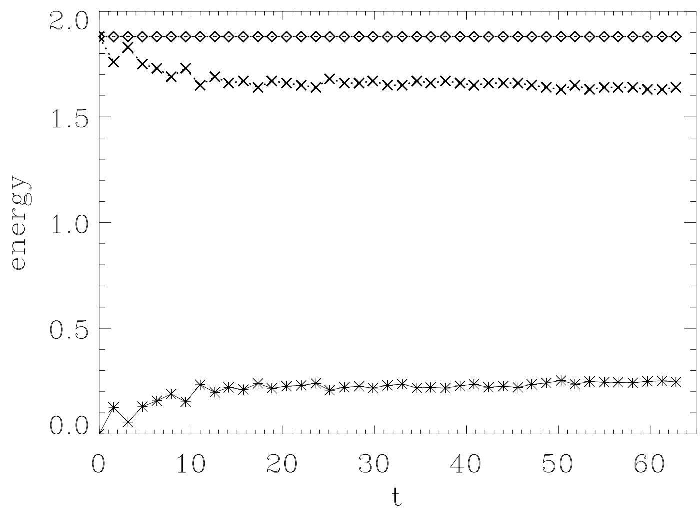

Figure 3: The wave heating for the case of $N = 10^4$ particles uniformly spread over one wavelength at the midway position $z/L = 0.5$ in the current sheet. The parameters for the initial wave amplitude is $\delta B = 1\,\mathrm{nT}$ with the reversed field being $B_0 = 20\,\mathrm{nT}$.