Stability of short wavelength tearing and twisting modes

F. L. Waelbroeck

Institute for Fusion Studies, The University of Texas at Austin

Austin, Texas 78712

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Abstract

The stability and mutual interaction of tearing and twisting modes in a torus is governed by matrices that generalize the well-known Δ' stability index. The diagonal elements of these matrices determine the intrinsic stability of modes that reconnect the magnetic field at a single resonant surface. The off-diagonal elements indicate the strength of the coupling between the different modes. We show how the elements of these matrices can be evaluated, in the limit of short wavelength, from the free energy driving radially extended ballooning modes. We apply our results by calculating the tearing and twisting Δ' for a model high-beta equilibrium with circular flux surfaces.

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1 Introduction

Small-wavelength resistive modes are of interest in connection with edge-localized modes (ELMs) [1], edge transport [2], sawtooth oscillations [3, 4], and beta-limit disruptions [5, 6]. In this paper, we consider the effect of the differential rotation of the plasma on these small-wavelength resistive modes.

In the absence of differential rotation, the most unstable small-wavelength modes are ballooning modes [7, 8, 9]. These modes have the property that the amount of reconnected flux is, to first approximation, equal on neighboring resonant surfaces. In a differentially rotating plasma, however, the mode frequency can only match the rotation frequency at a single resonant surface. At all other resonant surfaces, reconnection is inhibited by the rapid fluctuation of the mode amplitude in the plasma frame [10]. Thus, small-amplitude resistive modes in differentially rotating plasma reconnect the magnetic field at a single resonant surface, although they generally contain many poloidal harmonics.

Previous authors have investigated the stability of small-wavelength resistive modes in differentially rotating plasmas by using large aspect-ratio, low-beta expansions for circular or nearly circular equilibria [11, 12, 13, 14]. In this paper, we show how to obtain the required stability parameters, in the small wavelength limit, in terms of the ballooning solutions. Our results extend the well-known asymptotic expression for the stability index of tearing modes in a cylindrical plasma [15, 16], $\Delta' \sim -2k_\perp$, to the case of toroidal, high-beta plasmas. The analysis also sheds light on the relationship between ballooning and radial or Fourier eigenmodes [17].

We illustrate our method by applying it to the Shafranov equilibrium with locally enhanced pressure gradients and shear [18, 19]. Our calculation complements codes such as PEST-III [20] and T7 [21] that evaluate the scattering matrix for long wavelength modes. The short wavelength Δ', however, has the advantage of requiring only the knowledge of quantities determined locally near the magnetic surface of interest. This makes it possible to consider the effect of modifying the local gradients, such as the pressure or current gradient, without recalculating the entire equilibrium [19].

The paper is organized as follows. We begin in Sec. II by introducing the stability matrix and by describing its role in determining the growth rates of resistive eigenmodes. In Sec. III we consider the limit of small wavelength. We calculate the eigenvalue and eigenvectors of the stability matrix and describe their relationship to ballooning modes. We then show how to invert these relationships to obtain the elements of the stability matrix in terms of the properties of the ballooning solutions. In Sec. IV, we apply our analysis to Shafranov's shifted-circle equilibrium. We conclude and discuss the significance of our results in Sec. V.

2 The stability matrix

The theory of resistive stability divides the plasma into two complementary regions where different physical processes dominate. The first region consists of thin resonant layers surrounding surfaces, called mode-rational surfaces, where the magnetic field lines close upon themselves. Dissipative effects are important in the resonant layers, but the eigenmode enjoys approximate helical symmetry.

The second region consists of all the plasma lying between the resonant layers. Since the growth rate for resistive modes is large compared to the skin time τ_R, $\gamma \gg \tau_R = \eta/a^2$, the nonresonant plasma obeys ideal magnetohydrodynamics (MHD). Since the growth rate is small compared to the Alfvén time τ_A, by contrast, the nonresonant region must also obey force-balance (here $\tau_A = R/V_A$, a is the minor radius, R is the major radius, and $V_A = B/(\mu_0\rho)^{1/2}$ is the Alfvén velocity). We will thus refer to the second region as the magnetostatic region. It is the source of the free energy driving resistive instabilities [16].

Resistive instabilities consist of low-frequency, shear-Alfvén standing waves that transmit the free energy of the equilibrium to the resonant layers. These waves can be viewed as superpositions of incoming and outgoing waves. We will see that shear-Alfvén waves are evanescent in a plasma free of ideal instabilities, so that they do not propagate in the sense of geometric-optics. They do, however, carry information across the magnetic field through diffraction, so we may think of them as propagating waves in a broad sense.

We will thus refer to the wave which decays in the direction of the resonant surface as the incoming wave, and to the wave which decays in the opposite direction as the outgoing wave. These two waves have traditionally been labeled "small solution" and "large solution." The traditional labels reverse roles, however, depending on wether one considers the resonant layer or the magnetostatic region. We will thus avoid using them in this paper.

The dispersion relation for resistive eigenmodes follows by matching the ratios of the amplitudes of the incoming and outgoing waves on either side of the resonant layer to the reflection and transmission coefficients calculated for the layer. Alternatively, we may view the dispersion relation as expressing the matching of the impedances of the layers with those of the magnetostatic region. We will explore both points of view below.

We henceforth restrict consideration to toroidally symmetric equilibria. We may thus label eigenmodes with the toroidal mode number n, and mode-rational surfaces with the poloidal mode number of the resonant Fourier component, $m = nq(r_m)$. Here $q(r)$ is the safety factor and r_m is the radius of the mode-rational surface. We represent the modes alternatively in terms of the electrostatic potential ϕ and the poloidal flux $\psi = -A_\zeta$, where A_ζ is the toroidal component of the vector potential. In the magnetostatic region $\mathbf{b} \cdot \mathbf{E} = 0$ and Faraday's law implies that $-i\omega\psi = \mathbf{b}\mathbf{b} \cdot \boldsymbol{\nabla}\phi$, where $\mathbf{b}$ is the unit vector in the direction of the magnetic field $\mathbf{B}$.

Solution in the resonant layer

The solution in the resonant layer has three properties that inform the matching procedure. The first of these is that the resonant layers are physically separated and thus independent.

Within each layer, the mode is dominated by the Fourier component such that $\mathbf{k} \cdot \mathbf{B}(r_m) = 0$. The layer equations can thus be reduced to one-dimensional equations describing the variation of the resonant harmonic as a function of the distance to the mode-rational surface, $x = r - r_m$.

The second property of the solution in the resonant layer is that the layer equations admit independent even and odd solutions [15, 22]. This follows from their invariance under reflection about the mode-rational surface. The eigenmodes such that ψ is even (ϕ odd) are called tearing-parity modes, while those such that ϕ is even (ψ odd) are called twisting-parity modes. Tearing-parity modes give rise to magnetic islands, while twisting-parity modes interchange neighboring flux tubes.

The asymptotic behavior of the flux away from the mode-rational surface is

$$\psi_m^{te}(x) \sim \widehat{\psi}_{m,out}^{te} \left(\Delta_m^{\mathrm{te}}(\omega)|x|^{1+\nu} + |x|^{-\nu} \right) \tag{1}$$

for the tearing-parity solution and

$$\psi_m^{tw}(x) \sim sign(x) \widehat{\psi}_{m,out}^{tw} \left(\Delta_m^{\mathrm{tw}}(\omega)|x|^{1+\nu} + |x|^{-\nu} \right) \tag{2}$$

for the twisting-parity solution. Here,

$$\nu = -\frac{1}{2} + \sqrt{-D_I}$$

and D_I is the Mercier stability index [23, 24].

For the purpose of matching to the magnetostatic solution, the asymptotic form of the resistive layer solutions is more conveniently expressed by the relation linking the incoming wave amplitude to the outgoing wave amplitude,

$$\widehat{\psi}_{m,\mathrm{in}}^{\mathrm{te}} = \Delta_m^{\mathrm{te}}(\omega) \widehat{\psi}_{m,out}^{te}; \tag{3}$$

$$\widehat{\psi}_{m,\mathrm{in}}^{tw} = \Delta_m^{\mathrm{tw}}(\omega) \widehat{\psi}_{m,out}^{tw}. \tag{4}$$

The two functions $\Delta_m^{\rm te}(\omega)$ and $\Delta_m^{\rm tw}(\omega)$ are the inverses of the reflection coefficients for waves incident on the layer. They contain all the information concerning the layer solutions that enters into the dispersion relation.

At large x the effect of resistivity becomes negligible and $\lim_{|x|\to\infty} E_{||} = 0$. The electrostatic potential ϕ and the flux ψ are thus related asymptotically by $\omega\psi \sim -k_{||}\phi \simeq -(k_\theta x/L_s)\phi$, where $k_\theta = m/r_m$, $L_s = Rq/s$ is the magnetic shear length, and $s = r_m d\log q/dr$ is the magnetic shear. The asymptotic behavior of the electrostatic potential ϕ is thus readily obtained from that of the flux ψ,

$$\phi_m^{\rm te}(x) \sim sign(x)\hat{\phi}_{m,out}^{te}\left(\Delta_m^{\rm te}(\omega)|x|^\nu + |x|^{-\nu-1}\right), \tag{5}$$

$$\phi_m^{tw}(x) \sim \hat{\phi}_{m,out}^{\rm tw}\left(\Delta_m^{\rm tw}(\omega)|x|^\nu + |x|^{-\nu-1}\right). \tag{6}$$

It is instructive to consider the WKB representation of the asymptotic expressions (1),

$$\psi = |k_x|^{-1/2}\left[\widehat{\psi}_{m,\rm in}e^{-i\int k_x\, dx} + \widehat{\psi}_{m,\rm out}e^{i\int k_x\, dx}\right],$$

where we have suppressed the parity label. One readily verifies that

$$k_x = i\,\frac{\sqrt{-D_I}}{x}$$

reproduces (1)-(2). This shows that the fundamental solutions are evanescent in a plasma that satisfies the Mercier criterion, $D_I < 0$, as stated above.

The last and most important property of the solution in the resonant layer is that the matching parameters $\Delta(\omega)$ are very large unless the mode frequency is very close to one of the natural frequencies. In the drift-MHD model, for example, the matching parameters are [13]

$$r_m\Delta_m^\sigma(\omega) = (\omega - \omega_m^\sigma)\tau_\gamma, \tag{7}$$

where σ labels the parity of the mode and τ_γ is the characteristic growth time,

$$\tau_\gamma = \tau_A^{1/3}\tau_V^{-1/6}\tau_R^{5/6}.$$

Here $\tau_V = r_m^2/\mu$ is the viscous time, and μ is the specific viscosity. The natural frequency for tearing modes, in the drift-MHD model, is the electron drift frequency. The natural frequency for twisting modes, by contrast, is the electric drift frequency. The $\Delta(\omega)$ are thus of order $\omega_{*e}\tau_\gamma \gg 1$ unless the mode frequency matches one of the natural frequencies.

Solution in the magnetostatic region

To characterize the nature of the solutions in the magnetostatic region, it is helpful to think of each resonant layer as containing a pair of antennae capable of launching shear-Alfvén waves towards either the interior or the exterior of the plasma. In practice the role of these antennae is played by current structures in the resonant region such as those associated with magnetic islands or with twisted (interchanged) lines of flux. In the magnetostatic region, all the Fourier components of the mode are coupled by curvature and shaping. Furthermore, the tearing and twisting parities are also coupled together. An outgoing wave launched, for example, towards the exterior of the plasma from the ℓth mode-rational surface will thus give rise to incoming waves on both side of all the mode-rational surfaces.

Consider for definiteness a mode with toroidal mode-number n such that there are M mode-rational surfaces in the plasma. The corresponding antennae can launch $2M$ different outgoing waves. Each of these $2M$ outgoing waves gives rise to $2M$ incoming waves, one on each side of every mode-rational surface. For the purpose of obtaining a dispersion relation, the magnetostatic region is completely characterized by the matrix containing the $2M \times 2M$ amplitudes of the incoming waves excited by outgoing waves of unit amplitude. We can evaluate the elements of this matrix by solving the magnetostatic wave equations for each of the specified outgoing-wave boundary conditions.

In order to express the solution of the magnetostatic wave equations in convenient form, we define the pairs of vectors $\widehat{\mathbf{\Psi}}_{in}^{+}$ and $\widehat{\mathbf{\Psi}}_{in}^{-}$ containing the M amplitudes of the incoming waves traveling in the positive and negative directions at each rational surface. We likewise define a pair of vectors $\widehat{\mathbf{\Psi}}^+_{out}$ and $\widehat{\mathbf{\Psi}}^-_{out}$ containing the M amplitudes of the outgoing waves traveling in the positive and negative directions at each rational surface. The solutions of the magnetostatic wave equations specify the following linear relationships between these vectors,

$$\widehat{\mathbf{\Psi}}^+_{in} = \mathbf{Y}_{++}\widehat{\mathbf{\Psi}}^+_{out} + \mathbf{Y}_{+-}\widehat{\mathbf{\Psi}}^-_{out}; \tag{8}$$

$$\widehat{\mathbf{\Psi}}^-_{in} = \mathbf{Y}_{-+}\widehat{\mathbf{\Psi}}^+_{out} + \mathbf{Y}_{--}\widehat{\mathbf{\Psi}}^-_{out}. \tag{9}$$

The self-adjointness of the ideal-MHD equations implies that the matrices $\mathbf{Y}_{++}$ and $\mathbf{Y}_{--}$ are hermitean, and that $\mathbf{Y}_{+-} = (\mathbf{Y}_{-+})^\dagger$, the adjoint of the matrix $\mathbf{Y}_{-+}$ (this is known as the reciprocity property of antennae). Note that in the case of up-down symmetric equilibria, the $\mathbf{Y}$ matrices are real.

In the limit of vanishing pressure, the matrices $\mathbf{Y}$ are proportional to the antennae's admittance matrices. To see this, note that $D_I = -1/4$ in a pressureless plasma. The waves thus take the asymptotic form

$$\psi(x) \sim \widehat{\psi}^+_{m\,\mathrm{out}} + \widehat{\psi}^+_{m\,\mathrm{in}}x, \quad x > 0; \tag{10}$$

$$\psi(x) \sim \widehat{\psi}^-_{m\,\mathrm{out}} + \widehat{\psi}^-_{m\,\mathrm{in}}|x|), \quad x < 0. \tag{11}$$

The admittance of an antenna launching a wave from r_m towards the exterior of the plasma is

$$\frac{B_{\theta,m}}{\mu_0 E_{z,m}} = \frac{(Y_{++})_{mm}}{i\omega\mu_0} = \frac{1}{i\omega L},$$

where

$$L = \frac{\mu_0}{(Y_{++})_{mm}}$$

is the antenna inductance. Similarly, the coupling coefficients are

$$\frac{B_{\theta,\ell}}{\mu_0 E_{z,m}} = \frac{1}{i\omega M_{\ell,m}},$$

where the

$$M_{\ell,m} = \frac{\mu_0}{(Y)_{\ell m}}$$

are the mutual inductances between pairs of antennae.

In order to match the magnetostatic solution to the even and odd solutions inside the layers, it is more convenient to express the propagation matrix in terms of the even and odd components of the waves. To this end, we define

$$\widehat{\mathbf{\Psi}}^{te}_{out} = \frac{1}{2}(\widehat{\mathbf{\Psi}}^{+}_{out} + \widehat{\mathbf{\Psi}}^{-}_{out}); \qquad \widehat{\mathbf{\Psi}}^{te}_{in} = \frac{1}{2}(\widehat{\mathbf{\Psi}}^{+}_{in} + \widehat{\mathbf{\Psi}}^{-}_{in}); \tag{12}$$

$$\widehat{\mathbf{\Psi}}^{tw}_{out} = \frac{1}{2}(\widehat{\mathbf{\Psi}}^{+}_{out} - \widehat{\mathbf{\Psi}}^{-}_{out}); \qquad \widehat{\mathbf{\Psi}}^{tw}_{in} = \frac{1}{2}(\widehat{\mathbf{\Psi}}^{+}_{in} - \widehat{\mathbf{\Psi}}^{-}_{in}). \tag{13}$$

The amplitudes $\widehat{\mathbf{\Psi}}^{te}_{out}$ (respectively, $\widehat{\mathbf{\Psi}}^{te}_{in}$) are proportional to the currents driven (induced) when the antennae coils on either side of the resonant region are connected in series, or equivalently when the resonant region contains a rotating magnetic island. The amplitudes $\widehat{\mathbf{\Psi}}^{tw}_{out}$ (respectively, $\widehat{\mathbf{\Psi}}^{tw}_{in}$), by contrast, are proportional to the currents driven (induced) when the series connection on the antennae coils facing the core of the plasma are reversed, or equivalently when the resonant region contains twisted lines of flux.

The magnetostatic response equations relating these vectors are

$$\widehat{\mathbf{\Psi}}^{te}_{\rm in} = \mathbf{E}^{te}\widehat{\mathbf{\Psi}}^{te}_{\rm out} + \mathbf{H}\widehat{\mathbf{\Psi}}^{tw}_{\rm out}; \tag{14}$$

$$\widehat{\mathbf{\Psi}}^{tw}_{\rm in} = \mathbf{H}^{\dagger}\widehat{\mathbf{\Psi}}^{te}_{\rm out} + \mathbf{E}^{tw}\widehat{\mathbf{\Psi}}^{tw}_{\rm out}, \tag{15}$$

where $\mathbf{H}^t$ is the transpose of $\mathbf{H}$ and

$$\mathbf{E}^{te} = \frac{1}{2}(\mathbf{Y}_{++} + \mathbf{Y}_{+-} + \mathbf{Y}_{-+} + \mathbf{Y}_{--}); \tag{16}$$

$$\mathbf{E}^{tw} = \frac{1}{2}(\mathbf{Y}_{++} - \mathbf{Y}_{+-} - \mathbf{Y}_{-+} + \mathbf{Y}_{--}); \tag{17}$$

$$\mathbf{H} = \frac{1}{2}(\mathbf{Y}_{++} - \mathbf{Y}_{+-} + \mathbf{Y}_{-+} - \mathbf{Y}_{--}). \tag{18}$$

In the case of a cylindrical plasma, the $\mathbf{E}$ and $\mathbf{H}$ matrices are diagonal. When only one resonant surface is present, the asymptotic parity of a magnetostatic solution can be switched simply by changing the sign of the wavefunction on either side [12], Thus, $E^{te}_{mm} = E^{tw}_{mm} = \Delta'_m$, the well-known tearing-stability index. The H_{mm} describe the coupling between the tearing and twisting modes. It should be noted, however, that the cylindrical limit is singular for the twisting-parity index: that is, the twisting parity index does not approach the cylindrical value in the limit of large aspect-ratio and low beta [12].

Matching and dispersion relation

The matching of the layer and magnetostatic solutions is achieved by eliminating the incoming wave amplitudes from (3) and (14). This leads to the following equations between the outgoing wave amplitudes:

$$\left[\mathbf{E}^{te} -^{te}(\omega)\right] \widehat{\mathbf{\Psi}}^{te}_{\mathrm{out}} + \mathbf{H}\,\widehat{\mathbf{\Psi}}^{tw}_{\mathrm{out}} = 0, \tag{19}$$

$$\left[\mathbf{E}^{tw} - \mathbf{\Delta}^{tw}(\omega)\right] \widehat{\mathbf{\Psi}}^{tw}_{\mathrm{out}} + \mathbf{H}^{\dagger}\widehat{\mathbf{\Psi}}^{te}_{out} = 0, \tag{20}$$

where the $\mathbf{\Delta}(\omega)$ are diagonal matrices formed from the resistive layer parameters (3). The solubility condition for this system of equations is the dispersion relation.

The dispersion relation is easily solved in the case of a differentially rotating plasma. In this case, the $\Delta^{te}_m(\omega)$ and $\Delta^{tw}_m(\omega)$ parameters are very large, and the corresponding ψ^{te}_m and ψ^{tw}_m are very small unless the frequency matches one of the natural frequencies [10]. Since all the natural frequencies are distinct in general, only a single $\Delta(\omega)$ can be finite for any given mode and the amplitude of the outgoing waves at every other mode-rational surface must be negligible. It follows that the dispersion relation for the tearing modes resonant at surface m is simply

$$\Delta^{te}_m(\omega) = E^{te}_{mm}. \tag{21}$$

For this mode the amplitude of the resonant tearing harmonic $\widehat{\psi}^{te}_{m,out}$ dominates all the other components of the eigenvector $\widehat{\mathbf{\Psi}}^{te}_{out}$. Similarly, the twisting mode resonant at surface m obeys

$$\Delta^{tw}_m(\omega) = E^{tw}_{mm}, \tag{22}$$

and the amplitude of the resonant twisting harmonic $\widehat{\psi}^{tw}_{m,out}$ dominates all the other components of $\widehat{\mathbf{\Psi}}^{tw}_{out}$.

The dispersion relations (21) are extremely general. In particular, they apply independently of the magnitude of the mode number. The calculation of the parameters E^{te}_{mm} and E^{tw}_{mm}, however, is difficult: it requires the solution of at least M coupled, singular differential equations. For small mode numbers, the E_{mm} parameters can be calculated numerically with the codes PEST-III [20] and T7 [21]. For large mode-numbers, however, they have only been calculated in the limit of large aspect-ratio and low beta. In the following section we show how to calculate these parameters for finite aspect-ratio, high-beta configurations.

3 The limit of small wavelengths

In this section we show how to calculate the stability matrices for large n in terms of the properties of the small wavelength resistive instabilities occurring in *uniformly rotating* plasma. In uniformly rotating plasma, the resonance condition can be satisfied at all the rational surfaces simultaneously and the dispersion relation involves the determinant of the matrices in (19). The eigenmodes and eigenvalues of (19) can be deduced, however, from the symmetry properties of the solution.

The approximate radial symmetry enjoyed by short-wavelength instabilities in a uniformly rotating plasma results from the smallness of the variation of the equilibrium quantities between two neighboring mode-rational surfaces. Thus, in the large-n limit the coefficients of the **E** and **H** matrices depend only on the distance from the diagonal:

$$\mathbf{E} = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ \ldots & E_2 & E_1 & \Delta'_\infty & E_1 & E_2 & \ldots & \ldots & \\ \ldots & \ldots & E_2 & E_1 & \Delta'_\infty & E_1 & E_2 & \ldots & \ldots \\ & \ldots & \ldots & E_2 & E_1 & \Delta'_\infty & E_1 & E_2 & \ldots \\ & & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

where $E_j = E_{m,m+j} = E_{m,m-j}$ and we have suppressed the parity labels. Similar relations pertain for **H**.

In uniformly rotating plasma, the $\mathbf{\Delta}^{te}$ and $\mathbf{\Delta}^{tw}$ matrices describing the resistive layer enjoy the same radial-translation symmetry as the **E** and **H** matrices. That is, their elements $\Delta_m(\omega)$ are almost indistinguishable for neighboring mode-rational surfaces (i.e. for $m_i - m_j \ll (m_i + m_j)/2$). The natural frequencies for tearing and twisting modes remain separated, however, by the diamagnetic frequency. The tearing and twisting eigenmodes are thus uncoupled even in uniformly rotating plasmas. Their respective eigenmode equations are

$$\left[\mathbf{E}^{te} - \Delta^{te}(\omega)\mathbf{I}\right] \widehat{\mathbf{\Psi}}^{te}_{\rm out} = 0; \qquad \widehat{\mathbf{\Psi}}^{tw}_{out} \ll \widehat{\mathbf{\Psi}}^{te}_{out} \tag{23}$$

for the tearing mode and

$$\left[\mathbf{E}^{tw} - \Delta^{tw}(\omega)\mathbf{I}\right] \widehat{\mathbf{\Psi}}^{tw}_{\rm out} = 0; \qquad \widehat{\mathbf{\Psi}}^{te}_{out} \ll \widehat{\mathbf{\Psi}}^{tw}_{out} \tag{24}$$

for the twisting mode. Here **I** is the unit matrix.

Equations (23)–(24) state that the eigenmodes are the eigenvectors of the E matrices. By virtue of the translation invariance, they must also be eigenvectors of the translation operator. They are thus given by

$$\begin{pmatrix} \ldots \\ \widehat{\psi}_{m-1} \\ \widehat{\psi}_m \\ \widehat{\psi}_{m+1} \\ \ldots \end{pmatrix} = \begin{pmatrix} \ldots \\ e^{i(m-1)\theta_0} \\ e^{im\theta_0} \\ e^{i(m+1)\theta_0} \\ \ldots \end{pmatrix} \widehat{\psi}, \tag{25}$$

where θ_0 is the phase shift between the modes on consecutive rational surfaces and $\widehat{\psi}$ is the global mode amplitude. Solutions with this radial translation invariance are known as ballooning modes.

We may substitute the solutions directly into the matrix equation to verify that they are eigenvectors. The m-th element of the matrix product $\mathbf{E\Psi}$ is

$$\begin{aligned}(\mathbf{E\Psi})_m &= \widehat{\psi} \sum_{l=-\infty}^{\infty} E_{m,l} e^{il\theta_0} \\ &= \widehat{\psi} e^{im\theta_0} \sum_{k=-\infty}^{\infty} E_{m,m+k} e^{ik\theta_0} \\ &= \Delta'(\theta_0)\widehat{\psi}_m. \end{aligned} \tag{26}$$

Here $\Delta'(\theta_0)$ is the eigenvalue of the matrix $\mathbf{E}$,

$$\Delta'(\theta_0) = \sum_{k=-\infty}^{\infty} E_k e^{ik\theta_0}. \tag{27}$$

We will always write the $\Delta'(\theta_0)$ function with its argument in order to distinguish it from the other Δ' parameters used elsewhere in this paper. Equation (27) shows that the $\Delta'(\theta_0)$ function is the Fourier transform of the rows (or columns) of the $\mathbf{E}$ matrix. We can thus obtain the elements of the $\mathbf{E}$ matrix directly from this function by inverting the Fourier transform. The result is

$$E_\ell = \frac{1}{2\pi} \oint d\theta_0 \, \Delta'(\theta_0) e^{-i\ell\theta_0}. \tag{28}$$

Equation (28) is a key result: it allows the elements of the $\mathbf{E}$ matrix to be calculated, for small wavelength, by solving the one-dimensional ballooning magnetostatic equation. This represents a considerable simplification over the direct solution of M coupled, singular differential equations in the radial variable r.

The case of ideal instabilities

It is interesting to compare the formula for the resistive stability indices derived above, Eq. (28), with that giving the growth rate for pressure-driven ideal instabilities in a differentially rotating plasma [30, 31, 32],

$$\gamma = \oint \frac{d\theta_0}{2\pi} \gamma_B(\theta_0). \tag{29}$$

Here $\gamma_B(\theta_0)$ is the growth rate for ballooning instabilities in the equivalent rigidly rotating equilibrium.

We first point out that Eq. (28), while completely general, is only useful when considering instabilities that grow slowly compared to the Alfvén time. For more rapidly growing modes, inertia is important throughout the plasma and the asymptotic matching approach is inapplicable. Eq. (29), by contrast, applies to modes with Alfvénic growth rates as well as to weakly growing modes. It requires primarily that $\tau_A d\Omega/dq \ll 1$, where Ω is the rotation frequency. We note that the existing analytic derivations also require that the condition $\gamma_B(\theta_0) > 0$ be satisfied for all θ_0, but numerical results [32] suggest that Eq. (29) remains valid even when this condition is violated.

For weakly unstable ideal modes, we may clarify the differences between Eqs. (28) and (29) by rederiving the latter in terms of the stability matrix. The analysis of the inertial layer shows that in the large aspect-ratio, low-beta limit ($D_I = -1/4$),

$$\Delta(\omega_r + i\gamma) = \frac{i\pi k_\perp s \omega_H}{\omega_r - n\Omega(r_m) + i\gamma}.$$

Here $\omega_H = V_A/qR$ is the Alfvén frequency and ω_r is the real part of the eigenfrequency. We have assumed that the rotation is toroidal. Contrary to resistive modes, the inertial layer index $\Delta(\omega)$ is a smooth function of the frequency. It follows that ideal modes have comparable amplitude at all the rational surfaces, so that (21) does not apply.

The dispersion relation may nevertheless be solved as follows. We first express the amplitudes of all the outgoing waves in terms of the amplitudes of the incoming waves in the layer and magnetostatic regions by inverting the matrices $\mathbf{E}$ and $\boldsymbol{\Delta}$. We then perform the matching by equating the amplitudes of the incoming waves in both regions. Neglecting the tearing-parity amplitudes for simplicity, we obtain the matching equation

$$\mathbf{E}^{-1}\widehat{\boldsymbol{\Psi}}_{in} = \boldsymbol{\Delta}^{-1}\widehat{\boldsymbol{\Psi}}_{in}. \tag{30}$$

The matrices $\mathbf{E}^{-1}$ and $\mathbf{E}$ have the same eigenvectors, and their eigenvalues are in inverse relationship to each other. This leads us to substitute a linear superposition of the ballooning solutions (25) in (30). We set

$$(\widehat{\boldsymbol{\Psi}}_{\rm in})_m = \oint d\theta_0 e^{i(m-m_0)\theta_0}\widehat{\psi}(\theta_0),$$

where m_0 is the dominant poloidal mode number, and substitute $\widehat{\boldsymbol{\Psi}}_{in}$ into the mode equation. There follows

$$\oint d\theta_0 e^{i(m-m_0)\theta_0}\left[\pi k_\perp s\omega_H \Delta'(\theta_0)^{-1} - \gamma + i\omega_r - in\Omega(r_m)\right]\widehat{\psi}(\theta_0) = 0. \tag{31}$$

In a uniformly rotating plasma, $\Omega(r_m) = \Omega_0$, the solutions are the ballooning modes with growth rates

$$\gamma_B(\theta_0) = \pi k_\perp s\omega_H \Delta'(\theta_0)^{-1}. \tag{32}$$

In a differentially rotating plasma, by contrast, we may expand the Doppler shift about the dominant poloidal mode number m_0,

$$n\Omega(r_m) = n\Omega(r_{m_0}) + i\left(\frac{d\Omega}{dq}\right)_{r_{m_0}}(m-m_0).$$

After performing an inverse Fourier transform, we obtain the eigenmode equation

$$\left(\frac{d\Omega}{dq}\right)_{r_{m_0}}\frac{d\widehat{\psi}}{d\theta_0} = \left[\gamma - i\omega_r + in\Omega(r_{m_0}) - \pi k_\perp s\omega_H\Delta'(\theta_0)^{-1}\right]\widehat{\psi}(\theta_0). \tag{33}$$

The solubility condition for this equation yields $\omega_r = n\Omega(r_{m_0})$ and

$$\gamma = \pi k_\perp s\omega_H \oint \frac{d\theta_0}{2\pi}\Delta'(\theta_0)^{-1}. \tag{34}$$

which is equivalent to Eq. (29) after substitution of the ballooning growth rate given in Eq. (32).

We next show how to calculate the function $\Delta'(\theta_0)$ by solving the ballooning equation in the magnetostatic region.

The magnetostatic ballooning solutions

In order to obtain the eigenvalue function $\Delta'(\theta_0)$ we use the ballooning transformation to solve the mode equations in the magnetostatic region. We begin by briefly reviewing the derivation of the ballooning representation for the purpose of establishing the relationship between the radial and the ballooning wavefunctions. The derivation presented here is adapted from that of Hazeltine et al. [25].

We start from the Fourier series representation of the electrostatic potential,

$$\phi(r,\theta,\zeta) = e^{in\zeta}\sum_m e^{-im\theta}\phi_m(r), \tag{35}$$

and note the symmetry property of poloidal harmonics,

$$\phi_m(r) = \overline{\xi}(nq-m)e^{im\theta_0}. \tag{36}$$

Here, $\overline{\xi}$ is a cookie-cutter function for the poloidal waveform. The ballooning transformation follows by expressing the $\phi_m(r)$ poloidal harmonics in terms of the inverse Fourier transform $\xi(\eta)$ of the cookie-cutter function,

$$\overline{\xi}(nq) = \int_{-\infty}^{\infty} d\eta\, e^{-i\eta nq}\xi(\eta). \tag{37}$$

Substituting this in (35) and (36) leads to

$$\phi(r,\theta,\zeta) = e^{in\zeta}\sum_m \int_{-\infty}^{\infty} d\eta\, e^{-im(\theta-\theta_0-\eta)}\xi(\eta)e^{-i\eta nq(r)}.$$

We may evaluate the integral by inverting the order of the integration and summation, and using the identity

$$\sum_m e^{imy} = \sum_m \delta(y+2\pi m).$$

There follows

$$\phi(r,\theta,\zeta) = \sum_m \xi(\theta-\theta_0+2\pi m) e^{in[\zeta - q(\theta-\theta_0+2\pi m)]}. \tag{38}$$

Equation (38) is known as the ballooning representation. It expresses $\phi(r,\theta,\zeta)$, a periodic function of θ, in terms of the inverse Fourier transform ξ of its poloidal Fourier harmonics.

Using the ballooning representation in the magnetostatic mode equations yields the ordinary differential equation

$$\frac{1}{J}\frac{d}{d\theta}\left(\frac{|\boldsymbol{\nabla}\alpha|^2}{JB^2}\frac{d\xi(\theta)}{d\theta}\right) + 2\frac{(\boldsymbol{\kappa}\times\mathbf{B})\cdot\boldsymbol{\nabla}\alpha}{B^2}\frac{dp}{d\psi}\,\xi(\theta) = 0, \tag{39}$$

where J is the Jacobian for the field-line coordinates, $\boldsymbol{\kappa}$ is the field-line curvature and $\alpha = q(\theta-\theta_0) - \zeta$ is the field line label. This equation may be solved numerically in general, and analytically in some special limits [7, 26]. Note that due to the Fourier transform relation between ϕ_m and ξ, the ballooning representation maps the small-x resistive layer onto the large-θ asymptotic region and the magnetostatic region to the central part $\theta \sim 1$ of the real line. Accordingly, outgoing waves become incoming waves in ballooning space and vice-versa.

The asymptotic solutions of the ballooning equation, for large θ, follow from a two-scale analysis. They are

$$\xi(\theta) \overset{\theta\to+\infty}{\sim} \hat{\xi}^+_{in}|s\theta|^{\nu} + \hat{\xi}^+_{out}|s\theta|^{-\nu-1}; \tag{40}$$

$$\xi(\theta) \overset{\theta\to-\infty}{\sim} \hat{\xi}^-_{in}|s\theta|^{\nu} + \hat{\xi}^-_{out}|s\theta|^{-\nu-1}. \tag{41}$$

The magnetostatic ballooning equation, unlike the resistive layer equations, are asymmetric about the origin $\theta = 0$ (except in the special cases $\theta_0 = 0$ and $\theta_0 = \pi$ for equilibria symmetric about the midplane). We adopt as our basic set of solutions the two waves $\xi_+(\theta)$ and $\xi_-(\theta)$ obtained by launching incoming waves of unit amplitude from $+\infty$ and $-\infty$, respectively, towards $\theta = 0$. Part of these waves is reflected from, and part is transmitted through the magnetic well at finite θ. Their asymptotic coefficients are related by equations similar to (8),

$$\hat{\xi}^{+}_{out} = \mathbf{y}_{++}\hat{\xi}^{+}_{in} + \mathbf{y}_{+-}\hat{\xi}^{-}_{in}; \tag{42}$$

$$\hat{\xi}^{-}_{out} = \mathbf{y}_{-+}\hat{\xi}^{+}_{in} + \mathbf{y}_{--}\hat{\xi}^{-}_{in}. \tag{43}$$

We may express the above relations in terms of the tearing-parity and twisting-parity amplitudes as in (12),

$$\hat{\xi}^{\mathrm{t}e}_{out} = \frac{1}{2}(\hat{\xi}^{+}_{out} - \hat{\xi}^{-}_{out}); \qquad \hat{\xi}^{\mathrm{t}e}_{in} = \frac{1}{2}(\hat{\xi}^{+}_{in} - \hat{\xi}^{-}_{in}); \tag{44}$$

$$\hat{\xi}^{\mathrm{t}w}_{out} = \frac{1}{2}(\hat{\xi}^{+}_{out} + \hat{\xi}^{-}_{out}); \qquad \hat{\xi}^{\mathrm{t}w}_{in} = \frac{1}{2}(\hat{\xi}^{+}_{in} + \hat{\xi}^{-}_{in}). \tag{45}$$

The transfer matrix for the tearing-parity and twisting-parity amplitudes is the analog of (14),

$$\hat{\xi}^{\mathrm{t}e}_{out} = \Delta'^{\mathrm{t}e}_{B}(\theta_0)\hat{\xi}^{\mathrm{t}e}_{\mathrm{in}} + H_B(\theta_0)\hat{\xi}^{\mathrm{t}e}_{\mathrm{in}}; \tag{46}$$

$$\hat{\xi}^{\mathrm{t}w}_{out} = H_B(\theta_0)\hat{\xi}^{\mathrm{t}w}_{\mathrm{in}} + \Delta'^{\mathrm{t}w}_{B}(\theta_0)\hat{\xi}^{\mathrm{t}w}_{\mathrm{in}}. \tag{47}$$

Substitution of these results in the asymptotic form of $\xi(\theta)$, Eq. (40), yields

$$\xi_{\mathrm{t}e}(\theta) = \sigma\hat{\xi}^{\mathrm{t}e}_{in}(|s\theta|^{\nu} + \Delta'^{\mathrm{t}e}_{B}(\theta_0)|s\theta|^{-\nu-1}) + \hat{\xi}^{\mathrm{t}w}_{in}H_B|s\theta|^{-\nu-1}$$

for the tearing-parity solution and

$$\xi_{\mathrm{t}w}(\theta) = \hat{\xi}^{\mathrm{t}w}_{in}(|s\theta|^{\nu} + \Delta'^{\mathrm{t}w}_{B}(\theta_0)|s\theta|^{-\nu-1}) + \sigma\hat{\xi}^{\mathrm{t}e}_{in}H_B|s\theta|^{-\nu-1}$$

for the twisting-parity solution. Here $\sigma = sign(\theta)$. The matching parameters $\Delta'^{\mathrm{t}e}_{B}$, $\Delta'^{\mathrm{t}w}_{B}$, and H_B are given in terms of the reflection and transmission coefficients by

$$\Delta'^{\mathrm{t}e}_{B} = \frac{1}{2}(\mathbf{y}_{++} + 2\mathbf{y}_{+-} + \mathbf{y}_{--}); \tag{48}$$

$$\Delta'^{\mathrm{t}w}_{B} = \frac{1}{2}(\mathbf{y}_{++} - 2\mathbf{y}_{+-} + \mathbf{y}_{--}); \tag{49}$$

$$H_B = \frac{1}{2}(\mathbf{y}_{++} - \mathbf{y}_{--}). \tag{50}$$

A common error is to evaluate the Δ'_B by imposing the condition that the *ratio* of large and small coefficients be identical for $\theta \to +\infty$ and $\theta \to -\infty$. This condition is fulfilled when the ratio of large and small coefficients is equal to one of the eigenvalues of the matrix y or, equivalently, of the matrix

$$\begin{pmatrix} \Delta_B'^{tw} & H_B \\ H_B & \Delta_B'^{te} \end{pmatrix}.$$

The eigenvalues are

$$\Delta'^{\pm}_{pseudo} = \frac{1}{2}\left(\Delta_B'^{te} + \Delta_B'^{tw} \pm \sqrt{(\Delta_B'^{te} - \Delta_B'^{tw})^2 + H_B^2}\right).$$

This definition of Δ' is only appropriate in the special case where $H_B = 0$.

The matching parameters for the poloidal component $\phi_m(r)$, and thus for $\psi_m(r)$, are found from the Fourier transform relation (37):[22]

$$\Delta'^{te}(\theta_0) = \frac{\pi k_\perp^{1+2\nu} \Delta_B'^{te}(\theta_0)}{2\cos^2(\frac{\pi}{2}\nu)\,,\,{}^2(1+\nu)}; \tag{51}$$

$$\Delta'^{tw}(\theta_0) = \frac{\pi k_\perp^{1+2\nu} \Delta_B'^{tw}(\theta_0)}{2\sin^2(\frac{\pi}{2}\nu)\,,\,{}^2(1+\nu)}; \tag{52}$$

$$H(\theta_0) = \frac{i\pi k_\perp^{1+2\nu} H_B(\theta_0)}{\sin(\pi\nu)\,,\,{}^2(1+\nu)}, \tag{53}$$

where $k_\perp = nq/r_m$.

It follows that the diagonal elements of the $\mathbf{E}$ matrices, corresponding to the generalization of the conventional Δ's, are given by

$$E_0^{te} = \frac{k_\perp^{1+2\nu}}{4\cos^2(\frac{\pi}{2}\nu)\,,\,{}^2(1+\nu)} \oint d\theta_0\, \Delta_B'^{te}(\theta_0); \tag{54}$$

$$E_0^{tw} = \frac{k_\perp^{1+2\nu}}{4\sin^2(\frac{\pi}{2}\nu)\,,\,{}^2(1+\nu)} \oint d\theta_0\, \Delta_B'^{tw}(\theta_0); \tag{55}$$

$$H = \frac{ik_\perp^{1+2\nu}}{2\sin(\pi\nu)\,,\,{}^2(1+\nu)} \oint d\theta_0\, H_B(\theta_0). \tag{56}$$

This is the principal result of this paper. In the limit $dp/dr \to 0$, $\nu \to 0$ and $\Delta_B'^{te}(\theta_0) = -4/\pi$ so that we recover the well-known result $\Delta'^{te} = -2k_\perp$. In the following section we shall apply

Eqs. (54) by evaluating the parameters Δ'^{te} and Δ'^{tw} from the solutions of the ballooning equation.

4 Numerical Results

To demonstrate the method developed in the previous sections we apply it to Shafranov's shifted-circle equilibrium with locally enhanced pressure gradient and magnetic shear [18, 19]. The ballooning equation for this equilibrium is

$$\frac{d}{d\theta}\left\{[1+h(\theta,\theta_0)^2]\frac{d\xi(\theta)}{d\theta}\right\}+[\overline{\kappa}+\alpha(\cos(\theta)+h(\theta,\theta_0)\sin(\theta)]\xi(\theta)=0, \tag{57}$$

where

$$h(\theta,\theta_0)=s(\theta-\theta_0)-\alpha\sin(\theta).$$

Here $\alpha=-(2Rq^2/B^2)dp/dr$ is the driving term proportional to the pressure-gradient, and $\overline{\kappa}$ is proportional to the average of the normal curvature. The asymptotic solutions of this equation can be found by the method of averages [7].

The matching parameters $\Delta_B'^{te}(\theta_0)$, $\Delta_B'^{tw}(\theta_0)$, and H_B are evaluated by using the following procedure. First, we integrate the ballooning equation from $-\infty$ to $+\infty$ with initial conditions corresponding to an outgoing wave at $-\infty$ (Fig. 1). Matching the solution at $\theta\to+\infty$ to the incoming and outgoing wave solutions, we obtain the reflection and transmission coefficients, and from these the y_{--} and y_{+-}. We then repeat the procedure, inverting the direction of integration and adopting the boundary condition corresponding to an outgoing wave at $+\infty$. This yields the reflection and transmission coefficients, and from these the y_{++} and y_{-+}. The matching parameters are then evaluated with Eqs. (48)–(54).

Our results are plotted in Fig. 2. The solid curve corresponds to the stability limit for ideal ballooning modes. The region to the left of this curve, corresponding to comparatively weak pressure gradients, is known as the first-stability region. The region to the right is known as the second-stability region.

The most interesting feature of our results is the domain of positive Δ'_B for tearing-parity ballooning modes. This domain, shaded in the Figure, lies to the left of the ideal stability curve in the first stability region. Although the stability index is comparatively small in this region, typically a fraction of $k_\perp$, it may be supplemented by other sources of free energy such as the bootstrap current, which is known to overcome the stabilizing effect of compressibility at long wavelengths [27]. The key point is that the widespread assumption that $\Delta' \sim -2k_\perp$ is grossly violated in this region.

The dependence of the stability index on the ballooning angle θ_0 is shown in Fig. 3. We see that $\theta_0 = \pi$ is the most unstable ballooning angle for tearing parity modes. This reflects the fact that it is more energetically favorable for magnetic islands to align themselves so that the X-point of one island faces the O-point of the neighboring islands [13].

By contrast, the stability index E_0^{tw} for radially localized tearing modes, which is proportional to the average of $\Delta_B'^{\text{te}}(\theta_0)$, is negative in the region where $\Delta_B'^{\text{te}}(\pi) > 0$ for moderate shear. For magnetic shear $s > 2$, the curve corresponding to $\oint d\theta\, \Delta_B'^{\text{te}}(\theta_0) = 0$ emerges slowly from the region of ideal instability. For low and moderate shear, the tearing-parity ballooning instability is thus stabilized by rotation. If the rotation is arrested, however, rapid growth of the tearing-parity ballooning modes will result. This may occur under the following sets of circumstances:

1. Locking of the plasma to the wall due to error fields,
2. Mutual locking of resonant surfaces following the slow growth of an ideal mode when the ideal stability threshold is crossed, or
3. Mutual locking of resonant surfaces following the slow growth of a tearing mode when $\oint d\theta\, \Delta_B'^{\text{te}}(\theta_0) > 0$.

Given the known properties of mode-locking, all of these circumstances constitute possible trigger mechanisms for rapid nonlinear destabilization.

Strauss [7] and Fu and Van Dam [26] have shown that for $\theta_0 = 0$ the tearing-parity ballooning stability index $\Delta_B'^{te}$ remains finite as the ideal stability threshold is crossed. This surprising feature is explained by the set of curves in Fig. 4 showing the $\Delta_B'^{te}(\theta_0)$ index as a function of the pressure parameter α for various values of the ballooning angle and for fixed shear s. These curves may be understood as follows: For sufficiently large s (greater than $s = .75$ where the tearing-parity ballooning stability curve crosses the ideal stability curve), a critical $\theta_0 = \theta_{0c}$ exists such that the marginal ideal stability curve for that particular θ_0 is tangent to the line s =constant ($\theta_{0c} \simeq 1.1$ for the case shown in Fig. 4. For $|\theta_0| > \theta_{0c}$, ideal modes are stable for all α and $\Delta_B'^{te}(\theta_0)$ is bounded. For $|\theta_0| < \theta_{0c}$ the $\Delta_B'^{te}(\theta_0)$ curves have two singularities corresponding to the marginal stability points for that particular value of θ_0. As $\theta_0 \to 0$, the ideal marginal-stability points $\Delta_B'^{te}(\theta_0) = \infty$ merge with the tearing-parity resistive marginal-stability points $\Delta_B'^{te}(\theta_0) = 0$ in order to produce the bounded curve corresponding to $\theta_0 = 0$.

Another feature that deserves comment is the sequence of singularities of the Δ' stability indices for values of the Mercier ideal stability parameter $D_I = -\ell^2/4$, $\ell = 0, 1, 2, \ldots$. These singularities correspond to the circumstance that the Froebenius series expansion for the *large* solution becomes degenerate with that for the small solution. The degeneracy manifests itself as a divergence of the coefficient of the large solution when a regular solution is expressed as a combination of large and small solutions. Note that since the degeneracy manifests itself in the large solution, and since the large and small solutions exchange roles in the magnetostatic and layer regions, the singularity affects the stability index for the magnetostatic region in a way opposite to that for the singular layer. For $D_I = -1$, for example, the magnetostatic stability index diverges while the layer $\Delta(\omega)$ vanishes for all ω. The treatment of these singularities will be described in a forthcoming paper.

We conclude the discussion of our numerical results by describing the marginal stability curves for twisting-parity modes. In agreement with Sykes et al. [28] and Connor et al. [29], we find that the entire region of first-stability is unstable to twisting-parity ballooning modes. We also find that $\Delta_B'^{tw}(\theta_0) > 0$ depends only weakly on the ballooning angle even for large shear. As a result, the marginal stability curve for radial twisting modes in a differentially rotating plasma, $E_0^{tw} = 0$, lies very close to the corresponding curve for ballooning modes (Fig. 2).

5 Summary

We have developed a method for evaluating the stability parameters for resistive modes in the large wavelength limit. We find that the stability parameters for radial eigenmodes in a differentially rotating plasma are proportional to the average over the ballooning angle of the stability index for ballooning modes.

We have applied our method by calculating the stability parameters for a model equilibrium. Our numerical results show the existence of a region where tearing-parity *ballooning* modes (the appropriate eigenmodes for rigidly rotating plasma) are unstable but where the *radial or Fourier* modes (the appropriate eigenmodes in differentially rotating plasma) are stable. Our results also show that at weak shear the beta limit coincides with the ideal ballooning mode threshold, while at large shear the maximum beta achievable in the region of first stability is slightly reduced compared to the ideal limit.

The stabilizing effect of rotation on resistive instabilities suggests the following mechanism: as the amplitude of a tearing mode grows, the torque it exerts on neighboring resonant surfaces increases quadratically. This torque leads to a decrease in the difference between the rotation velocities of neighboring resonant surfaces. When the mode amplitude exceeds a certain threshold, the rotation of nearby resonant surfaces is suddenly forced into synchronism [33, 13]. This causes the ballooning modes, whose stability threshold lies at lower pressures than the tearing modes, to grow vigorously. The above mechanism offers an explanation for the suddenness of the onset of beta-limit disruptions.

An important application of the method developed here is to the calculation of the stability limit in the plasma edge, where the pressure gradient is controlled by edge localized mode (ELM). These modes are thought to be peeling modes driven by the strong current gradients in the edge. Recent calculations show that the combination of peeling and ballooning modes can deny access to the region of second stability [34]. The considerations outlined in the present paper, while not directly applicable to peeling modes due to our neglect of current gradients, nevertheless point to the importance of the effects of differential rotation in assessing the stability and coupling of peeling and ballooning modes, and provides a method for investigating these effects.

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Figure 1: Example of a wave incident upon the magnetostatic region from the right. The (matched) asymptotic forms of the reflected and transmitted waves are shown with dashed lines.

Figure 2: Stability diagram for the shifted-circle model equilibrium. The solid line represents the marginal stability curve for ideal ballooning modes. The dashed line is the marginal stability curve for tearing-parity resistive ballooning modes, neglecting compressibility. The dash-dotted lines are the marginal stability curves for twisting parity resistive ballooning modes (lower curve) and for radial twisting modes in a plasma with sheared rotation (upper curve). The dotted parabolas correspond to the Froebenius singularities where the stability indices diverge.

Figure 3: Variation of the stability index $\Delta_B'^{te}(\theta_0)$ with ballooning angle for the tearing-parity ballooning mode. The mode is most unstable for $\theta_0 = \pi$. The rapid variation around $\theta_0 = 0$ is a remnant of the cancellation of pole and zero described in the text and in Fig. 4.

Figure 4: Variation of the stability index $\Delta_B'^{te}$ for the tearing-parity ballooning mode as a function of α for $s = 1.5$. The angle $\theta_0 = 1.1$ corresponds to the critical value at which the ideal marginal stability line is tangent to the line $s = 1.5$. The singularities evident in the curve for $\theta_0 = \pi/4$ merge with the nearby zeros as $\theta_0 \to 0$ resulting in the bounded curve for $\theta_0 = 0$.