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Neutron spin quantum plasmas – Ferromagnetism as a relaxed state

Swadesh M. Mahajan, Felipe A. Asenjo *

Institute for Fusion Studies, The University of Texas at Austin, TX 78712, USA

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ABSTRACT

It is shown that a ferromagnetic "minimum energy relaxed state" is accessible to a neutron fluid. We model the neutron fluid as a spin quantum plasma where the electromagnetic interaction is trough the magnetic moment of the neutron. The neutron ferromagnetism results from the macroscopic spin alignment that occurs due to a profound interplay between the classical and spin quantum vorticities carried by the charge-less neutron fluid. The simplest manifestation of a neutron superfluidity comes about by an exact cancellation of the quantum and classical vorticities to create a helicity free system.

1. Introduction

Search for constrained minimum energy states, as a paradigm for self-organization, has been a very creative enterprise in plasma physics. A particular class of such states, where energy is minimized subject to helicity constraints, are known as relaxed states, and have been invoked to model plasma equilibria in a variety of settings, laboratory as well as astrophysical [1–7]. Most of this search was confined to classical, non-relativistic plasmas with the exception of Ref. [7] that investigates a fully relativistic electron–positron system.

This search, can be naturally extended to what are known as spin quantum plasmas [8–11], a relatively new field in a state of rapid development. Spin quantum plasmas are fluid systems consisting of particles that obey the Schrödinger–Pauli equation, and are being investigated to study the collective motions that a many-body quantum system can execute. Instead of dwelling on the binary interactions, a plasma description emphasizes the motion of a test particle in a mean field generated by all other particles.

Since the Schrödinger–Pauli equation (describing the non-relativistic dynamics of spin-half particles) may be used to model even uncharged particles, the general formalism [8,9,11–13] could just as well be applied to study the collective behavior of a neutron fluid. Though uncharged, the neutron fluid does interact with the electromagnetic field through its non-zero magnetic moment. In this work we demonstrate that an electromagnetically active spin quantum neutron fluid can self-organize into what may be viewed as a "ferromagnetic minimum energy relaxed state" – a state that is endowed with a macroscopic spin field, a magnetization current, and, consequently, a macroscopic magnetic field. The relaxed state is derived through a constrained variational principle involving the conserved dynamical energy, and the conserved total helicity of the neutron fluid.

Derivation and analysis of relaxed states in a charged spin system (an electron spin quantum plasma, for example) will be the subject of an upcoming paper. It aught to be pointed out that ferromagnetic behavior may also be induced via instabilities [14]. The relaxed states, investigated here, are, however, equilibrium ferromagnetic configurations.

2. Neutron plasma: spin quantum fluid model

To model the neutron fluid, we will invoke the recently developed vortical formalism of spin quantum plasmas [12,13]. This formalism, derived from the standard works on the subject [8,9,15], is particularly suitable for unifying and revealing phenomena connected with the deeper structure of plasma dynamics. The neutron plasma, defined as a fluid composed of particles with neutron mass, null charge but a finite intrinsic magnetic moment, is just a special case of the general quantum plasma; its electrodynamics is manifested through its spin-magnetic moment.

The dynamical equations describing a neutron fluid are abstracted from the general spinning quantum plasma system [12,13] by letting the electric charge e go to zero while retaining the magnetic moment μ (to appropriately reflect its observed value). The relevant system consists of the continuity

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0, \tag{1}$$

and the momentum balance equations

* Corresponding author.
E-mail addresses: mahajan@mail.utexas.edu (S.M. Mahajan), faz@physics.utexas.edu (F.A. Asenjo).

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$$mn\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\mathbf{v}=n\mu S_i\nabla\hat{B}^i-\nabla p+\frac{n\hbar^2}{2m}\nabla\left(\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right)+\frac{n\hbar^2}{8m}\nabla\left(\partial_j S_i\partial^j S^i\right),\tag{2}$$

where n is the density, $\mathbf{v}$ is the fluid velocity, $\mathbf{S}$ is the spin vector, m is the neutron mass, $\mu\simeq-9.66\times10^{-24}$ erg/G is the neutron magnetic moment, $\hbar$ is the reduced Planck constant and c is the speed of light. The cursive letters $(i,j=1,2,3)$ label the vector components. The effective magnetic field

$$\hat{\mathbf{B}}=\mathbf{B}+\frac{\hbar^2}{4mn\mu}\partial_i\left(n\partial^i\mathbf{S}\right)\tag{3}$$

adds a spin stress part to the magnetic field $\mathbf{B}$. The last three terms on the right hand side of the momentum equation are, respectively, the force produced by the classical fluid pressure p, the Bohm potential and the effective spin pressure. These gradient forces will not play much of a role in the vortical dynamics that we will develop in order to investigate self-organized ferromagnetism in a quantum neutron plasma. Lastly the macroscopic spin vector evolves via

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\mathbf{S}=\frac{2\mu}{\hbar}(\mathbf{S}\times\hat{\mathbf{B}}).\tag{4}$$

The spin field represents the normalized ($\mathbf{S}\cdot\mathbf{S}=1$), ensemble averaged macroscopic spin of the neutron fluid, and couples to the electromagnetic field through the magnetization $\mathbf{M}=\mu n\mathbf{S}$. For the neutron fluid, the spin magnetization current is the only source for the electromagnetic fields; the Maxwell equation takes the form

$$\nabla\times\mathbf{B}=4\pi\nabla\times\mathbf{M}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t},\tag{5}$$

where $\mathbf{E}$ is the electric field.

For an incompressible, barotropic, neutron fluid, the set of Eqs. (1)–(4) can be manipulated to the equivalent evolution equations [12,15]

$$\frac{\partial\boldsymbol{\Omega}_c}{\partial t}=\nabla\times(\mathbf{v}\times\boldsymbol{\Omega}_c)+\frac{\mu}{m}\nabla S_j\times\nabla\hat{B}^j,\tag{6}$$

$$\frac{\partial\boldsymbol{\Omega}_-}{\partial t}=\nabla\times(\mathbf{v}\times\boldsymbol{\Omega}_-)\tag{7}$$

for the two vorticities $\boldsymbol{\Omega}_c=\nabla\times\mathbf{v}$, and

$$\boldsymbol{\Omega}_-=\boldsymbol{\Omega}_c-\frac{\hbar}{2m}\boldsymbol{\Omega}_q=\nabla\times\left(\mathbf{v}-\frac{\hbar}{2m}\mathbf{P}_q\right)=\nabla\times\mathbf{P}_-.\tag{8}$$

The former is the standard classical fluid vorticity, while the latter is a recently introduced (and explored) hybrid vorticity [12] straddling classical and quantum physics. The quantum part of the vorticity, $\boldsymbol{\Omega}_q=\nabla\times\mathbf{P}_q$ is derivable from the vector potential constructed from the spin components $\mathbf{P}_q=-S_3\nabla\eta$, where $\eta=\arctan(S_2/S_1)$ (see Refs. [12,13,15] for details about quantum vorticity).

Eq. (7) for $\boldsymbol{\Omega}_-$ is in the Helmholtz form and constitutes an ideal vortex dynamics. When manipulated along with its uncurled counterpart

$$\frac{\partial\mathbf{P}_-}{\partial t}=\mathbf{v}\times\boldsymbol{\Omega}_-,\tag{9}$$

it yields the conserved helicity ($dh_-/dt=0$)

$$h_-=\langle\boldsymbol{\Omega}_-\cdot\mathbf{P}_-\rangle,\tag{10}$$

where $\langle\rangle\equiv\int d^3x$. Extracting a helicity constant of motion was the raison d'etre for constructing the hybrid vorticity $\boldsymbol{\Omega}_-$ [12]. In addition to the purely classical part $\boldsymbol{\Omega}_c\cdot\mathbf{v}$, the conserved helicity

$$h_-=\langle\boldsymbol{\Omega}_-\cdot\mathbf{P}_-\rangle=\left\langle\left(\boldsymbol{\Omega}_c-\frac{\hbar}{m}\boldsymbol{\Omega}_q\right)\cdot\mathbf{v}\right\rangle\tag{11}$$

has a classical–quantum hybrid part $\hbar\boldsymbol{\Omega}_q\cdot\mathbf{v}/m$; note that the purely quantum contribution to the helicity density ($\boldsymbol{\Omega}_q\cdot\mathbf{P}_q\equiv0$) vanishes due to the Clebsch form of the potential $\mathbf{P}_q$. In going from (10) to (11), we have performed a partial integration and dropped the surface term.

The neutron fluid also allows an energy invariant. Manipulating the momentum equation (2) and the spin evolution equation (4), we can readily extract the conserved quantity ($d\Sigma/dt=0$),

$$\Sigma=\left\langle\frac{mn}{2}v^2-n\mu\mathbf{S}\cdot\mathbf{B}+\frac{\hbar^2n}{8m}\partial_jS_i\partial^jS^i+\frac{B^2+E^2}{8\pi}\right\rangle,\tag{12}$$

that is the sum of the kinetic energy, the spin–magnetic interaction energy, the energy stored in the effective spin pressure, and the energy associated with the electromagnetic field.

Since thermal part is missing, Σ is not the total energy of the fluid. It is, however, the only part that is relevant for this study since thermal energy, contributing purely potential force, does not figure in the incompressible vortical dynamics. Throughout the rest of the Letter, although we will use the word energy for Σ, it must be understood that it signifies only the dynamically pertinent part of the energy. For example $\Sigma=0$ does not, by any means, imply that the total energy of the fluid is zero.

3. Relaxed states

The helicity and the energy invariants provide the foundation for explorations into a possible neutron ferromagnetic state. Following a strong plasma physics tradition, we seek a relaxed state (classical–quantum in this case) by a constrained minimization of the energy Σ; the constraint being the conservation of the helicity h_-. Formally, the relaxed states are derived from the condition

$$\delta\Sigma-\Lambda\delta h_-=0,\tag{13}$$

where Λ is a Lagrange multiplier needed to impose the constant helicity constraint.

Before continuing with the minimization process, we note that in the absence of a charged current, the equilibrium Maxwell equation (5)

$$\nabla\times\mathbf{B}=4\pi\mu n\nabla\times\mathbf{S},\tag{14}$$

relates the variations of the magnetic and the spin field, i.e. $\delta\mathbf{B}=4\pi\mu n\delta\mathbf{S}$. This extra constraint allow us to calculate the variation of energy (12) in a straightforward way

$$\delta\Sigma=\left\langle mn\mathbf{v}\cdot\delta\mathbf{v}-\left(\frac{n\hbar^2}{4m}\nabla^2\mathbf{S}+4\pi\mu^2n^2\mathbf{S}\right)\cdot\delta\mathbf{S}\right\rangle+\left\langle\frac{\mathbf{E}\cdot\delta\mathbf{E}}{4\pi}\right\rangle.\tag{15}$$

The helicity variation δh_- requires more care. We first express the total variation of Eq. (11) in terms of $\delta\mathbf{v}$ and the variation in the quantum potential $\delta\mathbf{P}_q$,

$$\delta h_-=\left\langle2\boldsymbol{\Omega}_-\cdot\delta\mathbf{v}-\frac{\hbar}{m}\boldsymbol{\Omega}_c\cdot\delta\mathbf{P}_q\right\rangle,\tag{16}$$

and then transform $\delta\mathbf{P}_q$ into the variation of the spin field $\langle\boldsymbol{\Omega}_c\cdot\delta\mathbf{P}_q\rangle=\langle-\boldsymbol{\Omega}_c\cdot\nabla\eta\delta S_3+\nabla S_3\cdot\boldsymbol{\Omega}_c\delta\eta\rangle$ to, finally, arrive at

$$\delta h_-=\left\langle2\boldsymbol{\Omega}_-\cdot\delta\mathbf{v}+\frac{\hbar}{m}\Omega_c\cdot\nabla\eta\delta S_3-\frac{\hbar}{m}\nabla S_3\cdot\boldsymbol{\Omega}_c\delta\eta\right\rangle.\tag{17}$$

Now, the condition (13) for arbitrary variations yields the equations determining the equilibrium relaxed states. These are $\mathbf{E}=0$,

$$\frac{mn}{2\Lambda}\mathbf{v}=\boldsymbol{\Omega}_-, \tag{18}$$

$$\frac{n\hbar}{4}\nabla\cdot(S_2\nabla S_1-S_1\nabla S_2)=-\Lambda\nabla S_3\cdot\boldsymbol{\Omega}_c, \tag{19}$$

$$\frac{n\hbar}{4}\left(S_3\frac{S_1\nabla^2 S_1+S_2\nabla^2 S_2}{S_1^2+S_2^2}-\nabla^2 S_3\right)=\Lambda\nabla\eta\cdot\boldsymbol{\Omega}_c, \tag{20}$$

where we have used the normalization of the spin to cast the variations of its components $\delta S_1=-S_2\delta\eta-S_1S_3\delta S_3/(S_1^2+S_2^2)$ and $\delta S_2=S_1\delta\eta-S_2S_3\delta S_3/(S_1^2+S_2^2)$ in terms of the variation of the two independent variables δS_3 and $\delta\eta$.

There are two particularly noteworthy features for these relaxed states of the neutron fluid. First, the magnetic field is purely spin generated [Eq. (14)] as it should be since neutrons, being charge neutral, carry no "electrical" current. Second, on the relaxed states, the Lagrange multiplier is related to the energy and helicity as $\Lambda=\Sigma/h_-$.

It is straightforward to show that the relaxed state, contained in Eqs. (18) to (20), is, indeed, an equilibrium state. The main step in the demonstration is that Eqs. (19) and (20) can be combined to yield

$$\frac{n\hbar}{4}\nabla S_i\nabla^2 S^i=\Lambda\boldsymbol{\Omega}_c\times\boldsymbol{\Omega}_q. \tag{21}$$

4. Exact Relaxed State (ERS)

Our next and main objective is to explore the possibility of "ferromagnetism" in the system of Eqs. (18) to (20), representing a minimum energy relaxed state accessible to a neutron spin quantum plasma; the ferromagnetism will be a consequence of a non-zero macroscopic spin field. It is hard not to notice that Eqs. (18) to (20) constitute a rather complicated set of highly nonlinear, constrained, partial differential equations. We have just begun a program to investigate their general solubility. Fortunately the system does allow a very simple but non-trivial two-dimensional exact solution. The reader can verify, by direct substitution, that

$$S_1=\sin\left(\frac{x}{L_x}\right)\cos\left(\frac{y}{L_y}\right),$$

$$S_2=\sin\left(\frac{x}{L_x}\right)\sin\left(\frac{y}{L_y}\right),$$

$$S_3=\cos\left(\frac{x}{L_x}\right), \tag{22}$$

$$\frac{\mathbf{v}}{c}=-\frac{\lambda_c}{2L_y}\cos\left(\frac{x}{L_x}\right)\hat{e}_y, \tag{23}$$

and

$$\frac{1}{\Lambda}=0, \tag{24}$$

does, indeed, solve Eqs. (18)–(20), where $\lambda_c=\hbar/mc$ is the Compton length, and L_x and L_y are the variation length scales along x and y directions. The solution preserves the spin normalization condition $\mathbf{S}\cdot\mathbf{S}=1$ that implies only two independent components of the spin vector. The velocity (and the theory) will remain non-relativistic as $\lambda_c\ll L_y$.

The condition (24) implies that the total helicity of this solution is zero. Notice that $h_-=\langle\mathbf{P}_-\cdot\boldsymbol{\Omega}_-\rangle=0$ is an integral condition and should not, necessarily, require the helicity density to vanish. However for the velocity field (23), the helicity density $\boldsymbol{\Omega}_-$ does vanish (on the solution $\mathbf{v}/c=\lambda_c\mathbf{P}_q/2$). The exact cancellation of the spin induced quantum part of the helicity by the classical part is what sets the stage for this remarkably simple and elegant solution. We will refer to this as ERS.

Despite its simplicity, the ERS is a totally non-trivial solution; it represents structured, non-zero spin and velocity fields, and consequently a non-trivial equilibrium magnetic field $\mathbf{B}=B_1\hat{e}_x+B_2\hat{e}_y+B_3\hat{e}_z$ given by Eq. (14). The components are readily calculated to be

$$B_1=\frac{4\pi\mu nL_x}{L_x^2+L_y^2}\left[L_y\cos\left(\frac{x}{L_x}\right)+L_x\sin\left(\frac{x}{L_x}\right)\right]\cos\left(\frac{y}{L_y}\right), \tag{25}$$

$$B_2=\frac{4\pi\mu nL_y}{L_x^2+L_y^2}\left[L_y\sin\left(\frac{x}{L_x}\right)-L_x\cos\left(\frac{x}{L_x}\right)\right]\sin\left(\frac{y}{L_y}\right), \tag{26}$$

$$B_3=4\pi\mu n\cos\left(\frac{x}{L_x}\right). \tag{27}$$

To avoid confusion, we should state here that if we had naively solved Eq. (14) to obtain $\mathbf{B}=4\pi\mu n\mathbf{S}$, we would be violating $\nabla\cdot\mathbf{B}=0$. The magnetic field, displayed above, is the exact solution consistent with the divergence condition.

To further explore the characteristics of this solution, let us assume that the two length scales are quite disparate with $L_y\ll L_x$. We could then consider L_x to be a measure of some appropriate macroscopic length of the system (for instance the radius of a neutron star if one were studying the origin of magnetic fields in such objects). Under this choice, the components B_1 and B_2 will have, in addition to the slow macroscopic x variation, a very fast y variation while the component B_3 will vary only on the macroscopic length scale L_x. The neutron fluid, thus, can sustain a global macroscopic magnetic field aligned along the direction of S_3.

The maximum global macroscopic field $B_3\sim 4\pi\mu n$ could attain values $\sim 1.2\times 10^{13}$ G in a neutron star with $n=10^{35}$ cm^{-3} (mass density $\sim 1.7\times 10^{11}$ g/cm^{-3}). In the extreme case of a neutron star plasma with $n=10^{37}$ cm^{-3} (corresponding to a mass density $\sim 1.7\times 10^{13}$ g/cm^{-3}), the maximum magnetic field could reach $\sim 10^{15}$ G. These intensities are close to the critical Schwinger limit, $B_S=m_e^2c^3/(e\hbar)\approx 4.4\times 10^{13}$ G (m_e and e are the electron mass and charge respectively) where quantum electrodynamical (QED) effects, like creation of electron–positron pairs, can take place. In regions close to the core, the density of the neutron star plasma is higher, and the magnetic field (27) is much larger than the Schwinger magnetic field. To deal with systems with such high magnetic fields, one must include electron–positron dynamics into the model; a simplified model based on a pure neutron fluid may not be sufficient. The results derived in the current work are based on the assumptions that the percentage of electron population in a neutron fluid (for example in the neutron star outer regions) could only introduce marginal QED effects for densities of $\sim 10^{35}$ cm^{-3}, that the treatment will remain non-relativistic as long as $L_y\gg\lambda_c$, and the expected average kinetic energy of the neutron is much less than a GeV.

To correct both these shortcomings, the theory of neutron star ferromagnetism should be extended through a fully quantum relativistic treatment of the neutron matter [16,17], and through including appropriate QED effects pertinent at such enormous magnetic fields. We, however, believe, that despite these considerations, the result for the equilibrium magnetic field (27) can be quite robust (even at the leading order behavior in the relativistic quantum regime).

5. Conclusions

We have just demonstrated the existence of ferromagnetism in a neutron fluid; we found an equilibrium relaxed state, accessible to a neutron fluid, that has a finite net spin field, which, in turn, yields a finite net magnetic field. For the very simple exact solution (ERS), the dynamical conserved energy is finite whereas the total conserved generalized helicity is zero. The most interesting and important feature of the ERS is the non-zero spin field and non-zero quantum vorticity – in fact the quantum vorticity must be large enough to cancel the classical part. The spin field (22) of the neutron fluid, via the associated magnetization current ($= \nabla \times \mu n \mathbf{S}$), generates an equilibrium magnetic field even when there are no charged particle currents.

Depending on the length scales of variations of the spin field, the magnetic field can show a macroscopic alignment in one direction [Eq. (27)]. This global ferromagnetism occurs when one of the spin length scales is a macroscopic length of the system. The origin of the spin–ferromagnetism displayed by the dynamic neutron fluid comes about through a deep interaction between classical and quantum features under specific conditions. In order to make the total vorticity zero the classical and spin quantum parts must balance. Since helicity is a measure of the topological complexity of the "flow", its vanishing is almost a precondition for superfluid-like behavior of the neutron fluid. The neutron superfluid carries a net non-zero spin field along with its concomitant magnetic field.

We believe that the Exact Relaxed State solution captures the essence of neutron ferromagnetism, though one should look for more general nonlinear solutions and general formalisms.

The theory presented in this Letter could certainly help in the development of a theoretical framework for a new creation mechanisms for large magnetic fields in neutron stars [18–20]. This will be the subject of a forthcoming paper.

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