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Generating vorticity and magnetic fields in plasmas in general relativity: Spacetime curvature drive

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Using the generally covariant magnetofluid formalism for a hot plasma, a spacetime curvature driven mechanism for generating seed vorticity/magnetic field is presented. The "battery" owes its origin to the interaction between the gravity modified Lorentz factor of the fluid element and the inhomogeneous plasma thermodynamics. The general relativistic drive is evaluated for two simple cases: seed formation in a simplified model of a hot plasma accreting in stable orbits around a Schwarzschild black hole and for particles in free fall near the horizon. Some astrophysical applications are suggested. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4792257]

I. INTRODUCTION

Just as the motion of a charged fluid in spacetime generates a magnetic field, it stands to reason that if spacetime were distorted in the region occupied by a charged fluid, a magnetic field would emerge. In a special relativistic context, it was recently demonstrated[1,2] that a generalized vorticity (GV)

$$\hat{\mathbf{B}} = \mathbf{B} + \frac{m}{q}\nabla \times (f\gamma \boldsymbol{v}), \tag{1}$$

consisting of magnetic and kinetic-thermal parts, may be generated, *ab initio*, in an ideal perfect fluid with inhomogeneous entropy. In Eq. (1), $\boldsymbol{v}$ is the velocity, m (q) are, respectively, the mass (charge) of the fluid particle, $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor, $v^2 = \boldsymbol{v}\cdot\boldsymbol{v}$, c is the speed of light, and f is the relativistic thermal factor related to the fluid density enthalpy $h = nmc^2 f$, with n as the fluid density. For a relativistic Maxwell distribution, $f \equiv f(x) = K_3(x)/K_2(x)$,[3,4] where K_j are the modified Bessel functions of order j, and $x = mc^2/k_BT$ is the inverse normalized temperature with the Boltzmann constant k_B. It is to be emphasized that the GV generation in special relativity proposed in Refs. 1 and 2 is entirely due to a distortion of *space* (as distinct from spacetime) caused by the special relativistic γ-factor. It is well known that in the non-relativistic dynamics of an ideal fluid, a topological constraint would forbid the emergence of GV from a zero initial value. Of course, motion in one frame need not be motion in another, and so the distortion is frame-dependent. If the astrophysical choice of "rest-frame" is clear the frame-dependence need not worry one.

In this paper we explore the possible role of general relativity in the generation of magnetic fields in plasmas. We will derive an appropriate curved spacetime generalization of the special relativistic formulation. Similar attempts studying the production of magnetic fields and vorticity in general relativistic plasmas may be found in the recent papers.[5–7]

A somewhat different interpretation of the special relativistic effect will be helpful in casting light on the extension to curved spacetime. The Poincarè group, $SO(1,3)\otimes_s \Re^4$, where $\otimes_s$ is the semi-direct product, guarantees the conservation of energy and momentum via spacetime translational invariance ($\Re^4$). Since $SO(1,3)\cong SO(3)\otimes SO(3)$, the first, $SO_L(3)$, can yield angular momentum conservation, and the second, $SO_S(3)$, will give the conservation of Dirac's *spin angular momentum conservation*. The former corresponds to spatial rotational invariance and the second to proper Lorentz invariance. Of course, what is conserved is the total angular momentum ($\mathbf{J} = \mathbf{L} + \mathbf{S}$), and it is *this* that provides the "seed" for magnetic field generation.[1]

Note that though the rotation can be undone over the entire spacelike hypersurface in a homogeneous spacetime (hence the effect is frame dependent for a homogeneous stress-energy tensor), it will persist in an inhomogeneous system; undoing the rotation at one place will simply push the twist elsewhere. Even then, there would have been no "seed" creation if there were no charge to induce non-homogeneity in the spacetime; the plasma is needed to provide the effect. It is also worth pointing out that the distortion is purely in the spacelike section (as the spacetime remains flat) and could be locally undone by a change of frame. However, it cannot be globally undone because of the inhomogeneity.

In general relativity, however, the curvature of spacetime will provide an effective motion at one point relative to a "rest-frame" at another. More precisely, we can take the local rest-frame at one point, as given by the tangent space using Riemann normal coordinates[8] and compare it with the local rest-frame at another point. There will be a definable local Lorentz factor there, giving the above special relativistic effect produced by gravity. The frame chosen is a special Fermi-Walker frame, which gives the geodesics *as if* they were straight lines bent due to an (appropriately modified) force of gravity. The general relativistic effects open up the

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exciting possibility of generation of magnetic fields near gravitating sources.

In the present calculation we do not consider the back-reaction of the plasma on spacetime. A complete self-consistent analysis would require the inclusion of the plasma contribution to the stress-energy tensor that drives the Einstein equations. Detailed evolution equations for the spacetime geometry have been recently developed.[5,6] We expect, however, that the simpler model invoked here will be enough to extract essential features of magnetic field generation in the vicinity of strongly gravitating bodies.

We begin, in Sec. II, by writing down the general relativistic plasma equations in a unified form; the word "unified" is used in the spirit of Refs. 1 and 2. In Sec. III we will derive an equation for the generalized vorticity (GV) that includes the general relativistic (GR) drives for the seed magnetic field. The extension of the special relativistic vortical dynamics derived in Ref. 9 (and investigated for vorticity generation in Refs. 1 and 2) to GR will be accomplished via a 3 + 1 decomposition of the plasma equations onto timelike and spacelike hypersurfaces. In Sec. IV we estimate the "value" of the generated vorticity (magnetic field) seed, and finally in Sec. V we provide a perspective for the results.

II. PLASMA DYNAMICS

The dynamics of an ideal plasma (charged fluid) is obtained using the conservation equation for the energy-momentum tensor $T^{\mu\nu}$ (using the usual symbol; for covariant derivatives)

$$T^{\mu\nu}{}_{;\nu} = qnF^{\mu\nu}U_\nu, \tag{2}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor and U^μ is the normalized plasma four-velocity ($U^\mu U_\mu = -1$). Here, we use $c = 1$. The charge q and the mass m of the fluid particles are invariants. The energy-momentum tensor for an ideal plasma

$$T^{\mu\nu} = hU^\mu U^\nu + pg^{\mu\nu} \tag{3}$$

involves two thermodynamic scalars, i.e., the enthalpy density h and the pressure p.

The equation of motion (2) could be written in terms of unified fields[9] (see also Ref. 10). In addition to facilitating calculations, this approach will help us to identify the GV in general relativity. Invoking the continuity equation $(nU^\mu)_{;\mu} = 0$ and introducing the auxiliary thermodynamic function $f = h/mn$, Eq. (2) is written as

$$mnU^\nu (fU^\mu)_{;\nu} = qnF^{\mu\nu}U_\nu - p_{,\nu}g^{\mu\nu}. \tag{4}$$

Following Ref. 9, we define the fully antisymmetric fluid tensor $S^{\mu\nu} = (fU^\nu)^{;\mu} - (fU^\mu)^{;\nu}$ and manipulate Eq. (4) to derive

$$q\, U_\nu M^{\mu\nu} = T\sigma^{,\mu}, \tag{5}$$

which is the unified covariant equation of motion in terms of the magnetofluid field $M^{\mu\nu} = F^{\mu\nu} + (m/q)S^{\mu\nu}$. All kinematic and thermal (through f) aspects of the fluid are now represented by $S^{\mu\nu}$. The function σ is the entropy density of the fluid, and it is related to pressure through

$$\sigma^{,\mu} = \frac{p^{,\mu} - mnf^{,\mu}}{nT}, \tag{6}$$

where T is the temperature. The antisymmetry of $M_{\mu\nu}$ guarantees that the fluid is isentropic $U_\mu \sigma^{,\mu} = 0$.

Inclusion of the Maxwell equations

$$F^{\mu\nu}{}_{;\nu} = 4\pi qnU^\mu \tag{7}$$

completes the system description.

III. GENERATION OF VORTICITY AND MAGNETIC FIELDS

Following the standard plasma procedure of Refs. 11–13, we will invoke the spacetime decomposition. The 3 + 1 formalism allows us to obtain a set of equations that is similar to those found in special relativity (see for example Refs. 12–21). The metric tensor in the canonical formalism is[23]

$$ds^2 = -\alpha^2 dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j, \quad (i,j = 1,2,3), \tag{8}$$

where α is the lapse function, β_i the shift vector, and γ_{ij} is the 3-metric of the spacelike hypersurfaces of metric $g_{\mu\nu}$. Since the square of the lapse function is the metric component $-g_{00}$, it essentially corresponds to the gravitational potential. More precisely, it has been shown that in a particular preferred frame, called the pseudo-Newtonian frame[24,25] (essentially a special choice of a Fermi-Walker frame), the gravitational potential comes out to be $\ln\sqrt{\alpha}$. The shift vector corresponds to the momentum. Of course, in the rest-frame (which can be obtained by an appropriate choice of gauge) the momentum is zero. Assuming that we can still obtain a global coordinate basis (which will not be possible for the Kerr metric, for example), we use the rest-frame so as to eliminate the shift vector. Though the more general discussion is physically very relevant, in this paper we will limit our investigations to spacetimes in which the shift vector can consistently be set to zero; a more complete analysis will be taken up in a future paper. Note that we could have chosen a frame of reference (gravitational gauge) to make the lapse function unity,[23] but this would "throw the baby out with the bath-water" as it would not display the gravitational potential for us to see the physics of its effect on the plasma. We would then be in the freely falling rest-frame and locally see Minkowski space around us. This is the frame of the fiducial observer. We would need to fit these local Minkowski spaces together and would then get the curved spacetime.

The normalized timelike vector field n^μ, obeying $n^\mu n_\mu = -1$ and $n^\mu \gamma_{\mu\nu} = 0$, is constructed in terms of the lapse function, $n_\mu = (\alpha, 0, 0, 0)$ and $n^\mu = (-1/\alpha, 0, 0, 0)$ (the shift vector is zero). Thus, the 3 + 1 decomposition is achieved by projecting every tensor onto n^μ in timelike hypersurfaces and onto $\gamma_{\mu\nu}$ in spacelike hypersurfaces. For example, the metric is decomposed as $g_{\mu\nu} = \gamma_{\mu\nu} - n_\mu n_\nu$. We, now, proceed to decompose the relevant tensors in terms of n^μ and $\gamma_{\mu\nu}$.

We first deal with the four-velocity $U^\mu = (\Gamma, \Gamma v^i)$, where $v^i = dx^i/dt$ corresponds to the i-component of the fluid velocity $\boldsymbol{v}$ and Γ is the Lorentz factor. Since $n_\mu U^\mu = \alpha\Gamma$, the decomposition

$$U^\mu = -\alpha\Gamma n^\mu + \Gamma\gamma^\mu{}_\nu v^\nu \tag{9}$$

allows us to write the Lorentz factor as

$$\Gamma = (\alpha^2 - \gamma_{\mu\nu} v^\mu v^\nu)^{-1/2}. \tag{10}$$

In flat space using Cartesian coordinates $\alpha = 1$, $\gamma_{ij} = \delta_{ij}$, and the well-known Lorentz factor of special relativity $\Gamma = (1 - v^2)^{-1/2}$ is recovered.

With the nomenclature straightened out and neglecting the plasma back-reaction on spacetime, one may readily write down the decomposition of the field equations. To illustrate the procedure for subsequent calculations, we begin with the Maxwell equations (7). Several authors[12–22] have expressed the electromagnetic tensor in terms of the electric (E^μ) and the magnetic (B^μ) fields, defined as ($\epsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric tensor)

$$E^\mu = n_\nu F^{\nu\mu}, \quad B^\mu = \frac{1}{2} n_\rho \epsilon^{\rho\mu\sigma\tau} F_{\sigma\tau}. \tag{11}$$

Both fields are spacelike, $n_\mu E^\mu = 0$ and $n_\mu B^\mu = 0$, and allow the electromagnetic tensor to be decomposed as

$$F^{\mu\nu} = E^\mu n^\nu - E^\nu n^\mu - \epsilon^{\mu\nu\rho\sigma} B_\rho n_\sigma. \tag{12}$$

Substituting Eq. (12) into Eq. (7) and projecting it onto n_μ we find $E^\mu{}_{;\mu} = 4\pi q n \alpha\Gamma$ that translates into the scalar form as[11–13]

$$\nabla \cdot \boldsymbol{E} = 4\pi q n \alpha \Gamma, \tag{13}$$

where ∇ is the spatial covariant derivative derived from $\gamma_{\mu\nu}$. Projecting Eq. (7) onto $\gamma^\beta{}_\mu$, we find the spacelike equation $\gamma^\beta{}_\mu E^\mu{}_{;\nu} n^\nu - \epsilon^{\beta\nu\rho\sigma}(B_\rho n_\sigma)_{;\nu} = 4\pi q n \Gamma v^\beta$ which, using $n_{\mu;\nu} = -n_\nu \alpha_{,\mu}/\alpha$ (Ref. 13), yields

$$\frac{1}{\alpha}\nabla \times (\alpha \boldsymbol{B}) = 4\pi q n \Gamma \boldsymbol{v} + \frac{1}{\alpha}\frac{\partial \boldsymbol{E}}{\partial t}, \tag{14}$$

the GR modified Maxwell law.[11–13]

Note that from the preceding two equations, we can derive the continuity equation

$$\frac{\partial}{\partial t}(\alpha n \Gamma) + \nabla \cdot (\alpha n \Gamma \boldsymbol{v}) = 0, \tag{15}$$

which could, just as well, be obtained when the decomposition (9) is introduced in the covariant equation $(nU^\mu)_{;\mu} = 0$.

For the homogeneous Maxwell equations, one defines the dual electromagnetic tensor

$$F^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\tau} F_{\rho\tau} = B^\mu n^\nu - B^\nu n^\mu + \epsilon^{\mu\nu\rho\tau} E_\rho n_\tau, \tag{16}$$

which satisfies $F^{*\mu\nu}{}_{;\nu} = 0$ by its antisymmetry. When projected onto n_μ, we find the timelike decomposition $B^\nu{}_{;\nu} = 0$, alternatively written as[11–13]

$$\nabla \cdot \boldsymbol{B} = 0. \tag{17}$$

The spacelike projection, $\gamma^\beta{}_\mu B^\mu{}_{;\nu} n^\nu + \epsilon^{\beta\nu\rho\tau}(E_\rho n_\tau)_{;\nu} = 0$, has the vectorial equivalent[11–13]

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times (\alpha \boldsymbol{E}), \tag{18}$$

the GR version of Faraday's law. Equations (13), (14), (17), and (18) constitute the Maxwell's equations in curved spacetime in the 3 + 1 decomposition. They are rather similar to Maxwell's equations in flat space: the spacetime curvature effects enter explicitly through the lapse function α.

Now we turn to the 3 + 1 formulation of the total unified dynamics of the magnetofluid embedded in curved spacetime [Eq. (5)]. Because of the antisymmetry of $M^{\mu\nu}$, the decomposition will be analogous to that for $F^{\mu\nu}$. In terms of the generalized electric (ξ^μ) and magnetic (Ω^μ) fields

$$\xi^\mu = n_\nu M^{\nu\mu}, \quad \Omega^\mu = \frac{1}{2} n_\rho \epsilon^{\rho\mu\sigma\tau} M_{\sigma\tau}, \tag{19}$$

both spacelike ($n_\mu \xi^\mu = 0$ and $n_\mu \Omega^\mu = 0$), the magnetofluid tensor reads similar to Eq. (12)

$$M^{\mu\nu} = \xi^\mu n^\nu - \xi^\nu n^\mu - \epsilon^{\mu\nu\rho\sigma}\Omega_\rho n_\sigma. \tag{20}$$

The detailed form of the vector fields ξ^μ and Ω^μ can be worked out using the definition $M^{\mu\nu} = F^{\mu\nu} + (m/q)S^{\mu\nu}$ and the four-velocity (9); the three-vector generalized electric and magnetic fields

$$\boldsymbol{\xi} = \boldsymbol{E} - \frac{m}{\alpha q}\nabla(f\alpha^2\Gamma) - \frac{m}{\alpha q}\frac{\partial}{\partial t}(f\Gamma\boldsymbol{v}), \tag{21}$$

$$\boldsymbol{\Omega} = \boldsymbol{B} + \frac{m}{q}\nabla \times (f\Gamma\boldsymbol{v}), \tag{22}$$

are the curved spacetime generalization of the corresponding vector fields defined in Refs. 1 and 2. We remind the reader that in our usage, the generalized magnetic field $\boldsymbol{\Omega}$ is synonymous with the generalized vorticity, GV. Evidently, general relativity enters the definition of GV through Γ.

Substituting the fields (ξ^μ and Ω^μ) into Eq. (20), the covariant equation of motion (5) converts to

$$\alpha\Gamma\xi^\mu - \Gamma v_\nu \xi^\nu n^\mu - \Gamma v_\nu \epsilon^{\mu\nu\rho\sigma}\Omega_\rho n_\sigma = \frac{T}{q}\sigma^{,\mu}, \tag{23}$$

from which the 3 + 1 equations are obtained by appropriate projections on the timelike and spacelike hypersurfaces. The n^μ projection, $\Gamma v_\mu \xi^\mu = (T/q) n_\mu \sigma^{,\mu}$, is the equation for energy conservation

$$q\alpha\Gamma\boldsymbol{v}\cdot\boldsymbol{\xi} = -T\frac{\partial\sigma}{\partial t}, \tag{24}$$

while the $\gamma^\beta{}_\mu$ projection, $\alpha\Gamma\xi^\beta + \Gamma n_\tau \epsilon^{\tau\beta\nu\rho} v_\nu \Omega_\rho = (T/q)\sigma^{,\beta}$, yields the momentum evolution equation

$$\alpha\Gamma\boldsymbol{\xi} + \Gamma\boldsymbol{v}\times\boldsymbol{\Omega} = \frac{T}{q}\nabla\sigma. \tag{25}$$

The charge q (and mass m), referring to the constants attributes of the "particles" that make up the fluid, pose no conceptual problems when we ignore the back reaction. Thermodynamic quantities like temperature, T, entropy density, σ, and f are more problematic. One would need to formulate more clearly what they signify in the strong field region near, for instance, the surface of a black hole. In the current work, we assume that thermodynamical properties belong to the "test matter" plasma, where the normal definitions are adequate in the chosen frame.

Equations (24) and (25) may look somewhat unfamiliar. However, it is possible to show that they are equivalent to the usual 3 + 1 plasma equations[12,13] invoked in plasma literature. The effects of the interaction of the fluid with the local gravitational acceleration are hidden in the definition of the unified fields. This is spelled out in Appendix A.

There is a very strong reason for the use of Eqs. (24) and (25) instead of other extant formalisms. The unified magnetofluid approach, epitomized in Eqs. (24) and (25), is a very powerful tool that leads us directly to the general vortical form of the plasma equations. It is in this form that the sources of general vorticity (magnetic fields being a part) are explicitly revealed, and it becomes relatively easy to develop an encompassing theory for the generation of general vorticity. Equations (24) and (25) are expected to be as effective in isolating the sources of vorticity in curved spacetime as their special relativistic antecedents.

We have yet to derive the promised "vortical" plasma system in curved spacetime. The antisymmetry of the unified tensor $M^{\mu\nu}$, in analogy with $F^{\mu\nu}$, implies that its dual must obey $M^{*\mu\nu}{}_{;\nu} = 0$. The 3 + 1 decomposition of this equation, equivalent to Eqs. (17) and (18), will lead to a spacelike projection

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = -\nabla \times (\alpha \boldsymbol{\xi}). \tag{26}$$

When the constraint (26) is used in Eq. (25), we arrive at

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{\Omega}) = \boldsymbol{\Xi}_B + \boldsymbol{\Xi}_R, \tag{27}$$

which has, precisely, the standard vortical form. $\boldsymbol{\Xi}_B$ and $\boldsymbol{\Xi}_R$, explicitly displayed on the right hand side, are the possible sources of the vorticity $\boldsymbol{\Omega}$. Both these drives are nonzero only for inhomogeneous thermodynamics. The first one is the traditional baroclinic term[1,2]

$$\boldsymbol{\Xi}_B = -\left(\frac{1}{q\Gamma}\right) \nabla T \times \nabla \sigma, \tag{28}$$

corrected by curvature. The non-relativistic limit of this term, called the Biermann battery, has been extensively studied. The second term, the general relativistic drive

$$\begin{aligned} \boldsymbol{\Xi}_R &= \frac{T}{q\Gamma^2} \nabla \Gamma \times \nabla \sigma \\ &= \frac{T\Gamma}{2q} [-\nabla \alpha^2 + \nabla(\gamma_{ij} v^i v^j)] \times \nabla \sigma \end{aligned} \tag{29}$$

is the principal object of this search. The terms $\boldsymbol{\Xi}_B$ and $\boldsymbol{\Xi}_R$ are the non-magnetic thermodynamic source terms that create the conditions for driving the linear growth of the magnetic fields from a zero initial value. In this sense, these drives act as batteries.

Although the baroclinic term given in Eq. (28) is somewhat modified by the curved spacetime metric $g_{\mu\nu}$ (through Γ and ∇), there is no dramatic or qualitative change. The relativistic drive $\boldsymbol{\Xi}_R$, however, is radically transformed from its flat space antecedent[1,2] to which it reduces in the appropriate limit. The striking result is that the gravitational potential, through the gradients of g_{00} (or α), can produce a magnetic field in any region populated by charged particles even if their local velocities are negligible. The origin of this effect is the interaction of the gradient of the gravity-modified Lorentz factor Γ with the inhomogeneous thermodynamics of the plasma. This phenomenon could be called a gravitomagnetic battery.

We expect this result to have many astrophysical consequences. In particular, we can compare the strengths of the baroclinic term and the general relativistic drive. If the baroclinic drive is nonzero, then

$$\frac{|\boldsymbol{\Xi}_R|}{|\boldsymbol{\Xi}_B|} \approx \frac{l\,\Gamma^2}{\sqrt{1 - r_0/r}} |\nabla \alpha^2 - \nabla(\gamma_{ij} v^i v^j)|, \tag{30}$$

where $l/\sqrt{1 - r_0/r}$ is the scale length of variation of the temperature corrected by the curvature. If l is similar to the scale length of the variations of the relativistic effects of the plasma, then the GR drive can be much more important than the baroclinic term when $\Gamma^2 \gg 1$ (i.e., when $\alpha^2 - \gamma_{ij} v^i v^j \ll 1$). On the other hand, note that the GR drive can be relevant even if the plasma velocities are negligible in some special case configuration. In conclusion, when the plasma is under strong gravitational fields and/or high relativistic effects, the GR drive is the more relevant source for the magnetic field generation. In flat spacetime, the relativistic drive will be important only for relativistic velocities.[1]

Before estimating the magnitude of this drive, we would like to emphasize that this first conceptual paper will be limited to demonstrate the existence of it for the creation of vorticity/magnetic field. More detailed and rigorous calculations and their consequences will be submitted in a subsequent paper. It is true that a simple-minded extension of special relativistic notions to curved spacetime can cause conceptual problems. There are two ingredients required for the special relativistic mechanism to work: (a) an inhomogeneous stress-energy tensor and (b) a preferred direction provided by the "boost." When we go to, say, a Schwarzschild spacetime, these ingredients are missing. The following prescription provides the proper framework for extending the formalism to curved spacetime. We can put the inhomogeneity into some non self-gravitating "test matter" that has been neglected compared with the mass of the Schwarzschild entity. We must, similarly, rely on the "test matter" to provide the second ingredient. Both would be provided, for example, by the plasma near the black hole.

IV. ESTIMATES FOR VORTICITY GENERATION

Though the main result of this paper is the analytic expression (29), we will now estimate the strength of the relativistic drive and vorticity/magnetic field in a very simplified model of a plasmas in curved spacetime. Consider an accretion plasma disk around a Schwarzschild black hole. The relevant space-time metric elements are $\alpha^2 = 1 - r_0/r$, $\gamma_{rr} = \alpha^{-2}$, $\gamma_{\theta\theta} = r^2$, and $\gamma_{\phi\phi} = r^2 \sin^2\theta$, where $r_0 = 2MG/c^2$ is the Schwarzschild radius, M is the mass of the black hole, G is the gravitational constant, and r is the radial distance to the plasma matter (from now on we will put c explicitly in the calculations). We will estimate the GV in two representative cases: (1) for a plasma element in a stable orbit at $5r_0$ and (2) for a free-falling plasma element near the Schwarzschild radius.

A. Seed generation in an accretion plasma

We assume that the plasma is in a thin accretion disk and moves in the equatorial plane ($\theta = \pi/2$) of the disk with zero azimuthal speed, $\dot{\theta} = 0$. For an in-spiral motion for thin disks, the orbital velocity can be estimated to be Keplerian, $v^\phi = r\dot{\phi} = c\sqrt{r_0/2r}$; we will assume it to be larger than the radial velocity at which matter falls into the black hole, $v^\phi \gg v^r$.[26,27] We can only ensure this sufficiently far from the gravitational source and would, therefore, miss the really strong-field effects. To ensure relatively stable orbits about the black hole, we will locate the plasma disk at about $5r_0$, where we could neglect the radially inward component of the velocity.

In addition to the spatial variations of the metric tensor, the drive $\mathbf{\Xi}_R$ depends on the gradients of the entropy density. At $5r_0$, the usual definition for entropy[9] is valid since the nonlinearity of the gravitational field is not dominant, If, in addition, the plasma obeys a barotropic equation of state, i.e., the pressure is a function of density, $\sigma = F(T)$, then $(T/c)\nabla\sigma \equiv \zeta k_B \nabla T$, where ζ is of order unity.

Note that for a plasma with this kind of an equation of state, the baroclinic drive $\mathbf{\Xi}_B$ vanishes because $\nabla\sigma \propto \nabla T$; the only source left for generating a magnetic field is the general relativistic drive.

In the 3 + 1 decomposition, the gradient of a scalar field P, $\nabla P = (1 - r_0/r)^{1/2}\,\partial_r P\,\hat{e}_r + (1/r)\partial_\theta P\,\hat{e}_\theta + (1/r\sin\theta)\,\partial_\phi P\,\hat{e}_\phi$, has the factor $(1 - r_0/r)^{1/2}$ coming from the radial metric coefficient. For the model described above, the GR drive (in the equatorial plane $\theta = \pi/2$) becomes

$$\mathbf{\Xi}_R = \frac{3\zeta c k_B r_0 \alpha}{4\,e\,r^3}\left(1 - \frac{3r_0}{2r}\right)^{-1/2}\frac{\partial T}{\partial\phi}\hat{e}_z\,, \tag{31}$$

where the variations of the temperature have been taken in cylindrical geometry, we have used the electron charge $q = -e$, and we have simplified the thin disk model by neglecting the toroidal temperature gradients compared with the poloidal variations, $\partial_\theta T \ll \partial_\phi T$.

All the charged matter of the accretion disk contributes to $\mathbf{\Xi}_R$ and therefore acts as a source for $\mathbf{\Omega}$. Since $\mathbf{\Xi}_R \to 0$ for $r \to \infty$, the contribution from matter relatively close to the compact object will be dominant. Notice that the relativistic drive has a net flux in the $\hat{e}_z$ direction. For the stable orbit at $r = 5r_0$, the relativistic drive (31) simplifies to

$$\mathbf{\Xi}_R \approx \frac{3\zeta c k_B}{500\,e\,r_0^2}\frac{\partial T}{\partial\phi}\,\hat{e}_z \tag{32}$$

and is proportional to the temperature of the disk. We can assume that the complete accretion disk radiates like a black-body with an average temperature $\bar{T} = \int \partial_\phi T d\phi \approx 5\times 10^7\,(M_\odot/M)^{1/4}$K (Ref. 26), where $M_\odot$ is the solar mass. It is easy to see that as long as the black hole mass $M \geq 10^{-2}M_\odot$, $x = mc^2/k_B T \gg 1$, and the plasma temperature remains non-relativistic.

Under these conditions, the total relativistic drive (32) produced by the plasma in the thin ring of matter centered around $r = 5r_0$ can be estimated as

$$\mathbf{\Xi}_{R\text{total}} = \int_0^{2\pi} d\phi\,\mathbf{\Xi}_R \approx 3\times 10^{-2}\zeta\left(\frac{M_\odot}{M}\right)^{9/4}\hat{e}_z. \tag{33}$$

Substituting the simplified drive into Eq. (27), the GV generated by the space-time curvature can be calculated. Let us begin with an initial state with zero GV. For some short enough time ς (the initial seed generation phase), when the nonlinear terms involving $\mathbf{\Omega}$ are negligible, $\mathbf{\Omega}$ grows linearly with time: $\mathbf{\Omega}_{\text{total}} \approx \mathbf{\Xi}_{R\text{total}}\varsigma$. To estimate the growing time ς, we notice that the linear proportionality cannot hold when the nonlinear term in Eq. (27) is comparable to $\mathbf{\Xi}_{R\text{total}}$. A good measure of ς is provided by the relation $|\mathbf{\Omega}_{\text{total}}|\varsigma^{-1} \simeq |\nabla\times(\boldsymbol{v}\times\mathbf{\Omega}_{\text{total}})|$ implying that $\varsigma \simeq L/|\boldsymbol{v}|$, where L is the length of variation of the $|\boldsymbol{v}\times\mathbf{\Omega}|$ force. Taking the length L on which $|\boldsymbol{v}|$ varies to be of the order of the (curvature corrected) variation scale $5r_0/\alpha$, the time for initial linear phase of GV seed formation may be approximated as

$$\varsigma = \frac{5r_0}{|\boldsymbol{v}|\alpha} \approx 1.7\times 10^{-4}\left(\frac{M}{M_\odot}\right), \tag{34}$$

measured in seconds, where we have assumed that the velocity is of order v^ϕ. Thus, the total strength of the magnetic field generated (in gauss) for the "test" plasma matter accreting at a distance $5r_0$ is

$$|\mathbf{\Omega}_{\text{total}}| \approx 5\times 10^{-6}\zeta\left(\frac{M_\odot}{M}\right)^{5/4} \tag{35}$$

and lies in the $\hat{e}_z$ direction. For a black hole of stellar mass ($M \approx M_\odot$), the maximum generated magnetic field seed is found to be of the order of $|\mathbf{\Omega}_{\text{total}}| \approx 5\times 10^{-6}$G.

It is important to realize that this initial seed is supposed to be small. It is what is created in a very short initial time in a state where there was, precisely, no magnetic field to begin with. The existence of this seed is crucial to the very startup of the standard processes of long-time magnetic field generation, like the dynamo process or the magneto-rotational instability. The dynamo process that converts short scale fluid vorticity into long term magnetic field (electromagnetic vorticity) can operate only when it has some initial magnetic field to amplify; we have just shown that the general relativistic drive can, precisely, provide the needful.

B. Strong field generation near the horizon

One expects that the GR drive will get considerably stronger as our test plasma moves closer and closer to the event horizon at r_0. To get an idea of the strength of the drive, here we do a very simple, somewhat crude, calculation. A more sophisticated treatment, including various QED plasmas effects,[30] is left for future work.

When the plasma is near the horizon (there are no stable particle orbits), we may approximate it as a fluid in free fall with a purely radial velocity v^r. As the fluid element approaches the horizon (in $r = r_0$), the radial velocity and the Lorentz factor roughly go, respectively, as $v^r \approx c\alpha^2 \sqrt{r_0/r}$ (measured in the universal time) and $\Gamma \approx 1/\alpha^2$, so that $\Gamma v^r \approx c\sqrt{r_0/r}$.[27] Then, the relativistic drive (29) is

$$\Xi_R \approx \frac{\alpha\zeta c k_B}{e r_0^2}\frac{\partial T}{\partial \phi}\hat{e}_z. \qquad (36)$$

Note that the drive, as always, is inhomogeneity-driven and needs a non-radial gradient of the plasma temperature. The growing time ς may be estimated like we did in the farther accretion region. As the plasma location approaches the horizon, the growing time $\varsigma \approx r_0/(c\alpha^3)$, leading to a simple estimate

$$|\mathbf{\Omega}_{\rm total}| \approx \frac{\zeta k_B T}{\alpha^2 e r_0}, \qquad (37)$$

for the total GV generated. Despite the crude approximations invoked to obtain the total GV (37), we have found quite a spectacular result. Since $\alpha^2 = 1 - r_0/r \to 0$ near the horizon, enormous GV (magnetic field) can be generated by the mechanism investigated in this paper. This mechanism, if it survives more thorough examination (via, perhaps, detailed numerical calculations), could provide just the strong guide field that could collimate escaping plasma particles and advance our understanding of the formation of astrophysical jets. The temperature T in Eq. (37) is only a perturbation of the homogenous spherical symmetric temperature of the free-falling plasma.[27] We assume that this kind of perturbations will always be present. Even for small temperature perturbations, the result (37) shows that the GV can be very large near the horizon.

V. DISCUSSION

We have demonstrated the existence of a gravito-magnetic battery mechanism [with strength given by a source term (29)] for generating the seed vortex/magnetic field in astrophysical and cosmic settings. The battery action is created by a fundamental interaction between two inhomogeneities; inhomogeneity in gravity (through the Lorentz factor of the fluid modified by spacetime curvature) and in plasma thermodynamics; both elements are essential.

The current theory is quite unlike other classical theories that invoke, for instance, the difference in the *e*/*m* ratio between protons and electrons to create initial currents or those that introduce drag effects in the electron motion (Compton drag) to create initial currents in the context of cosmology.[26] Besides, the gravito-magnetic battery mechanism presented here has the advantage that it is nonzero when the standard Biermann battery is null. The Biermann battery, driven by the baroclinic term, is rather difficult to operate because the variations of temperature and entropy tends to align in thermodynamical equilibrium, $\nabla T \times \nabla\sigma = 0$. It is likely that in most cases of interest, the general relativistic drive would be the only source to produce seed vorticity and magnetic fields.

Though the most important result of this paper is contained in the analytic formula (29), we have chosen to explicitly estimate the strength of the generated vorticity for two representative cases: (1) The plasma is an accretion disk located around $5r_0$ from the black hole. In this relatively weak field region with stable particle orbits, one is interested in calculating the seed field that could be a progenitor, for instance, of a dynamo action; (2) the plasma is in free fall near the horizon; the idea is to see if a strong enough magnetic field can be created for jet formation.

For the first scenario, two explanatory remarks are in order: (1) Parity breaking in the gravitational field, involved in generating a magnetic field, need not be worrisome because of the opposite parities of the gravitational and electromagnetic fields; (2) there is no guarantee that the particles in the stable orbit (at $r = 5r_0$) will not fall into the black hole. We follow here the standard assumption made in astrophysics that though some matter will escape from the stable region, other matter will replace it. In that sense one could think of invoking above estimates for orbits closer than this limit. However, the timescale for infall and the breakdown of the assumption that the speed of infall is negligible prevents such an extension.

It is also worthwhile to contrast our mechanism with relativistic effects like the Blandford-Znajek (B-Z),[28] used in modeling active galactic nuclei, quasars, and gamma ray bursters; the latter deals with strong fields, highly energetic events. By contrast, this seed creating mechanism pertains to a test plasma in a relatively stable orbit around the black hole. Further, in our analysis it is the plasma that is "rotating" while in the B-Z case it is the gravitational source that is spinning.

The vorticity/magnetic field obtained in Eq. (35) is rather small. However, it is more than adequate as a crucial seed field to drive a dynamo amplification. Gravity, in this, case just gets the process started, the eventual energy for field generation in the accretion disk comes from short scale velocity turbulence.

The literature is full of mechanisms, explored only for the purpose of producing small seeds of magnetic fields: the rotation of black holes[34,38] and the radiation force on electrons,[39] being two examples. As we said before, during the short phase where nonlinear effects can be neglected, the magnetic field can grow (from a state of zero field) linearly with time. Once created, these seeds can grow further by a variety of nonlinear processes. The dynamo mechanism is, of course, one of the most investigated mechanisms for black holes,[40,41] where the rotation of the black hole can introduce a new effect which is added to the known $\alpha - \Omega$ dynamo.[42,43] Nonlinear effects can, in addition, provide long range order to the generated magnetic fields. Well known examples are the *shearing* of the magnetic field, and Parker's mechanism.[29] Both these ideas pertain to rotating objects in which poloidal (toroidal) magnetic field lines transform into toroidal (poloidal) ones.

The motivation for the second part of the calculation, where we deal with a test plasma in the vicinity of the horizon, is entirely different. A very rough estimate shows that the gravito-magnetic drive (37) turns out to be very strong in this neighborhood. A detailed study, however, is needed: (1) to calculate the long time evolution of the growth of GV because the initial stage of linear growth will soon yield to the nonlinear stage and (2) to incorporate other effects due to the horizon than those considered in this work.[30] Based on our rough estimates, we can argue that the general relativistic drive can be a source of large magnetic fields. Here, unlike the accretion disk case, it is gravity that is directly feeding GV and the magnetic field. The curvature driven magnetic field (very near the horizon) may be just what we need for collimating jets of charged matter emitted from the accretion disk of compact objects. Again there are a variety of mechanisms proposed to explain jet collimation; these mechanisms make varied assumptions about the plasmas, the compact objects or the inertial effects of the jet.[26,44] Discussing the jet collimation within the framework of this model will be taken up in future work.

The final magnetic field [the general relativistic drive (29)] is created intrinsically by the effect of the curvature in combination with the properties of the plasma matter accreting onto the black hole. However, we have not, yet, examined the possibility of a black hole acquiring a magnetic field due to the magnetized matter falling into it (see the membrane paradigm[11]). Even more exciting is the possibility of the magnetic field being generated without an accretion disk plasma. There is reason to believe that astrophysical black holes spin.[31–33] A spinning black hole would provide the preferred direction and cause neutral matter to get ionized and become a magnetic plasma. The procedure adopted here, of using the rest frame, would no longer be available due to frame-dragging.[23] The generalization of the previous $3+1$ decomposition to the Kerr metric will be used in that case. We find that the rotation of the black hole contributes to the general relativistic drive. Similar results for magnetohydrodynamics have been suggested.[34] On the other hand, it would not be necessary to use the full Kerr metric for a slowly rotating black hole. We could use the Lense-Thirring effect[35–37] for a semi-classical analysis without too much additional complication. This is left for future work.

Finally, we believe worth to mention that Giovannini and Rezaei[6] have recently proposed a similar mechanism in cosmological context, where gravity is the source for a primordial vorticity.

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APPENDIX A: EQUATIONS IN FLUID VARIABLES

We can write the energy conservation equation (24) and the momentum equation (25) in terms of fluid variables instead of unified fields. Using the definition (21) for ξ, the energy conservation equation becomes

$$\frac{1}{\alpha}\frac{\partial e}{\partial t}+\frac{1}{\alpha}\nabla\cdot(h\alpha^2\Gamma^2\boldsymbol{v})=qn\Gamma\boldsymbol{E}\cdot\boldsymbol{v}-h\Gamma^2\boldsymbol{v}\cdot\nabla\alpha\,, \qquad \text{(A1)}$$

where the energy density is $e=h\alpha^2\Gamma^2-p$. Note that the last term is the interaction of the fluid with the local gravitational acceleration. In the same way, using Eq. (22) for $\boldsymbol{\Omega}$, the momentum equation could be written as

$$\frac{1}{\alpha}\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)(\alpha h\Gamma^2\boldsymbol{v})=qn\alpha\Gamma\boldsymbol{E}+qn\Gamma\boldsymbol{v}\times\boldsymbol{B}-\frac{\nabla(\alpha p)}{\alpha}-h\Gamma^2\boldsymbol{v}(\nabla\cdot\boldsymbol{v})-\frac{e\nabla\alpha}{\alpha}. \qquad \text{(A2)}$$

This equation resembles the form of the plasma fluid dynamical equation in special relativity, where now the effects of general relativity are introduced via the lapse function and the Γ factor. Again, the gravitational acceleration effect is in the last term. The preceding two equations can as well be obtained using the formalism developed in Refs. 12 and 13. In this case, the starting point is the $3+1$ decomposition of the plasma energy-momentum tensor

$$T^{\mu\nu}=en^\mu n^\nu-n^\mu s^\nu-n^\nu s^\mu+W^{\mu\nu}\,, \qquad \text{(A3)}$$

where $s^\mu=\alpha\Gamma^2\gamma^\mu{}_\nu v^\nu$ is the energy flux and $W^{\mu\nu}=h\Gamma^2\gamma^\mu{}_\beta\gamma^\nu{}_\phi v^\beta v^\phi+p\gamma^{\mu\nu}$ is the stress tensor. This energy-momentum tensor is also obtained when the decomposition for the four-velocity (9) is used in Eq. (3).

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