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Duality of the Lagrangian and Eulerian representations of collective motion—a connection built around vorticity

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Abstract
To allow a non-zero 'vorticity' associated with the generalized momentum, the Lagrangian describing general fluid-mechanical collective motions must incorporate a non-canonical structure. The canonical formalism, symbolized by the basic Hamilton–Jacobi equation $\boldsymbol{P} = \nabla S$ relating the momentum '$\boldsymbol{P}$' with the action 'S', does not permit finite vorticity. The Lagrangian in the Eulerian view (suited for coupling with other fields such as the electromagnetic) must include 'topological constraints' embodying this non-canonical feature. Analyzing the role of the abstract fields (introduced as Lagrange multipliers) constituting the constraints, we may unify the Lagrangians in both Eulerian and Lagrangian views. Relativistic (Lorentz-invariant) formulation reveals the natural meaning of the Clebsch parametrization.

1. Introduction

From a non-linear field theory perspective, a 'flow' is hard to categorize; adjectives invoked to describe it mostly tell us what it is not—non-linear, non-Hermitian, non-canonical, non-commutative (even non-Abelian etc). Classical fluid mechanics, thus, still constitutes a rich resource of basic concepts that need further elucidation; such concepts could be just as relevant to other fields.

The existence of 'vorticity' or that of the fluid 'helicity' creates the central problem that makes totally non-trivial the derivation of fluid mechanics by the standard methodology. The very basic relation $\boldsymbol{P} = \nabla S$ of the Hamilton–Jacobi formalism forces zero vorticity ($\nabla\times\boldsymbol{P} \equiv 0$), and consequently, an identically zero-helicity density for the vector field $\boldsymbol{P}$ (conventionally named the momentum). A potential flow $\boldsymbol{P} = \nabla S$ is just right for single-particle dynamics because it is devoid of a notion of a collection (bundle) of orbits (the helicity is a geometric index measuring the twists of multiple orbits). For $\boldsymbol{P}$ to be an appropriate representation for the 'fluid' momentum, its vorticity must be allowed to be non-zero, for instance, to allow the streamlines to make linkages. To affect the required departure from Hamilton–Jacobi type relations, one has to invoke appropriate 'topological constraints' so that the vector field $\boldsymbol{P}$ is imparted a finite helicity (vorticity).

Following the pioneering work of Serrin [1] on the Lagrangian formalism of a fluid, which could produce only a potential flow, different fluid-mechanical Lagrangians have been proposed. Based on different perspectives, we have two such groups: one invokes the use of a Lagrangian description of fluid motion, and the other represents fluids by Eulerian fields such as the momentum $\boldsymbol{P}$. Or, in mathematical terms, the former formulates fluid mechanics by an adjoint operator on a Lie algebra space, while the latter can be regarded as the dual of the former, i.e. a flow is a coadjoint operator on the dual vector space of the Lie algebra space [2]. The aim of this effort is to delineate an explicit relation between these two representations, and to introduce a unified, Lorentz-covariant Lagrangian that can be applied, inter alia, to construct an efficient and accurate scheme for numerical analysis or perturbation theory.

The classical Lagrangian description for particle motion may be readily extended for a continuum by invoking the *diffeomorphism* that describes the displacements of matter in the Lagrangian view (section 2). While particle classical mechanics is best framed in the Lagrangian view, standard field theories tend to be Eulerian (section 3). One of the most essential problems of the Lagrangian description is that the very existence of a diffeomorphism collapses when a dissipation is introduced as a singular perturbation (physically, the streamlines may annihilate/create, reconnect or diffuse). Yet, the Lagrangian description is, by its canonical Hamiltonian structure, very powerful, especially in the study of ideal instabilities [3, 4]. It is in the quest for an Eulerian representation that the incorporation of vorticity (and helicity) emerges as a problem.

As is well known, an ideal fluid obeys a 'non-canonical' Hamiltonian system [5]—the generator of the infinite-dimensional dynamics is defined by a degenerate symplectic operator, and its 'defect' (the cokernel of the symplectic operator) represents a topological constraint on the *helicity* (Casimir invariant) associated with the fluid vorticity. Interestingly, the use of the *Clebsch parametrization* of the flow field (with a finite vorticity) allows us to construct a 'canonical' system of equations. By introducing 'parasite variables' as Lagrange multipliers on 'abstract constraints', which alchemize into the Clebsch variables parametrizing the flow field, we can formulate a Lagrangian that is capable of describing a finite vorticity system [6–9].

However, the physical meaning of the 'abstract constraints' is not obvious; some authors leave them just as 'some constraints' pertinent to streamlines, and others suggest to connect one of them to entropy conservation; Kambe [10] proposed a gauge-field implication of the vorticity. Suspending the physical interpretation, the technical procedure is elegant (for mathematical interpretation of Clebsch variables, see [11–14]). Via the method of Lagrange multipliers for constrained variational principle (immersing the constrained manifold into an extended parameter space), we can convert implicit constraints into explicit (linear) contact conditions (the non-canonical Poisson bracket can be derived from a canonical bracket in an extended parameter space [15, 16]). However, we note that the formal 'canonicalization' by introducing parasite variables goes in the opposite direction from the naïve strategy of resolving the topological constraint by removing the constrained degree of freedom, i.e. separating the cokernel of the symplectic operator. By analyzing the *range* of the Clebsch parametrization [17], the 'pleat' hiding the superfluity of the seemingly canonical variables will become clear (section 4).

In this work, we will show that the 'constraints' are those of the initial condition (indeed, while introducing the constraints, Lin [6] intended them to be precisely the Lagrangian coordinates) by delineating an explicit relation between the Lagrangian label and the Clebsch variables. The former determine the initial identity of each infinitesimal fluid element while the latter turn out to be the Eulerian counterpart of Lagrangian label [18]. We will, then, establish the equivalence of the Lagrangians in the two views (section 5) as the 'dual' representation of the motion.

In section 6, we will use the Eulerian displacement as a field variable, and formulate a new type of variational principle. The Lorentz-invariant form of the Lagrangian takes a very simple form, revealing the meaning of the Clebsch parametrization (section 7).

2. Fluid mechanics in the Lagrangian view

To set the stage for our discussions on the comparison of the Lagrangian and Eulerian formalisms, we begin with the Lagrangian of a non-relativistic particle (mass m and charge e) in the presence of an electromagnetic (EM) field:

$$\mathcal{L} = \mathcal{L}_P + \mathcal{L}_{\mathrm{EM}}, \tag{1}$$

$$\mathcal{L}_P = \boldsymbol{P}\cdot\boldsymbol{v} - H, \tag{2}$$

$$\mathcal{L}_{\mathrm{EM}} = \int L_{\mathrm{EM}}\,\mathrm{d}^3x = \int -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,\mathrm{d}^3x, \tag{3}$$

where

$$H = \frac{p^2}{2m} + e\phi$$

is the Hamiltonian,

$$\boldsymbol{P} = \boldsymbol{p} + \frac{e}{c}\boldsymbol{A}$$

is the canonical momentum, and $\boldsymbol{p}$ is the mechanical momentum. The vector and the scalar potentials define the four potential $A^\mu = (\phi, \boldsymbol{A})$ whose curl is the Faraday (field strength) tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The velocity $\boldsymbol{v}$ is related to the particle orbit $\boldsymbol{q}(t)$ (along which $\mathcal{L}_P$ is to be evaluated) by

$$\boldsymbol{v} = \dot{\boldsymbol{q}}. \tag{4}$$

The relation (4) is the essential input that 'causes' the motion of the particle. Indeed, if we were to calculate the variation of $\int L\,\mathrm{d}t$ for a general $\delta\boldsymbol{v}$, we would find $\boldsymbol{P} = 0$, and it is only for the variation $\delta\dot{\boldsymbol{q}}$ (with fixed end points $\boldsymbol{q}(t_0)$ and $\boldsymbol{q}(t_1)$), the well-known Lagrange equation of motion follows.

Generalizing the single-particle orbit $\boldsymbol{q}(t)$ to a diffeomorphism $\boldsymbol{Q}(\boldsymbol{x}_0, t)$ in $\boldsymbol{R}^3$ ($\boldsymbol{x}_0$ is the initial position of each streamline), and introducing a particle-number density n, we construct the fluid Lagrangian

$$\mathcal{L}_F = \int L_F\,\mathrm{d}^3x = \int (\boldsymbol{P}\cdot\boldsymbol{V} - H_F)\,n\,\mathrm{d}^3x \tag{5}$$

by replacing the single-particle velocity $\dot{\boldsymbol{q}}$ in (4) by the flow velocity

$$\boldsymbol{V}(\boldsymbol{x}, t) = \dot{\boldsymbol{Q}}|_{\boldsymbol{x},t} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Q}(\boldsymbol{x}_0(\boldsymbol{x}, t), t). \tag{6}$$

Here, the time derivative $\mathrm{d}/\mathrm{d}t$ is evaluated along each streamline (orbit of the fluid element) staring from $\boldsymbol{x}_0$; we denote by $\boldsymbol{x}_0(\boldsymbol{x}, t)$ the initial position of the streamline being connected to the space-time position $(\boldsymbol{x}, t)$. To relate $(\boldsymbol{x}, t)$ to $(\boldsymbol{x}_0, 0)$, one needs the inverse map $\boldsymbol{x}_0(\boldsymbol{x}, t) := \boldsymbol{Q}^{-1}(\boldsymbol{x}, t)$ of the diffeomorphism $\boldsymbol{Q}(\boldsymbol{x}_0, t)$, which traces-back the streamlines. Since $\boldsymbol{x}_0$ is invariant on each streamline, $\mathrm{d}\boldsymbol{x}_0(\boldsymbol{x}, t)/\mathrm{d}t = 0$ when $\mathrm{d}/\mathrm{d}t$ denotes the *Lagrangian derivative* to be defined in (9), the time derivative on (6) is at fixed $\boldsymbol{x}_0$ (the relations among $\boldsymbol{x} = \boldsymbol{Q}(\boldsymbol{x}_0, t)$, $\boldsymbol{x}_0 = \boldsymbol{Q}^{-1}(\boldsymbol{x}, t)$, and $\boldsymbol{V}$ will be discussed more explicitly in section 5).

The fluid Hamiltonian (density) H_F consists of the kinetic and potential energies plus an 'internal (thermal) energy' $\varepsilon(n)$:

$$H_F = H + \varepsilon(n). \tag{7}$$

Here, we consider a 'barotropic fluid' where ε is a function of only n ('homentropic' fluid is a subclass). The fluid pressure $\mathcal{P}$ may be defined by $\mathrm{d}\mathcal{U} = -\mathcal{P}\mathrm{d}V$ where $\mathcal{U} = \varepsilon n V$ is the total internal energy, and V is the volume. The particle conservation law $\mathrm{d}(nV) = 0$ leads, then, to the relation $\mathrm{d}\varepsilon = (\mathcal{P}/n^2)\mathrm{d}n$. The density n must obey

$$n(\boldsymbol{x}, t) = n_0(\boldsymbol{x}_0(\boldsymbol{x}, t)) \cdot \frac{D(\boldsymbol{x}_0)}{D(\boldsymbol{x})}, \tag{8}$$

where $D(\boldsymbol{x}_0)/D(\boldsymbol{x})$ is the Jacobian of the transformation $\boldsymbol{x} \mapsto \boldsymbol{x}_0$. With the formal choice $n_0(\boldsymbol{x}_0) = \delta(\boldsymbol{q}_0 - \boldsymbol{x}_0)$, (1) is readily recovered with $n\mathrm{d}^3x\mathrm{d}t$ giving the integral along the orbit $\boldsymbol{q}(t)$.

In what follows we denote

$$D_t f = \partial_t f + \boldsymbol{V} \cdot \nabla f, \tag{9}$$

$$D_t^* f = \partial_t f + \nabla \cdot (\boldsymbol{V} f). \tag{10}$$

The particle conservation law

$$D_t^* n = 0 \tag{11}$$

is a direct consequence of (8). By the criticality of the action $\int(\mathcal{L}_F + \mathcal{L}_{\mathrm{EM}})\,\mathrm{d}t$, fixing the space-time boundaries, we obtain, from the variation $\delta\boldsymbol{p}$, $\boldsymbol{p} = m\boldsymbol{V}$ $(=m\dot{\boldsymbol{Q}})$, and from $\delta\boldsymbol{Q}$, the equation of motion

$$m D_t \boldsymbol{V} = -\nabla h + e\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{V} \times \boldsymbol{B}\right), \tag{12}$$

where $\boldsymbol{E} = -\nabla\phi - \partial_t \boldsymbol{A}/c$, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, and the molar enthalpy h is defined by

$$h = \frac{\partial(n\varepsilon)}{\partial n} = \varepsilon + \frac{\mathcal{P}}{n}. \tag{13}$$

With $(en, en\boldsymbol{V})$ representing the charge and current densities, the Maxwell equations follow from the variation δA_μ.

Although the derivation of (12) is well known, a review of its somewhat involved procedure will be useful for the later discussions on the relation between the variation of $\boldsymbol{Q}$ and its Eulerian counterpart (section 5): we first calculate the responses of $\boldsymbol{V}(\boldsymbol{x}, t)$ and $n(\boldsymbol{x}, t)$ to a perturbation $\boldsymbol{Q}(\boldsymbol{x}_0, t) \to \boldsymbol{Q}'(\boldsymbol{x}_0, t)$. A vexing complication arises from the gap between the Lagrangian view of the diffeomorphism $\boldsymbol{Q}(\boldsymbol{x}_0, t)$ (operating on the 'initial position' $\boldsymbol{x}_0$) and the Eulerian view of other fields to be evaluated at each space-time point $(\boldsymbol{x}, t)$. To merge different views, we have to evaluate the variation $\delta\boldsymbol{Q}$ in the Eulerian view:

$$\begin{aligned}\delta\boldsymbol{Q}|_{\boldsymbol{x},t} &= (\boldsymbol{Q}'(\boldsymbol{x}_0, t) - \boldsymbol{Q}(\boldsymbol{x}_0, t))|_{\boldsymbol{x}=\boldsymbol{Q}(\boldsymbol{x}_0,t),\ t}\\ &= \boldsymbol{Q}'(\boldsymbol{Q}^{-1}(\boldsymbol{x}, t), t) - \boldsymbol{x}\\ &:= \boldsymbol{x}' - \boldsymbol{x}.\end{aligned}$$

Note that this $\delta\boldsymbol{Q}$ evaluates the variations of the streamlines, originating from the common origins at $\boldsymbol{x}_0 = \boldsymbol{Q}^{-1}(\boldsymbol{x}, t)$. The corresponding variations of the fields (that are functions of $\boldsymbol{Q}$) are given by the 'Lie-derivatives':

$$\begin{aligned}\delta\boldsymbol{V}(\boldsymbol{x}, t) &= (\dot{\boldsymbol{Q}}' - \dot{\boldsymbol{Q}})|_{\boldsymbol{x},t}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Q}'(\boldsymbol{x}_0'(\boldsymbol{x}, t), t) - \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Q}(\boldsymbol{x}_0(\boldsymbol{x}, t), t)\\ &= \left[\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Q}'(\boldsymbol{x}_0'(\boldsymbol{x}', t), t) - \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Q}(\boldsymbol{x}_0(\boldsymbol{x}, t), t)\right]\\ &\quad - \left[\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Q}'(\boldsymbol{x}_0'(\boldsymbol{x}', t), t) - \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Q}'(\boldsymbol{x}_0'(\boldsymbol{x}, t), t)\right]\\ &= \partial_t\delta\boldsymbol{Q} + (\boldsymbol{V}\cdot\nabla)\delta\boldsymbol{Q} - (\delta\boldsymbol{Q}\cdot\nabla)\boldsymbol{V},\\ \delta n(\boldsymbol{x}, t) &= n'(\boldsymbol{x}, t) - n(\boldsymbol{x}, t)\\ &= [n'(\boldsymbol{x}', t) - n(\boldsymbol{x}, t)] - [n'(\boldsymbol{x}', t) - n'(\boldsymbol{x}, t)]\\ &= \left[n_0\frac{D(\boldsymbol{x}_0)}{D(\boldsymbol{x}')} - n\right] - \delta\boldsymbol{Q}\cdot\nabla n\\ &= [n\,(1 - \nabla\cdot\delta\boldsymbol{Q}) - n] - \delta\boldsymbol{Q}\cdot\nabla n\\ &= -n\nabla\cdot\delta\boldsymbol{Q} - \delta\boldsymbol{Q}\cdot\nabla n\\ &= -\nabla\cdot(n\delta\boldsymbol{Q}).\end{aligned}$$

After appropriate manipulations and integrations by parts to collect terms multiplied by $\delta\boldsymbol{Q}$, one obtains (12).

3. Fluid mechanics in the Eulerian view

The relative facility of the Lagrangian view to yield the equations of motion disappears in the Eulerian case where no *a priori* relation between the fluid velocity $\boldsymbol{V}$ and the streamlines is assumed, and the unrestricted variation $\delta\boldsymbol{V}$ yields $\boldsymbol{P} = 0$. To reproduce properly the evolution equations, we must appropriately 'constrain' $\boldsymbol{V}$.

The measure $n\mathrm{d}^3x\,\mathrm{d}t$, defined by (8), is the generalization of the path integral for single orbit to the collective orbits. It was argued, then, that imposing a physically motivated 'restriction' on n that leads to the conservation law (11), must be a step in the right direction. Serrin, in a pioneering paper [1], proposed the Lagrangian density

$$L_F = (\boldsymbol{P}\cdot\boldsymbol{V} - H_F)\,n + S D_t^* n, \tag{14}$$

in which the variation of the Lagrange multiplier S does exactly that. Note that, in the Lagrangian formulation, the particle conservation (11) is built in through (8).

The Serrin Lagrangian, however, falls short to describe general fluid mechanics; the momentum field is limited to be such that $\boldsymbol{P} = \nabla S$ (obtained by the variation $\delta\boldsymbol{V}$) which describes only an 'irrotational' flow.

To derive flows with vorticity, a variety of authors [6–8] have imposed additional constraints of the form $-n\sum_{j=1}^{\nu}\lambda^j D_t\sigma_j$ to the Serrin Lagrangian (14); depending on author, ν varies from 1 to 3. As to be shown in (17), the constrained Lagrangian yields a flow such that

$$\boldsymbol{P} = \nabla S + \sum_{j=1}^{\nu}\lambda^j\nabla\sigma_j, \tag{15}$$

corresponding to the so-called *Clebsch parametrization* of a vector field, which does acquire a finite vorticity. However, the correct number ν of additional constraints as well as their physical interpretation has been somewhat ambiguous. Mathematically, a general three-dimensional vector field can be cast into the form of (15) with $\nu = 2$; however, in formulating a variational principle, we have to vary each function λ^j or σ_j independently without violating the boundary conditions, and for this requirement we need $\nu = 3$; see [17]. In section 5, σ_j turns out to be the Eulerian representation of the Lagrangian coordinates; thus ν must be indeed three [18, 19].

Let us follow the formal calculus: we start from a Serrin–Lin Lagrangian density

$$L_F = \left[\boldsymbol{P} \cdot \boldsymbol{V} - H_F - \left(D_t S + \sum_{j=1}^{\nu} \lambda^j D_t \sigma_j \right) \right] n. \quad (16)$$

In what follows, we choose $\nu = 3$, and we apply the summation rule to omit $\sum_{j=1}^{\nu}$. For the symmetry of expression (and for an alternative interpretation of the formulation to be described later), we have replaced $S D_t^* n$ of the Serrin Lagrangian by $-n D_t S$. The variational principle $\delta \int (L_F + L_{\rm EM})\, {\rm d}^3x\, {\rm d}t = 0$ yields

$$\delta \boldsymbol{V} \Rightarrow \boldsymbol{P} = \nabla S + \lambda^j \nabla \sigma_j, \quad (17)$$

$$\delta \boldsymbol{p} \Rightarrow \boldsymbol{p} = m \boldsymbol{V}, \quad (18)$$

$$\delta S \Rightarrow D_t^* n = 0, \quad (19)$$

$$\delta \sigma_j \Rightarrow D_t^*(n\lambda^j) = 0 \Rightarrow D_t \lambda^j = 0, \quad (20)$$

$$\delta \lambda^j \Rightarrow D_t \sigma_j = 0, \quad (21)$$

$$\delta n \Rightarrow D_t S = \boldsymbol{P} \cdot \boldsymbol{V} - (H + h), \quad (22)$$

and, by δA_μ, Maxwell's equations with the currents $(en, en\boldsymbol{V})$.

The particle conservation law is already apparent in (19). The other equations combine to reproduce the equation of motion; using $D_t(\nabla f) = \nabla(D_t f) - \nabla \boldsymbol{V} \cdot \nabla f$, we obtain

$$\begin{aligned} D_t \boldsymbol{P} &= D_t(\nabla S + \lambda^j \nabla \sigma_j) \\ &= \nabla[-(H + h) + \boldsymbol{P} \cdot \boldsymbol{V}] - \nabla \boldsymbol{V} \cdot \boldsymbol{P} \\ &= -\nabla(e\phi + h) + \frac{e}{c}[\boldsymbol{V} \times \boldsymbol{B} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{A}], \end{aligned}$$

which is equivalent to (12). Therefore, the evolution of the *Serrin–Lin fields* n, S, λ^j, μ_j dictated by the Lagrangian density (16) is *consistent* to the fluid/plasma equations.

What do the Serrin–Lin fields signify? The role of S is best understood by referring to the original Serrin form ($\lambda^j = \sigma_j = 0$). Although Serrin's S is a Lagrange multiplier that imposes particle conservation (11), we may proffer a different interpretation. By moving (by integrating by parts) D_t^* from n to S, as we did in (16), one may think of n as a Lagrange multiplier demanding that L_F must be a complete derivative (evaluated through each streamline of $\boldsymbol{V}$) of some scalar field S—this is nothing but Hamilton's principle demanding the criticality of the action integral with S as the 'action'. Indeed, if the thermal energy ε is neglected and $\lambda^j = \sigma_j = 0$ in (17) and (22), we obtain the well-known Hamilton–Jacobi equations $\partial_t S = -H(\boldsymbol{x}, \boldsymbol{P}, t)$ and $\nabla S = \boldsymbol{P}$.

The additional fields λ^j and σ_j are, mathematically, the constraints producing a finite vorticity in the bundle of streamlines of $\boldsymbol{V}$, and violating (or generalizing) the Hamilton–Jacobi equations; remember the discussion in the introduction. Their physical meaning will be revealed in the following sections.

We end this section with actualizing the *canonical structure* in the governing equations of the Serrin–Lin variables. Plugging the Clebsch form (17) into the Serrin–Lin Lagrangian density L_F, we can simplify (16) as

$$L_F' = (n\dot{S} + \Lambda^j \dot{\sigma_j}) - nH_F|_{\boldsymbol{P} = \nabla S + (\Lambda^j/n)\nabla\sigma_j}, \quad (23)$$

where we denote $\dot{f} := \partial_t f$, and define $\Lambda^j = n\lambda^j$. We observe that this Lagrangian density consists of a canonical 1-form $(n\dot{S} + \Lambda^j \dot{\sigma_j})$ (i.e. the 2-form ${\rm d}n \wedge {\rm d}S + {\rm d}\Lambda^j \wedge {\rm d}\sigma_j$ defines the symplectic structure) and a Hamiltonian density nH_F, thus the pairs n-S and Λ^j-σ_j ($j = 1, 2, 3$) constitute 'canonical variables' obeying Hamilton's equations: defining the Hamiltonian

$$\mathcal{H}_F = \int nH_F|_{\boldsymbol{P} = \nabla S + (\Lambda^j/n)\nabla\sigma_j}\, {\rm d}^3x, \quad (24)$$

the forgoing Euler–Lagrange equations (19)–(22) can be cast into a system of canonical equations

$$\begin{cases} \dot{n} = \partial_S \mathcal{H}_F \\ \dot{S} = -\partial_n \mathcal{H}_F, \end{cases} \qquad \begin{cases} \dot{\Lambda^j} = \partial_{\sigma_j} \mathcal{H}_F \\ \dot{\sigma_j} = -\partial_{\Lambda^j} \mathcal{H}_F \end{cases} \qquad (j = 1, \ldots, \nu). \quad (25)$$

Here we choose $\nu = 3$ (the similar canonical system with a different ν will become a subject in the next section).

4. The relation between non-canonical and canonical formalisms

As is well known, the standard plasma equations (11)–(12) are *non-canonical* as a Hamiltonian system. However, we have seen that the Serrin–Lin parametrization of the system yields a canonical system (25); by the Serrin–Lin variables obeying the canonical equations, we can construct the solution of (11)–(12). Here, we analyze the exact relation between the original non-canonical system and the canonicalized system.

We start by reviewing what we call *non-canonical* [5]. A general Hamiltonian system (on a Hilbert space V) is endowed with a symplectic operator $\mathcal{J}$ and a Hamiltonian $H(u)$ ($u \in V$); the evolution equation is written as

$$\partial_t u = \mathcal{J} \partial_u H(u),$$

where $\partial_u H(u)$ is the gradient of a (smooth) functional $H(u)$. We define the Poisson bracket by $[F, G] := \langle \mathcal{J}\partial_u F, \partial_u G \rangle$, where $\langle u, v \rangle$ is the inner product of V. The symplectic operator $\mathcal{J}$ must be antisymmetric, and we demand Jacobi's law $[[F, G], H] + [[G, H], F] + [[H, F], G] = 0$. Generally, $\mathcal{J}$ may be a function of u (then, we denote $\mathcal{J}(u)$), and may have a non-trivial kernel, i.e. there may exist $v \neq 0$ such that $\mathcal{J}(u)v = 0$ (v may depend on u), and then, the system is said non-canonical. If there exists a functional $C(u)$ ($\neq$ constant) such that $\mathcal{J}(u)\partial_u C(u) = 0$, then, $[C, F] \equiv 0$ ($\forall F$). Such $C(u)$ is called a *Casimir invariant*; because $[H, C] = 0$ (H: Hamiltonian), C is a constant of motion.

A trivial example of Casimir can be made by adding an extra degree of freedom to a canonical system of a finite dimension: Let $\boldsymbol{u} = (u^1, \ldots, u^n)$ be a complete set of canonical variables governed by $\partial_t \boldsymbol{u} = \mathcal{J}\partial_u H$. We consider its extension $\tilde{\boldsymbol{u}} = (\boldsymbol{u}, u^{n+1})$. The superfluous component u^{n+1} cannot be independent to $\boldsymbol{u}$. For simplicity, let us assume that $u^n = u^{n+1}$. The equation of motion for $\tilde{\boldsymbol{u}}$ may be written as $\partial_t \tilde{\boldsymbol{u}} = \tilde{\mathcal{J}}\partial_{\tilde{u}}\tilde{H}$ with an extended Hamiltonian $\tilde{H}$ such that $\partial_{u^n}\tilde{H} = \partial_{u^{n+1}}\tilde{H} = \partial_{u^n} H$, and an extended symplectic operator $\tilde{\mathcal{J}}$ such that $\tilde{\mathcal{J}}^{i,j} = \mathcal{J}^{i,j}$ $(i, j = 1, \ldots, n-1)$, $\tilde{\mathcal{J}}^{i,n} = \tilde{\mathcal{J}}^{i,n+1} = \mathcal{J}^{i,n}/2$, $\tilde{\mathcal{J}}^{n,j} = \tilde{\mathcal{J}}^{n+1,j} = \mathcal{J}^{n,j}/2$ $(i, j = 1, \ldots, n)$ and $\tilde{\mathcal{J}}^{n+1,n+1} = \mathcal{J}^{n,n}/2$. The kernel of $\tilde{\mathcal{J}}$ is ${}^t(0, \ldots, 0, 1, -1)$, and hence, arbitrary $f(u^n - u^{n+1})$ is a Casimir representing the above mentioned constraint.

The plasma equations (11)–(12) do have Casimirs [5]. Choosing $\boldsymbol{u} = {}^t(n, \boldsymbol{P})$ as the state vector, the evolution equations can be cast into a Hamiltonian form

$$\partial_t \begin{pmatrix} n \\ \boldsymbol{P} \end{pmatrix} = \begin{pmatrix} 0 & -\nabla\cdot \\ -\nabla & -n^{-1}(\nabla \times \boldsymbol{P})\times \end{pmatrix} \begin{pmatrix} \partial_n \mathcal{H}_F \\ \partial_{\boldsymbol{P}} \mathcal{H}_F \end{pmatrix} \tag{26}$$

with the conventional Hamiltonian (total energy; see (7))

$$\mathcal{H}_F = \int H_F \, \mathrm{d}^3x = \int n \left[\frac{|\boldsymbol{P} - e\boldsymbol{A}/c|^2}{2m} + e\phi + \varepsilon(n) \right] \mathrm{d}^3x.$$

Note that this $\mathcal{H}_F$ is represented by the naïve state variables $\boldsymbol{u} = {}^t(n, \boldsymbol{P})$, which is compared with the foregoing Hamiltonian (24) in terms of the Serrin–Lin variables. The block operator on the right-hand side of (26) defines the symplectic structure. Evidently, two functionals

$$C_1 = \int n \, \mathrm{d}^3x, \quad C_P = \int (\nabla \times \boldsymbol{P}) \cdot \boldsymbol{P} \, \mathrm{d}^3x$$

are Casimirs. The constancy of C_1 is nothing but the particle conservation law; C_P is the so-called helicity.

Now we return to the question as to how the non-canonical system (26) and the canonical system (25) are related. From the forgoing practice of 'non-canonicalization' by 'adding' a constrained parameter, one may expect that some superfluous parameters have been 'subtracted' from the original non-canonical system (26) to purify it canonically, and (25) is the result of adjustment. Unfortunately, this is not the case. In fact, the canonical system (25) is keeping the variable n that is constrained by the Casimir C_1 (as to be shown, C_P can play an interesting role in canonicalizing the system). Since the relation between these two systems is rather involved, we start by examining the 'range' of the canonicalized dynamics (25) with varying ν (the number of the Clebsch pairs λ^j and σ_j).

Going back to (16), let us remember that ν can be (mathematically) arbitrary; for every ν, we obtain a closed canonical system that is consistent with the original non-canonical system (26). In this sense, the canonical system (25), with an arbitrary ν, is 'embedded' in the non-canonical system (26).

Evidently, neither $\nu = 0$ ($\boldsymbol{P} = \nabla S$) nor $\nu = 1$ ($\boldsymbol{P} = \nabla S + \lambda\nabla\sigma$) is sufficient for the canonical system to describe the general dynamics; the former cannot have vorticity ($\Omega = \nabla\times\ \boldsymbol{P} = 0$), while the latter can encompass flows with only integrable vortex lines ($\Omega \cdot \nabla\lambda = \Omega \cdot \nabla\sigma = 0$) [17]. However, one may claim that the canonical system (25) with $\nu \leqslant 1$ describes some restricted (but self-consistent) classes of dynamics. The $\nu = 0$ canonical system describes the dynamics in a subspace of *irrotational flows*; an irrotational flow remains irrotational for ever, thus the dynamics is closed in this subspace. The $\nu = 1$ case is somewhat tricky. The set of functions $\{\nabla S + \lambda\nabla\sigma\}$ (the domain of the dynamical system (25) for $\nu = 1$) is not a linear subspace, because a linear combination of two members may not be cast in a $\nu = 1$ Clebsch form (reflecting the nonlinear nature of the Clebsch parametrization including products $\lambda^j\nabla\sigma_j$). This subclass of dynamics is characterized by the helicity constraint. For $\boldsymbol{P} = \nabla S + \lambda\nabla\sigma$, the helicity reads as $C_P = \int \nabla \cdot (S\nabla\lambda \times \nabla\sigma)\mathrm{d}^3x$. This C_P, being an integral of an exact 3-form, is fixed, when we give boundary conditions on the Clebsch parameters S, λ, σ (a non-zero helicity is 'external' being caused by inhomogeneous boundary values of the Clebsch potentials). If C_P is fixed (decomposed from the dynamical variables), the kernel of the Poisson bracket is removed, and hence, we obtain a canonical system endowed with a regular symplectic 2-form; see Jackiw [20]. Hence, the $\nu = 1$ canonical system can be deemed as a result of purification pertinent to the helicity constraint (however, an over-purification eliminating the general non-integrable vortex-line dynamics).

On the other hand, we can also assume an arbitrarily large ν. Apparently, $\nu > 3$ (the coordinate-space dimension) is 'superfluous', i.e. the Clebsch parametrization (15) includes inter-related components. To be precise, we say that the set of variables $(S, \lambda^1, \sigma_1, \ldots, \lambda^\nu, \sigma_\nu)$ contains superfluous component, if the map

$$\mathcal{T} : (S, \lambda^1, \sigma_1, \ldots, \lambda^\nu, \sigma_\nu) \longmapsto \boldsymbol{P} := \nabla S + \sum_{j=1}^{\nu} \lambda^j \nabla\sigma_j$$

has a kernel, i.e. if there is a non-trivial element $(S, \lambda^1, \sigma_1, \ldots, \lambda^\nu, \sigma_\nu)$ such that $\nabla S + \sum_{j=1}^{\nu} \lambda^j\nabla\sigma_j = 0$. Then, the parametrization $\mathcal{T}^{-1}$ is not unique.

Interestingly, even for $\nu = 1$, 2 or 3, the Clebsch parametrization includes superfluous components [17]. In fact, for $\nabla \times (\sum_{j=1}^{\nu} \lambda^j\nabla\sigma_j) = 0$, one can find an appropriate S to satisfy this relation, and hence, any $\nu \geqslant 1$ yields superfluous component.

Superfluity does not guarantee the *completeness*; even with a superfluous components, the range of the map $\mathcal{T}$ can be smaller than the totality of physical variables, i.e. some $\boldsymbol{P}$ cannot be parametrized. As mentioned above, $\nu = 1$ ($\boldsymbol{P} = \nabla S + \lambda\nabla\sigma$) is insufficient. We can show that $\nu = 2$ suffices for the completeness [17]. However, for the Hamiltonian formalism, we need $\nu = 3$ to avoid the boundary conditions to connect the Clebsch parameters as a simultaneous equation (i.e. in order to impose the boundary conditions separately on every components).

In summary, the $\nu = 3$ canonical system (25) is *densely* embedded in the phase space of the original non-canonical system (26); yet, the former contains superfluous parameters that must be subject to some constraint. The non-canonical property (or the topological defect), represented by $\mathrm{Ker}(\mathcal{J}) = \mathrm{Coker}(\mathcal{J})$, of the original system (26) is subsumed into Ker($\mathcal{T}$). At the surface, i.e. in the space of the technical variables ${}^t(n, S, \Lambda^1, \sigma_1, \ldots)$, the governing equation (25) is canonical—there is no apparent constraint. Beneath this parameter space, however, there must be a constraint by which the parametrization $\mathcal{T}^{-1}$ can be evaluated. This constraint turns out to be the *initial condition*.

To see the role of initial condition, let us invoke the previous toy model of non-canonical system. We can formulate a 'canonicalized system' whose solution coincides with that of the non-canonical one for a selected initial condition. Instead of the previous $\tilde{\mathcal{J}}$, we define $\tilde{\mathcal{K}}$ such that $\tilde{\mathcal{K}}^{i,j} = \mathcal{J}^{i,j}$ $(i, j = 1, \ldots, n)$, $\tilde{\mathcal{K}}^{i,n+1} = 0$ $(i = 1, \ldots, n)$, $\tilde{\mathcal{K}}^{n+1,j} = \mathcal{J}^{n,j}$ $(j = 1, \ldots, n-1)$, $\tilde{\mathcal{K}}^{n+1,n} = 0$ and $\tilde{\mathcal{K}}^{n+1,n+1} = \mathcal{J}^{n,n}$. This $\tilde{\mathcal{K}}^{i,j}$ is regular, and hence, $\dot{\tilde{u}} = \tilde{\mathcal{K}}\partial_{\tilde{u}}\tilde{H}$ is canonical. Since $\dot{u^n} = u^{n+1}$, the condition $u^n = u^{n+1}$ is satisfied if we restrict the initial condition $u^n(0) = u^{n+1}(0)$.

From this observation, one may speculate that (i) the Lagrangian representation (labeling the streamlines by the initial positions) is free from the topological defects, and (ii) the Clebsch parametrization, leading to a seemingly canonical (defect-free) formulation, may be somehow connected to the Lagrangian representation. In the next section, we will see how the Clebsch parametrization is connected with the Lagrangian representation, and clarify the physical meaning of the variables λ^j and σ_j.

5. Connection between the Lagrangian and Eulerian views

As discussed in the previous section, we are assuming both λ^j and σ_j to be the components of three-vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}$. This interpretation of λ^j and σ_j will now be exploited to relate the Lagrangian and Eulerian formulations of the Lagrangians, respectively, given in sections 2 and 3. Through this analysis, we will also find a simpler form of the Lagrangian that turns out to be translatable to the most natural Lorentz-invariant form.

The key is the identity $\boldsymbol{V} \equiv D_t\boldsymbol{x}$ (the Cartesian parametrization of the vector field $\boldsymbol{V}$; D_t is the *symbol* of the flow, which will be generalized to $U^\nu\partial_\nu$ in the Lorentz-covariant formulation in section 7), by which one may write

$$\boldsymbol{P}\cdot\boldsymbol{V} - \boldsymbol{\lambda}\cdot D_t\boldsymbol{\sigma} = \boldsymbol{P}\cdot D_t\boldsymbol{\xi} - \boldsymbol{\mu}\cdot D_t\boldsymbol{\sigma},$$

where $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{\sigma}$ and $\boldsymbol{\mu} = \boldsymbol{\lambda} - \boldsymbol{P}$ [18]. The Serrin–Lin Lagrangian density (16), then, transforms to

$$L_F = (\boldsymbol{P}\cdot D_t\boldsymbol{\xi} - H_F - D_tS - \boldsymbol{\mu}\cdot D_t\boldsymbol{\sigma})\,n. \tag{27}$$

From this form, the equivalence between the Eulerian and the previous Lagrangian (5) formalism will be established when we connect $D_t\boldsymbol{\xi}$ with $\dot{\boldsymbol{Q}}$.

In (27), the variation of $\boldsymbol{\mu}$ forces $D_t\boldsymbol{\sigma} = 0$ implying

$$D_t\boldsymbol{\xi} = D_t\boldsymbol{x} \equiv \boldsymbol{V}. \tag{28}$$

This Eulerian representation of the flow velocity is compared with the Lagrangian representation (6) that invoked the diffeomorphism $\boldsymbol{Q}(\boldsymbol{x}_0, t)$. We will now show that $\boldsymbol{\sigma}$ (in its fundamental choice) is nothing but $\boldsymbol{x}_0$, and hence, $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{\sigma}$ is the 'displacement':

$$\boldsymbol{\xi} = \boldsymbol{Q}(\boldsymbol{x}_0, t) - \boldsymbol{x}_0. \tag{29}$$

Here, the left-hand side is represented in the Eulerian frame ($\boldsymbol{x}$ and t), while the right-hand side, in the Lagrangian frame ($\boldsymbol{x}_0$ and t). Both sides are connected by invoking the inverse map $\boldsymbol{x}_0 = \boldsymbol{Q}^{-1}(\boldsymbol{x}, t)$ and writing the right-hand side as $\boldsymbol{x} - \boldsymbol{Q}^{-1}(\boldsymbol{x}, t)$. We may regard the map $\boldsymbol{Q}^{-1}(\boldsymbol{x}, t)$ as the fundamental Eulerian field that is the counterpart of the diffeomorphism $\boldsymbol{Q}(\boldsymbol{x}_0, t)$, the fundamental Lagrangian field.

Let us show that $\boldsymbol{x}_0 = \boldsymbol{Q}^{-1}(\boldsymbol{x}, t) = \boldsymbol{\sigma}(\boldsymbol{x}, t)$. We note that $D_t\boldsymbol{\sigma} = 0$ implies that $\boldsymbol{\sigma}$ is constant along every streamline defined by $\boldsymbol{V}$ and, therefore, must be a function only of the *initial condition*. The simplest expression of such a $\boldsymbol{\sigma}$ is the initial condition itself. Formally, we can construct $\boldsymbol{\sigma}$ by solving

$$\partial_t\boldsymbol{\sigma} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{\sigma} = 0 \tag{30}$$

as a partial differential equation (PDE) for a given initial condition $\boldsymbol{\sigma}(\boldsymbol{x}, 0) = \boldsymbol{\sigma}_0(\boldsymbol{x})$. The 'characteristics' corresponding to the hyperbolic PDE (30) are determined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{V}(\boldsymbol{x}, t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{31}$$

Both (30) and (31) share the same $\boldsymbol{V}$. Solving (31) for every initial value $\boldsymbol{x}_0$, we may construct the diffeomorphism $\boldsymbol{x} = \boldsymbol{Q}(\boldsymbol{x}_0, t)$. The $\boldsymbol{V}$ in (30) is now common with that of (6). Therefore, by (28), we obtain

$$\dot{\boldsymbol{Q}} = \boldsymbol{V} = D_t\boldsymbol{\xi}. \tag{32}$$

The interpretation of the field $\boldsymbol{\xi}$ as the 'displacement', i.e. relation (29), is established by choosing an appropriate initial condition for $\boldsymbol{\sigma}$ in solving (30). Using the inverse map $\boldsymbol{Q}^{-1}(\boldsymbol{x}, t)$, we may write $\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{\sigma}_0(\boldsymbol{Q}^{-1}(\boldsymbol{x}, t))$. The simplest initial condition $\boldsymbol{\sigma}_0(\boldsymbol{x}) = \boldsymbol{x}$, i.e. $\boldsymbol{\sigma}_0$ is the identity, leads to

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{Q}^{-1}(\boldsymbol{x}, t) \equiv \boldsymbol{x}_0(\boldsymbol{x}, t). \tag{33}$$

We already mentioned that when $\boldsymbol{\sigma} = \boldsymbol{x}_0$, $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{\sigma}$ is the Eulerian representation of the 'displacement'.

Now, the unification of the Lagrangian and Eulerian representations is established: choosing n such that $D_t^*n = 0$, satisfying (8), one may omit the constraining term $(D_tS)n$ in (27). Also choosing $\boldsymbol{\sigma}$ such that $D_t\boldsymbol{\sigma} = 0$, the corresponding term vanishes in (27), and (32) allows us to write $\boldsymbol{P}\cdot D_t\boldsymbol{\xi} = \boldsymbol{P}\cdot\dot{\boldsymbol{Q}}$. Then, (27) is nothing but the Lagrangian density of (5).

6. Simplified Eulerian representation

In the preceding argument, we tried to connect the Eulerian formulation (27) to the Lagrangian view by interpreting the term $\boldsymbol{\mu}\cdot D_t\boldsymbol{\sigma}$ as restricting $\boldsymbol{\xi}$ so that $D_t\boldsymbol{\xi} = \dot{\boldsymbol{Q}}$. Now, we return to the Eulerian view, and proceed to derive a more elegant and perspicacious form of the Lagrangian.

The relation $\sigma(\boldsymbol{x}, t) = \boldsymbol{x}_0(\boldsymbol{x}, t)$, established in (33), will be applied in an alternative way to translate (27) in which $\boldsymbol{\xi}$ is treated as an independent Eulerian field (and varies freely) with fixing $\boldsymbol{V}$. Note that the 'velocity' $D_t\boldsymbol{\xi}$, constituting the coupling term $\boldsymbol{P}\cdot D_t\boldsymbol{\xi}$, and the physical velocity $\boldsymbol{V}$ are independent, and their relation is reserved open in calculating the variations—instead of connecting them, as we did in section 5, we relate $\boldsymbol{x}$ to $\boldsymbol{\xi}$ for the variation, i.e. we promise to evaluate all fields, excepting $\boldsymbol{V}$, for $\boldsymbol{x}=\boldsymbol{\xi}+\boldsymbol{\sigma}$ with fixed $\boldsymbol{\sigma}(=\boldsymbol{x}_0)$ such that $D_t\boldsymbol{\sigma}=0$. Under this protocol, we may remove the term $\boldsymbol{\mu}\cdot D_t\boldsymbol{\sigma}$ in (27), and write

$$L_F=(\boldsymbol{P}\cdot D_t\boldsymbol{\xi}-H_F-D_tS)\,n\,|_{\boldsymbol{x}=\boldsymbol{\xi}+\boldsymbol{\sigma}}\,, \tag{34}$$

where we have written

$$\boldsymbol{P}=m\boldsymbol{V}+\frac{e}{c}\boldsymbol{A},\qquad H_F=\frac{mV^2}{2}+\varepsilon+e\phi$$

with expressing the mechanical momentum $\boldsymbol{p}$ explicitly as $m\boldsymbol{V}$ for simplicity (if we reserve $\boldsymbol{p}$ as a variable, the variation $\delta\boldsymbol{p}$ yields $\boldsymbol{p}=mD_t\boldsymbol{\xi}=m\boldsymbol{V}$, because $\boldsymbol{\xi}=\boldsymbol{x}-\boldsymbol{\sigma}$ and $D_t\boldsymbol{\sigma}=0$).

The Euler–Lagrange equations are

$$\text{by}\quad \delta n,\qquad \frac{L_F}{n}=h-\varepsilon \tag{35}$$

$$\text{by}\quad \delta\boldsymbol{V},\qquad \boldsymbol{P}\cdot(\nabla\boldsymbol{\xi})-\nabla S=0, \tag{36}$$

$$\text{by}\quad \delta S,\qquad D_t^*n=0, \tag{37}$$

$$\text{by}\quad \delta\boldsymbol{\xi},\qquad D_t\boldsymbol{P}=-\nabla(e\phi+h)+\frac{e}{c}\nabla\boldsymbol{A}\cdot\boldsymbol{V}. \tag{38}$$

We easily verify that (38) yields the momentum equation (12). To derive (38), we have to take into account the variation of involved fields caused by the perturbation $\delta\boldsymbol{x}=\delta\boldsymbol{\xi}$ of the observation point; we find $\delta(n\,\mathrm{d}^3x)=(\nabla n\cdot\delta\boldsymbol{\xi}+n\nabla\cdot\delta\boldsymbol{\xi})\,\mathrm{d}^3x=\nabla\cdot(n\delta\boldsymbol{\xi})\,\mathrm{d}^3x$, which shows a reciprocal relation with $\delta n=-\nabla\cdot(n\delta\boldsymbol{Q})$ given in section 2. We also have $\delta\phi=\nabla\phi\cdot\delta\boldsymbol{\xi}$, and $\delta(\boldsymbol{A}\cdot\boldsymbol{V})=(\nabla\boldsymbol{A}\cdot\boldsymbol{V})\cdot\delta\boldsymbol{\xi}$. Here, $\boldsymbol{V}$ is freed from $\delta\boldsymbol{\xi}$ because we define the $D_t\boldsymbol{\xi}$ with fixing $\boldsymbol{V}$. We also note that the variation of $\int f|_{\boldsymbol{x}=\boldsymbol{\xi}+\boldsymbol{\sigma}}\,\mathrm{d}^3x$ due to $\delta\boldsymbol{\xi}$ vanishes for every scalar field (0-form) f because $(\nabla f\cdot\delta\boldsymbol{\xi})\,\mathrm{d}^3x$ and $f(\nabla\cdot\delta\boldsymbol{\xi})\,\mathrm{d}^3x$ cancel (it appears just a shift of the coordinate). By this fact, the variation of $\boldsymbol{\xi}$ does not propagate to $\mathcal{L}_{\mathrm{EM}}$.

Since explicit calculations are easier in the framework of tensor calculus for relativistic formulation, we will explain the details of variational principle in the next section.

7. Lorentz-invariant form

The Lagrangian density (34) can be generalized to be a Lorentz-invariant form that reveals the deep structure of the fluid Lagrangian.

Let us revisit the particle Lagrangian in the relativistic framework, and recall some basic notation. The action of a particle (in EM) is

$$\begin{aligned}S&=\int -mc\,\mathrm{d}s-\frac{e}{c}A^\mu\mathrm{d}x_\mu\\&=\int -\left[mcu^\mu+\frac{e}{c}A^\mu\right]\mathrm{d}x_\mu\\&=\int -\left[\gamma mc^2+e\phi-\left(mcu^j+\frac{e}{c}A^j\right)v_j\right]\mathrm{d}t\\&\equiv\int\mathcal{L}_P\,\mathrm{d}t\end{aligned} \tag{39}$$

($\mathrm{d}s=\sqrt{\mathrm{d}x^\mu\mathrm{d}x_\mu}=u^\mu\mathrm{d}x_\mu$, $u^\mu=\mathrm{d}x^\mu/\mathrm{d}s=(\gamma,\gamma\boldsymbol{v}/c)$). Following preliminaries set the stage for formulating a fluid Lagrangian:

(a) For a free-particle ($A^\mu=0$), the particle Lagrangian reads $-mc^2\sqrt{1-(v/c)^2}=-mc^2/\gamma$. The corresponding continuous Lagrangian density may be written as $-mc^2n/\gamma$ leading to the action $-\int mc^2n/\gamma\,\mathrm{d}^4x$.

(b) The density of the fluid element transforms with the frame. The rest-frame (with respect to the fluid element) density is a scalar denoted as n, while the density in an arbitrary frame is γn and transforms like the zero component of a four vector.

(c) In terms of the relativistic fluid four-velocity

$$U^\mu=(\gamma,\gamma\boldsymbol{V}/c),$$

the non-relativistic 'convective derivative' reads

$$D_t=c\gamma^{-1}U^\mu\partial_\mu,$$

where $U^\mu\partial_\mu$ is Lorentz covariant.

(d) The relativistic fluid four-momentum density may be defined as

$$p^\mu=(\mathcal{E}/c)U^\mu=(\mathcal{E}\gamma/c,\mathcal{E}\gamma\boldsymbol{V}/c^2), \tag{40}$$

where $\mathcal{E}$ is the energy density of the fluid (in the rest frame). The effective rest-mass of a particle composing the fluid is given by $\tilde{m}=\mathcal{E}/c^2$. The fluid energy (sum of the kinetic and thermal energies) is $n\gamma\mathcal{E}=n\gamma\tilde{m}c^2$ (in the non-relativistic limit, we may write $p^0=(\tilde{m}V^2/2+\varepsilon)/c$). The enthalpy is $n\partial(n\mathcal{E})/\partial n=n\mathcal{E}+\mathcal{P}$ (see (13)), where

$$\mathcal{P}=n^2\frac{\partial\mathcal{E}}{\partial n} \tag{41}$$

is the pressure. We note that this thermodynamic relation, based on the particle conservation $\mathrm{d}(nV_0)=0$ (V_0 is the proper volume), holds in the rest frame.

Now the relativistic version of the fluid Lagrangian (34) is written in a manifestly covariant form:

$$L_F=-\left[(\mathcal{E}U^\mu+eA^\mu)\,U^\nu\partial_\nu\Xi_\mu+cU^\nu\partial_\nu S\right]n\,\big|_{x_\mu=\xi_\mu+\sigma_\mu}\,. \tag{42}$$

Here, the x_μ is four-dimensionalized including $x_0=ct$. According to this, we set $\xi_0=c\tau$ with a function τ, and $\sigma_0=0$. A covariant vector Ξ_μ is defined as

$$\Xi_\mu=(c\tau,-\boldsymbol{\xi}), \tag{43}$$

which parametrizes the contravariant four-velocity:

$$U^\nu\partial_\nu\Xi_\mu|_{x_\mu=\xi_\mu+\sigma_\mu}=(\gamma,-\gamma\boldsymbol{V}/c)\equiv U_\mu, \tag{44}$$

$$U^\mu U^\nu\partial_\nu\Xi_\mu|_{x_\mu=\xi_\mu+\sigma_\mu}=U^\mu U_\mu=1. \tag{45}$$

The fluid Lagrangian (42) is the most natural generalization of the particle Lagrangian given in (39). In comparison with the non-relativistic formulation (34), however, one notes that an additional field $\xi_0 = c\tau$ has sneaked in the four-vector Ξ_μ. The role of this new field will be discussed later.

Let us show that the variational principle gives the relativistic fluid equations. By δS, we obtain the particle conservation law

$$\partial_\nu (nU^\nu) = 0. \tag{46}$$

By δn, we find

$$\frac{L_F}{n} = n\frac{\partial \mathcal{E}}{\partial n} U^\mu U^\nu \partial_\nu \Xi_\mu = n\frac{\partial \mathcal{E}}{\partial n} \tag{47}$$

$$= \frac{\mathcal{P}}{n}. \tag{48}$$

We note that the last equality (48), based on the thermodynamic relation (41), holds only in the rest frame.

By δU^ν, we obtain

$$P_\nu = P^\mu \partial_\nu \sigma_\mu + c\partial_\nu S + \mathcal{E} U_\nu, \tag{49}$$

where $P^\mu = \mathcal{E} U^\mu + eA^\mu$.

The variation with respect to Ξ_μ ($\mu = 0, 1, 2, 3$) consists of five components (as mentioned above, $\boldsymbol{V}$ and, hence, U^μ must be freed from $\delta\Xi_\mu$):

$$\int (X^\mu_{(1)} + X^\mu_{(2)} + X^\mu_{(3)} + X^\mu_{(4)} + X^\mu_{(5)})\delta\Xi_\mu \mathrm{d}^4x = 0,$$

$$X^\mu_{(1)} = \partial_\nu[n(\mathcal{E}U^\mu + eA^\mu)U^\nu],$$

$$X^\mu_{(2)} = -n(\partial^\mu \mathcal{E})U^\mu U^\nu \partial_\nu \Xi_\mu = -n(\partial^\mu \mathcal{E}),$$

$$X^\mu_{(3)} = -en(\partial^\mu A^\mu)U^\nu \partial_\nu \Xi_\mu = -en(\partial^\mu A^\mu)U_\mu,$$

$$X^\mu_{(4)} = \frac{L_F}{n}(\partial^\mu n) = n(\partial^\mu \mathcal{E}),$$

$$X^\mu_{(5)} = -\partial^\mu L_F.$$

In evaluating $X^\mu_{(4)}$ we have used (47). $X^\mu_{(5)}$ is due to $\delta(\mathrm{d}^4x) = \partial^\mu \delta\Xi_\mu \mathrm{d}^4x$. We want to use (48) to evaluate $X^\mu_{(5)}$. For this purpose, we have to calculate the perturbation of the proper time-space volume [21]. Denoting

$$q^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu,$$

which is the 'projector' orthogonal to U^ν, we may write

$$\delta(\mathrm{d}^4x) = q^{\mu\nu}\partial_\nu \delta\Xi_\mu\, \mathrm{d}^4x.$$

Hence, we obtain

$$X^\mu_{(5)} = -\partial_\nu\left(q^{\mu\nu}\mathcal{P}\right).$$

Combining all terms using (46) in $X^\mu_{(1)}$, we obtain the familiar equation of motion (see, for example, [22]):

$$\partial_\nu T^{\mu\nu} - enF^{\mu\nu}U_\nu = 0, \tag{50}$$

where

$$T^{\mu\nu} = (n\mathcal{E} + \mathcal{P})U^\mu U^\nu - \mathcal{P}g^{\mu\nu} \tag{51}$$

is the standard energy–momentum tensor.

We remark that the zero-component ($\mu = 0$) of (50) has been derived by the variation of the new field $\Xi_0 = c\tau$, which was fixed to ct in the non-relativistic version (34). In the present framework, indeed, the derivation of the zero-component (which primarily implies the energy conservation) by the variation of Ξ_0 was not necessary—we can deduce it from the other momentum equations ($\mu = 1, 2, 3$) and the particle conservation law (46); by contracting (50) with U_μ, we obtain the isentropic relation $nTU^\nu\partial_\nu\mathcal{S} = 0$ (T: temperature, $\mathcal{S}$: entropy); for example see [22, 23].

Hence, the introduction of the new field Ξ_0 was for the symmetry of the covariant formulation. In a more general matter-field coupling, however, the energy conservation law may not be a consequence of the momentum equations and the particle conservation law.

We end this section pointing out that the Eulerian field $\Xi_\mu = x_\mu - \sigma_\mu$ is the 'displacement' of the observation point x_μ from the initial point σ_μ of the fluid element. It is now evident that the Clebsch potential σ_μ is the initial coordinate, i.e. the Lagrangian label.

8. Concluding remarks

Analyzing how 'vortices' can be represented in Lagrangian formalisms, we have delineated an explicit connection between the Lagrangian and Eulerian descriptions—in an abstract language, they are in a 'duality relation' of adjoint and coadjoint representations of maps. More explicitly, the duality is that of the Lagrangian description of the diffeomorphism $\boldsymbol{Q}(\boldsymbol{x}_0, t)$ and the Eulerian description of the Lagrangian coordinates $\boldsymbol{Q}(\boldsymbol{x}, t)^{-1}$. The latter is nothing but the Clebsch parameter $\boldsymbol{\sigma}$, and by this translation, we could show the direct equivalence of the Lagrangians in both views.

The parameter $\boldsymbol{\sigma}$ is the creator of vorticity; this connection, however, is not apparent in the Lagrangian view in which every topological constraint is recognized as an initial condition (this naïveness can be a merit of the Lagrangian view). By analyzing the *range* of the Clebsch parametrizations of different degree (ν), we noted that the embedding of the given subclass of canonical dynamics into the general phase space is rather complicated; the $\nu = 0$ class is confined in the linear subspace of potential flows (giving the complete representation of every potential flow), the $\nu = 1$ class is confined in the zero-helicity leaf of the Casimir foliation (not complete to represent every zero-helicity flow, while containing superfluous components), $\nu = 2$ class is still not dense (while containing superfluous components), and $\nu = 3$ class is complete (containing superfluous components).

As noted, the canonicalization by Casimir parametrization (i.e. representation by Serrin–Lin fields) of the original non-canonical system is not simply the separation of the constrained degree of freedom (such as the helicity). The reader is referred to section 2 of Mills *et al* [24] for an interesting comparison of the Lagrangian and Eulerian views (the authors call the former 'Newcomb gauge'), in which the potential energy of the Beltrami field, in its Eulerian representation, is shown to be constrained directly by the helicity.

In this paper, we considered a single-species plasma (putting $e = 0$, the model degenerates into that of a neutral fluid). Generalization to multi-species plasma is straightforward. However, the singular perturbation of limiting the electron inertia and the ion skin depth to zero is not evident; the connection of the MHD model and two-fluid model seems to be discontinuous.

A variational principle may merit in formulating approximation schemes such as numerical or perturbative methods. Eulerian Clebsch representation will be useful when we consider an appropriate subclass of dynamics or structures (for example, the $\nu = 1$ integrable vortex-line structure).

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