Real beads on virtual strings: Charged particles on magnetic field lines

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We discuss a similarity between the drift of a charged particle inside a slowly moving solenoid and the motion of a fluid element in an ideal incompressible fluid. This similarity can serve as a useful instructional example to illustrate the concepts of magnetic field lines and magnetic confinement.
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I. INTRODUCTION

This paper is inspired by several discussions of the magnetic confinement concept with graduate students. We hope that it may therefore serve as useful addition to the collection of educational examples in plasma physics.

Explanation of plasma confinement by a magnetic field often involves a statement that particles are attached to the field lines.[1] This statement is usually accompanied by a figure in which the field lines look like fibers or strings that can be identified by their position with respect to the magnetic coils of the confinement device. Such an impression is in fact deceptive and can easily be misleading, because the magnetic field lines differ essentially from a bunch of material strings linked to the coils.

A less vibrant but more accurate description of plasma "attachment" to the field lines is known as the frozen-in constraint.[2–4] It is based on magnetic flux conservation and it is inherently immune to the temptation of seeing magnetic field lines as physical objects. The frozen-in constraint simply reflects the fact that a perfectly conducting loop preserves the magnetic flux through that loop. It is appropriate to emphasize that the frozen-in constraint applies to the conducting medium rather than to single-particle behavior. An apparent distinction between the conducting fluid (plasma) and the test particles is that the plasma itself can modify the magnetic field.

It is still interesting to take a closer look at a single-particle problem in order to illustrate the difference between a bunch of immaterial magnetic field lines and a bunch of material strings. We believe that the simple single-electron problem discussed below can help to visualize what particles are "attached to" when they are confined by the magnetic field.

II. PROBLEM FORMULATION

Consider an initially immobile test electron inside a very long superconducting solenoid of an arbitrary (not necessarily circular) cross section with a uniform magnetic field B_z. Imagine that the solenoid moves slowly in the xy plane as a rigid body. The question to answer is how the test electron will move in this case.

The simplistic idea that the magnetic field lines act like material strings and that the electron is attached to a field line would suggest that the electron travels together with the solenoid in a rigid body manner. This is generally not the case. The truth is that this naive prediction works fortuitously for a purely translational motion, but it fails when the motion involves rotation.

The case of a solenoid with a circular cross section makes it immediately clear that slow rotation of such a solenoid around its axis will leave our test electron at rest in the laboratory frame. The reason is that rotation of the axisymmetric solenoid does not affect the magnetic field at any spatial location. This particular example already shows that, generally speaking, the test electron will move with respect to the solenoid, in contrast with the rigid body expectation.

Having disproved the naive rigid body picture, we are now left with a question of whether there exists a valid and relatively simple alternative. The answer is affirmative. In what follows, we will show that the test electron actually moves as an element of ideal incompressible fluid within the volume bounded by the solid wall of the solenoid. The electron is thereby magnetically confined within the solenoid without being linked to any particular field line. This simple description is also fully applicable to an electron that moves initially along the solenoid axis, because the axial motion is completely decoupled from the transverse motion in our illustrative problem. Other configurations of the magnetic coils will generally make the parallel motion a nontrivial part of the problem, which would be an interesting issue to address separately.

III. TEST PARTICLE MOTION

The very slow motion of the solenoid (compared to the electron gyro-motion) suggests that inertia of the test electron is negligibly small in the problem of interest. In this case, the electron equation of motion,

$$m\frac{d\mathbf{V}}{dt} = e(\mathbf{E} + [\mathbf{V}\times\mathbf{B}]), \tag{1}$$

simplifies to

$$\mathbf{E} + [\mathbf{V}\times\mathbf{B}] = 0, \tag{2}$$

where $\mathbf{V}$ is the electron velocity and $\mathbf{E}$ is the electric field that arises due to the motion of the solenoid.

Given that the magnetic field $\mathbf{B}$ is spatially uniform and that it does not change in time inside the solenoid, we find from Faraday's law that $\nabla\times\mathbf{E} = 0$ inside the solenoid. To be more precise, the change of the uniform magnetic field inside the moving solenoid (with velocity u) is negligible because it is on the order of u^2/c^2 (as follows from Lorentz transformation of the field[5]), whereas the inductive electric field is proportional to u/c. The electric field can therefore be represented by an electrostatic potential Ψ as

$$\mathbf{E} = -\nabla\Psi. \tag{3}$$

In addition, we note that the potential must satisfy the Laplace equation,

$$\nabla^2\Psi = 0, \tag{4}$$

because there is no space charge (other than the test electron) inside the solenoid. It follows from Eqs. (2) and (3) that the electron drifts across the magnetic field with

$$\mathbf{V} = -\frac{1}{B^2}[\nabla\Psi \times \mathbf{B}]. \tag{5}$$

It is noteworthy that this velocity field is divergence-free, i.e.,

$$\nabla \cdot \mathbf{V} = 0, \tag{6}$$

when the magnetic field is spatially uniform. It also follows from Eqs. (4) and (5) that the velocity field is vortex-free, i.e.,

$$\nabla \times \mathbf{V} = 0, \tag{7}$$

in the area of spatially uniform magnetic field. The vortex-free feature makes it possible to introduce a potential Φ for the velocity field, so that

$$\mathbf{V} = \nabla\Phi. \tag{8}$$

Like the electric potential Ψ, the velocity potential Φ must satisfy the Laplace equation,

$$\nabla^2\Phi = 0, \tag{9}$$

which ensures that the velocity field is divergence-free [see Eq. (6)]. The two different representations of the two-dimensional velocity field are apparently linked by the Cauchy-Riemann relations:

$$\frac{\partial\Phi}{\partial x} = -\frac{1}{B}\frac{\partial\Psi}{\partial y}, \tag{10}$$

$$\frac{\partial\Phi}{\partial y} = \frac{1}{B}\frac{\partial\Psi}{\partial x}. \tag{11}$$

We now return to Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{12}$$

to determine the electric field at the solenoid interior surface. When the solenoid moves as a rigid body, its local velocity is

$$\mathbf{u} \equiv \mathbf{u}_0 + [\Omega \times \mathbf{r}], \tag{13}$$

where $\mathbf{u}_0$ is the center of mass velocity, $\Omega \equiv \Omega\mathbf{b}$ is the angular velocity, $\mathbf{b}$ is a unit vector directed along the solenoid axis (which is also the direction of the magnetic field), and the radius $\mathbf{r}$ is measured from the center of mass. Given that the motion is very slow, we consider the magnetic field to be quasistatic and determined by the Biot-Savart law for a given shape of the rigid coil, so that the field is time-independent in the reference frame associated with the solenoid. This approximation means that the entire field pattern moves in a rigid way together with the solenoid in the laboratory frame, i.e.,

$$\frac{\partial\mathbf{B}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{B} = 0. \tag{14}$$

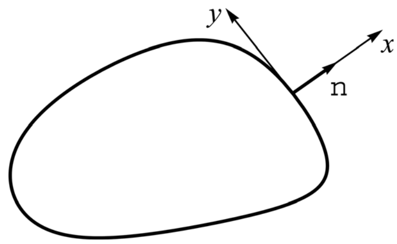

Fig. 1. Local coordinate system that is used to calculate the electric field inside the moving solenoid.

We next combine Eqs. (12) and (14) into

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = u_x\frac{\partial B_z}{\partial x} + u_y\frac{\partial B_z}{\partial y}. \tag{15}$$

It is convenient to assume, without loss of generality, that the x axis is locally normal to the coil, whereas the y axis is tangential to the coil, as shown in Fig. 1. With this assumption, integration of Eq. (15) over a small interval δx across the coil gives a tangential component of the electric field at the coil inside the solenoid

$$E_y = u_x B_z, \tag{16}$$

which is a general boundary condition at the surface of a moving conductor.

On the other hand, the y component of Eq. (2) gives

$$E_y = V_x B_z. \tag{17}$$

Compatibility of Eqs. (16) and (17) requires $V_x = u_x$ at the coil, which means that the normal component of the electron velocity must vanish at the coil in the solenoid rest frame. This condition can be rewritten in terms of either Ψ or Φ by taking into account that $V_x = -(1/B)(\partial\Psi/\partial y) = \partial\Phi/\partial x$. An immediate consequence of this condition is that the electron never crosses the wall and remains perfectly confined within the solenoid as it moves along its orbit (see Fig. 4).

The entire velocity field can now be determined by solving either Eq. (4) or Eq. (9) in the domain bounded by the solenoid wall and with the following boundary conditions for Ψ and Φ (in a coordinate-independent form):

$$\nabla\Psi \cdot [\mathbf{n} \times \mathbf{B}] = B^2(\mathbf{n}\cdot\mathbf{u_0} + \mathbf{n}\cdot[\Omega \times \mathbf{r}]), \tag{18}$$

$$(\mathbf{n}\cdot\nabla\Phi) = (\mathbf{n}\cdot\mathbf{u_0} + \mathbf{n}\cdot[\Omega \times \mathbf{r}]), \tag{19}$$

where $\mathbf{n}$ is a unit vector normal to the wall.

IV. VORTEX-FREE FLOW OF IDEAL FLUID BOUNDED BY THE MOVING SURFACE

In fluid dynamics, the Euler equation for isentropic flow has the form[6]

$$\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla w, \tag{20}$$

where w is the fluid enthalpy per unit mass. It follows from Eq. (20) that an initially vortex-free flow ($\nabla \times \mathbf{V} = 0$) remains vortex-free over time. Such flow is described by means of a velocity potential Φ so that the flow velocity equals the gradient of the potential, $\mathbf{V} = \nabla\Phi$. In the case of an incompressible flow, the velocity has zero divergence. As a result, the velocity potential has to satisfy Laplace's equation,[7] $\nabla^2\Phi = 0$. If the ideal fluid is bounded by a moving solid surface, then the normal component of the fluid velocity must be equal to the normal component of the surface velocity $\mathbf{u}$, i.e., $\mathbf{n}\cdot\nabla\Phi = \mathbf{n}\cdot\mathbf{u}$. We thus conclude that the electron velocity field in a moving solenoid and the velocity of an incompressible fluid do satisfy the same equations.

V. TRANSLATIONAL MOTION

In the absence of rotation ($\Omega = 0$), a straightforward constant velocity solution,

$$\nabla\Phi = \mathbf{u_0}, \tag{21}$$

satisfies Eq. (9) and boundary condition (19). This solution means that the interior of the solenoid moves as a rigid body and that the electric field is spatially uniform inside the solenoid, i.e.,

$$\mathbf{E} = -[\mathbf{u}_0 \times \mathbf{B}], \tag{22}$$

as follows from Eq. (2). The electric field apparently vanishes outside the solenoid, which implies that there appears an induced surface charge at the solenoid inner surface (see Fig. 2). The fact that the electric field and the velocity field are spatially uniform in the case of the purely translational motion explains why the naive idea that the particle is tied to a particular field line works fortuitously in this case.

VI. ROTATION

Let $R(\theta)$ be the shape of the solenoid wall in polar coordinates (r, θ). We choose the origin of the coordinate system inside the solenoid, so θ varies from 0 to 2π. The unit vector normal to the wall is then

$$\mathbf{n} = \frac{R(\theta)\dfrac{\mathbf{r}}{r} + \dfrac{dR(\theta)}{d\theta}\left[\dfrac{\mathbf{r}}{r}\times\mathbf{b}\right]}{\sqrt{R^2(\theta) + \left(\dfrac{dR(\theta)}{d\theta}\right)^2}}. \tag{23}$$

A regular (at the origin) solution to Laplace's equation (9) is

$$\Phi = \Omega\sum_{m=0}^{\infty}(a_m r^m \cos m\theta + b_m r^m \sin m\theta), \tag{24}$$

and the corresponding potential of the electric field is [see Eqs. (10) and (11)]

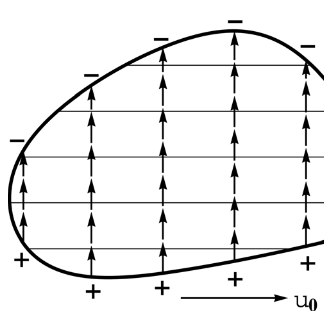

Fig. 2. Electric field lines (vertical) and equipotential lines (horizontal) in the solenoid moving with a constant velocity u_0. The + and − signs indicate induced surface charges at the solenoid inner surface.

$$\Psi = \Omega B\sum_{m=0}^{\infty}(-a_m r^m \sin m\theta + b_m r^m \cos m\theta). \tag{25}$$

In this expression, constants a_m and b_m need to be chosen to satisfy conditions (18) and (19) at the boundary. Given that

$$\nabla\Phi = \Omega\frac{\mathbf{r}}{r}\sum_{m=0}^{\infty} m r^{m-1}(a_m\cos m\theta + b_m \sin m\theta) + \Omega\left[\frac{\mathbf{r}}{r}\times\mathbf{b}\right]\sum_{m=0}^{\infty} m r^{m-1}(a_m \sin m\theta - b_m\cos m\theta), \tag{26}$$

we rewrite the boundary condition (19) as

$$-\frac{1}{2}\frac{d}{d\theta}R^2 = \frac{d}{d\theta}\sum_{m=0}^{\infty}R^m(a_m\sin m\theta - b_m\cos m\theta), \tag{27}$$

which incorporates Eq. (23) for $\mathbf{n}$. The boundary condition for Ψ [Eq. (18)] also reduces to Eq. (27). The electric field pattern is now more complicated than for the translational motion and depends on the solenoid shape (see Fig. 3).

VII. PARTICLE ORBITS

Equations (24) and (27) determine the velocity field and the ensuing particle orbits for a given shape of the rotating solenoid $R(\theta)$. Fourier transformation reduces Eq. (27) for a_m and b_m to a set of linear algebraic equations that can be solved numerically. All orbits are closed in the solenoid reference frame and they represent contour plots of the function

$$\frac{r^2}{2} - \frac{\Psi}{B\Omega} \equiv \frac{r^2}{2} + \sum_{m=0}^{\infty} r^m(a_m\sin m\theta - b_m\cos m\theta). \tag{28}$$

However, the periods of particle motion (in the solenoid frame) are generally different from the solenoid rotation period and they are different for different orbits, as seen from Fig. 4. It is noteworthy that the shape of the orbits is independent of the angular velocity Ω. The nested orbits in Fig. 4 exhibit perfect confinement of the particles within the solenoid. The particle motion along the orbit is governed by the following equations:

$$\frac{dr}{dt} = \frac{\partial\Phi}{\partial r}, \tag{29}$$

$$\frac{d\theta}{dt} = \frac{1}{r^2}\frac{\partial\Phi}{\partial\theta} - \Omega. \tag{30}$$

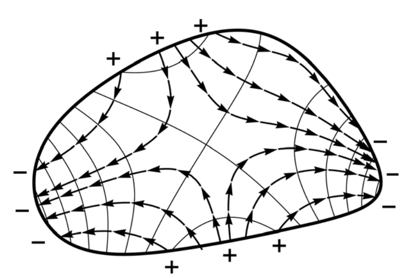

Fig. 3. Electric field lines, equipotential lines, and induced surface charges in the solenoid rotating with a constant angular velocity Ω.

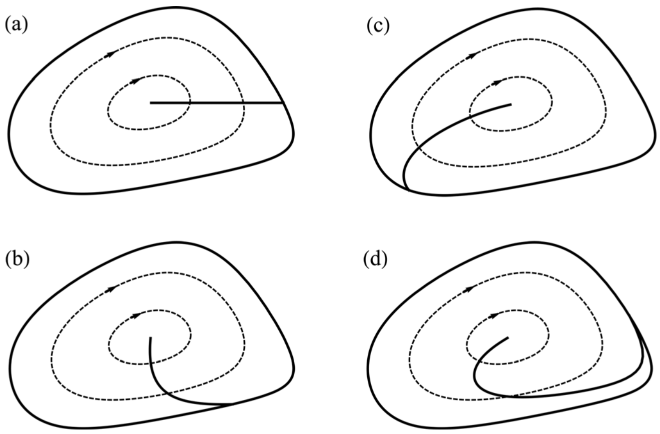

Fig. 4. Snapshots of the particle positions in the reference frame that rotates counterclockwise with the solenoid: (a) $t=0$; (b) $t=0.5\,\pi/\Omega$; (c) $t=\pi/\Omega$; (d) $t=12\,\pi/\Omega$. The outmost contour is the solenoid boundary. The dashed contours show particle orbits. The evolving curve marks the particles that are initially located on the straight horizontal line. This curve coils up over time because the particles have different rotation periods.

The same orbits look more complicated in the laboratory frame where they generally become helical, as shown in Fig. 5.

There are two particular cases in which Eq. (27) admits simple analytical solutions: (1) any elliptical solenoid (see Ref. 6); and (2) a nearly circular solenoid with an arbitrary but small deviation from the circular shape.

For an elliptical solenoid with a major semi-axis $A/\sqrt{1-\alpha}$ and minor semi-axis $A/\sqrt{1+\alpha}$, we have

$$R^2 = \frac{A^2}{1+\alpha\cos 2\theta}, \tag{31}$$

and it is then straightforward to check that

$$\Phi = -\frac{1}{2}\alpha\Omega r^2 \sin 2\theta \tag{32}$$

satisfies the Laplace equation with the boundary condition Eq. (27). The only nonvanishing coefficient in Eq. (24) in this case is

$$b_2 = -\alpha/2, \tag{33}$$

so that the velocity potential and the electrostatic potential scale as r^2 for the elliptical solenoid. As a result of this special feature, all particle orbits, defined by Eq. (28), have the same elliptical shape with an eccentricity $\varepsilon = \sqrt{2\alpha/(1+\alpha)}$ in the solenoid rest frame. Also, in this frame, the rotation period T is the same for all orbits, because the r^2 scaling of Φ eliminates the r dependence from Eq. (30) and thereby decouples Eq. (30) from Eq. (29). A straightforward integration of Eq. (30) now gives $T=(2\pi/\Omega)/\sqrt{1-\alpha^2}$. It is noteworthy that T is generally different from the solenoid rotation period $2\pi/\Omega$. For a solenoid other than elliptical the periods would generally be different for different orbits, as illustrated in Fig. 4.

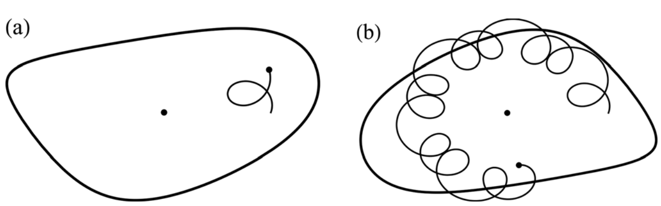

Fig. 5. Test particle orbit in the laboratory reference frame: (a) $t=\pi/\Omega$; (b) $t=12\,\pi/\Omega$. The dot on the orbit marks the instantaneous position of the particle. The closed curve shows the instantaneous position of the solenoid wall. The central dot marks the rotation axis of the solenoid. Although a part of the trajectory image is outside the solenoid (at $t=12\,\pi/\Omega$), the particle itself always stays inside of the solenoid.

A solution for any nearly circular solenoid can be constructed via a perturbation technique. We let

$$R = R_0 + \sum_{l=2}^{\infty}(c_l\cos l\theta + d_l \sin l\theta), \tag{34}$$

with $c_l \ll R_0$ and $d_l \ll R_0$. We can safely assume that $c_1 = d_1 = 0$ in Eq. (34), because c_1 and d_1 describe a displacement of the solenoid axis from the solenoid center of mass. By retaining only linear terms in c_l/R_0 and d_l/R_0, we reduce Eq. (27) to

$$R_0\sum_{l=2}^{\infty}(c_l\cos l\theta + d_l\sin l\theta) = \sum_{m=2}^{\infty} {R_0}^m (b_m \cos m\theta - a_m \sin m\theta). \tag{35}$$

We thus have $b_m = R_0^{1-m}c_m$ and $a_m = -R_0^{1-m}d_m$, which gives the following explicit expression for the velocity potential:

$$\Phi = \Omega\sum_{m=2}^{\infty} R_0^{1-m}(c_m r^m \sin m\theta + d_m r^m \cos m\theta). \tag{36}$$

The corresponding particle orbits are defined by Eq. (28). In particular, we have $\Phi = 0$ for the circular cross section of the solenoid, which is consistent with our earlier remark in Sec. II that the electron will remain at rest in the lab frame when such a solenoid rotates around its axis.

VIII. CONCLUSIONS

We have discussed an illustrative problem that establishes a link between two-dimensional motion of a magnetized test particle in a solenoid (of arbitrary shape) and vortex-free flow of an ideal incompressible fluid. In particular, the particle moves in step with the magnetic coils if the coil motion is translational, but such rigidity is lost when the coils rotate. However, the particle still remains perfectly confined within the solenoid as long as the solenoid motion is sufficiently slow. Moreover, the particle trajectory in the solenoid reference frame is independent of the rotation velocity and depends solely on the solenoid shape. This problem also shows that the concept of particle attachment to magnetic field lines does not provide an immediate solution to the problem of particle motion because the field lines themselves are not rigidly linked to the magnetic coils. The subtlety is that it is generally impossible to somehow mark a field line and trace its motion when the magnetic coils move together with the current they carry.[8]

It may be tempting to extend the described analogy with a fluid flow to three-dimensional magnetic configurations (say, a toroidal system with closed field lines). However, nonuniformity of the magnetic field and anisotropy of the magnetic stress tensor actually prevent such an extension. Our two-dimensional problem with spatially uniform magnetic field is apparently special in that regard. Yet, the problem of single particle motion in the field of slowly moving magnetic coils is still quite tractable, because the corresponding magnetic and electric fields are determined by a vector potential that is related to the current density in the coils by the Biot-Savart law. The corresponding current density is then constant in the reference frame associated with the rigid coils. As the coils move, the currents are displaced in the laboratory frame accordingly. An interesting aspect of this problem is that the particle will now move not just across but also along the magnetic field, because the induced electric field will generally have a parallel component. This feature deserves a separate discussion that goes beyond the scope of the present paper.

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Liquid Level Demonstration

This device, on display at the student lounge at the Creighton University physics department, is sometimes called Pascal's Vase. The pressure at a given depth is equal across the apparatus. Thus, if students understand the formula for pressure: $p = \eta gh$, where η is the density and h is the height of the water column, they will know that the height of the water level is independent of the volume or shape of the reservoir. This demonstration is also called the Hydrostatic Paradox. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)