Mode signature and stability for a Hamiltonian model of electron temperature gradient turbulence

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Stability properties and mode signature for equilibria of a model of electron temperature gradient (ETG) driven turbulence are investigated by Hamiltonian techniques. After deriving new infinite families of Casimir invariants, associated with the noncanonical Poisson bracket of the model, a sufficient condition for stability is obtained by means of the Energy-Casimir method. Mode signature is then investigated for linear motions about homogeneous equilibria. Depending on the sign of the equilibrium "translated" pressure gradient, stable equilibria can either be energy stable, i.e., possess definite linearized perturbation energy (Hamiltonian), or spectrally stable with the existence of negative energy modes. The ETG instability is then shown to arise through a Kreĭn-type bifurcation, due to the merging of a positive and a negative energy mode, corresponding to two modified drift waves admitted by the system. The Hamiltonian of the linearized system is then explicitly transformed into normal form, which unambiguously defines mode signature. In particular, the fast mode turns out to always be a positive energy mode, whereas the energy of the slow mode can have either positive or negative sign. A reduced model with stable equilibria shear flow that possess a continuous spectrum is also analyzed and brought to normal form by a special integral transform. In this way it is seen how continuous spectra can have signature as well. © *2011 American Institute of Physics*. [doi:10.1063/1.3569850]

I. INTRODUCTION

An important issue for the stability of equilibria of continuous media concerns the existence of *negative energy modes* (NEMs), spectrally stable modes of oscillation of a medium with negative energy. One reason NEMs are important is because equilibria with them, although linearly or spectrally stable, can be destabilized by arbitrarily small perturbations. For example, if dissipation is added to the dynamics so as to remove energy from a NEM, then it can be proven that the mode becomes spectrally unstable. This is the so-called Thompson–Tait theorem.[1–4] On the intuitive level, dissipation removes energy from the already negative energy of the mode, which makes it more negative and increases the amplitude of the mode. In nondissipative systems, NEMs can become unstable with the presence of positive energy modes (PEMs) through nonlinear coupling.[5–10] By this means the system can even develop finite-time singularities while conserving the energy of the nonlinear system.

In plasma physics, the study of NEMs has a long tradition dating to the early work of Sturrock[11] on streaming instabilities and Greene and Coppi[4] on magnetohydrodynamic (MHD) type dissipative instabilities in confinement systems. NEMs have been studied in many plasma contexts; for example, Vlasov–Maxwell dynamics,[12–17] Maxwell drift-kinetic[18] theories, wave-wave interaction in the two-stream instability,[8,9,19] magnetic reconnection,[20] ideal MHD in the presence of equilibrium flows,[21–23] magnetorotational instability,[24,25] and magnetosonic waves in the solar atmosphere.[26]

In order to find NEMs it is, of course, necessary to obtain the energy of the linear dynamics, and various means have been used for accomplishing this. For example, the linear equations of motion can be manipulated in order to obtain a quadratic conserved quantity, that is then deemed the energy. However, without proper physical intuition, this can give an incorrect answer, since energy signature cannot be determined by the linear equations of motion alone. Alternatively, one can appeal to the expression for the dielectric energy for media in terms of a dielectric function, a common practice. However, this also can give the wrong answer, as was shown in Ref. 14. When one is dealing with reduced fluid models that contain various physics, the situation becomes even more difficult. For these reasons we have argued in Ref. 27 that the only reliable way to define energy is within the Hamiltonian context. Indeed, once the Hamiltonian structure of the model under consideration is known, an unambiguous definition of the energy of the system becomes available: the total energy corresponds to the Hamiltonian of the system and the energy of the linear dynamics must come in a natural way from the second variation of this nonlinearly conserved quantity. Moreover, the normal form theory for linear Hamiltonian systems, provides a clear and systematic way for determining the signature of modes in the neighborhood of an equilibrium of the system. Indeed, for systems with discrete degrees of freedom, the Hamiltonian of the linearized system can always be cast, for stable modes, into the sum of Hamiltonians of decoupled harmonic oscillators, each of which possesses a characteristic frequency and a *characteristic signature*. Namely this signature, which, for each mode, can be positive or negative depending on whether the mode provides a positive or negative contribution to the total energy, provides a systematic way to identify PEMs or NEMs of the system. Finally, given the existence of the Hamiltonian structure, energy-based methods, akin to "δW" of MHD, can be used to obtain sufficient conditions for stability of equilibria or to indicate the presence of NEMs. More precisely, an equilibrium has a NEM, if it is spectrally stable but the second variation of its free energy functional, evaluated at that equilibrium, has indefinite sign. We also point out that, by virtue of its generality, the Hamiltonian approach, is applicable, unlike for instance traditional methods based on the plasma dispersion relation, to the case of arbitrary equilibria and is not restricted to the case of plane wave perturbations.

In this paper, we investigate the presence of NEMs and stability properties of a reduced model for electron temperature gradient (ETG) instabilities, in the Hamiltonian framework. ETG turbulence has been considered as one of the mechanisms that enhances anomalous particle and electron thermal fluxes in tokamaks.[28,29] The detection of NEMs, is therefore important in order to see what potentially unstable modes might lie dormant in the absence of dissipation. When destabilized by dissipation, such modes might enhance the anomalous transport across the confining magnetic field. We note, however, that the methods applied for the ETG model, can be applied, in principle, to any ideal plasma model.

The model for ETG turbulence considered here, has been previously investigated in Refs. 30 and 31, and in Ref. 30 it was described how this model possesses a noncanonical Hamiltonian formulation, with a Poisson bracket that turns out to be essentially identical to that for reduced MHD[32] (see also Refs. 33 and 34).

In the present paper we first provide further information about the Hamiltonian structure of the model by deriving explicitly its families of Casimir invariants. We then determine sufficient conditions for stability of generic equilibria by making use of the Energy-Casimir method. The investigation of the presence of NEMs is carried out for the case of homogeneous equilibria, for which we derive an explicit condition for the presence of NEMs. This condition warns us that NEMs are present if the value of the equilibrium pressure gradient lies in a given interval whose end points depend on the perpendicular wave vector and the magnetic field curvature. In particular, the length of this interval shrinks to zero as the perpendicular wave number goes to infinity. After obtaining the eigenvalues and eigenvectors of the system, we explicitly construct and carry out the transformation that puts the Hamiltonian for the ETG model into normal form. This is followed by considering inhomogeneous equilibria in a reduced system that has pure continuous spectrum. The transformation to normal form is obtained and signature is thus assigned to the continuum, for the first time in a plasma fluid model.

The paper is organized as follows. In Sec. II we review the model and its Hamiltonian formulation, and then derive the Casimir invariants. In Sec. III we apply the Energy-Casimir method and obtain conditions for energy stability. In Sec. IV, after reviewing the theory of mode signature for linear Hamiltonian systems we apply it to the ETG model. Explicit conditions for the existence of NEMs, their relationship to energy stability and spectral stability conditions coming from the dispersion relation are derived, and the normal form transformation is explicitly obtained. Next, in Sec. V, we consider the reduced model that contains only stable continuous spectra. We show how an integral transform can be used to map this system into normal form and thus define signature for the continuous spectrum. Finally, we conclude in Sec. VI.

II. HAMILTONIAN STRUCTURE OF THE ETG MODEL

The nondissipative ETG driven turbulence model of Refs. 30 and 31 is given by

$$\frac{\partial}{\partial t}(1-\nabla^2)\phi = [\phi, \nabla^2\phi + x] + \left[\frac{p}{\sqrt{r}}, \sqrt{r}x\right], \tag{1}$$

$$\frac{\partial}{\partial t}\frac{p}{\sqrt{r}} = \left[\frac{p}{\sqrt{r}}, \phi\right] + [\sqrt{r}x, \phi], \tag{2}$$

where two-dimensional slab geometry is assumed so that the field variables, ϕ the stream function and p the pressure, are functions of the Cartesian coordinates (x,y). The quantity $[f,g] = \partial f/\partial x \partial g/\partial y - \partial f/\partial y \partial g/\partial x$ is the canonical Poisson bracket. Equations (1) and (2) are written in the normalized form described in Ref. 30, where the constant parameter r is defined as

$$r = \frac{L_n^2}{L_B L_P}, \tag{3}$$

with L_n, L_B, and L_P being the characteristic length scales of variation of the background density, magnetic, and electron pressure fields, respectively. The parameter r is thus related to the mechanism providing the drive for the ETG instability. In particular, in the limit $r \to 0$, corresponding to a flattening of the electron pressure gradient, the linear dispersion relation indicates that ETG modes degenerate into marginally stable drift waves.[31]

In Ref. 30, the authors showed that the systems Eqs. (1) and (2) possess a Hamiltonian structure in terms of a noncanonical Poisson bracket. This means (see, e.g., Refs. 10 and 33) that the system can be cast in the form

$$\frac{\partial \chi^i}{\partial t} = \{\chi^i, H\}, \quad i = 1, \ldots, n \tag{4}$$

with $\chi^i(\mathbf{x},t)$ indicating a suitable set of n field variables (with $n=2$ in our case) and $H[\chi^1, \cdots, \chi^n]$ a Hamiltonian functional that is conserved by the dynamics. The Poisson bracket $\{,\}$ appearing in Eq. (4) is an antisymmetric bilinear binary operator satisfying the Leibniz rule and the Jacobi identity. For the model of Eqs. (1) and (2), it was shown[30] that, with the choice $\chi^1 = \phi$, $\chi^2 = p$, the Hamiltonian of the system is

$$H[\phi, p] = \frac{1}{2}\int d^2x \left(\phi^2 + |\nabla\phi|^2 - \frac{p^2}{r}\right) \tag{5}$$

and the Poisson bracket is

$$\{F,G\} = -\int d^2x\big((\phi-\nabla^2\phi - x)[\mathcal{L}^{-1}F_\phi,\mathcal{L}^{-1}G_\phi] + (p+rx)([\mathcal{L}^{-1}F_\phi,G_p]+[F_p,\mathcal{L}^{-1}G_\phi])\big), \qquad (6)$$

where the operator $\mathcal{L}$, and its inverse $\mathcal{L}^{-1}$ are formally defined so that $\mathcal{L}f=f-\nabla^2 f$, and $\mathcal{L}^{-1}\mathcal{L}f=\mathcal{L}\mathcal{L}^{-1}f=f$, for a generic field f. The subscripts on F and G in Eq. (6) denote functional derivatives with respect to the fields ϕ or p.

Noncanonical Poisson brackets such as Eq. (6) are characterized by the presence of so called Casimir invariants (see, e.g., Ref. 10), due to degeneracy in the cosymplectic operator of the bracket. More precisely, a Casimir invariant of a Poisson bracket is a functional $C(\chi^1,\ldots,\chi^n)$ that satisfies

$$\{C,F\}=0 \qquad (7)$$

for any functional F of the field variables. Because they commute in particular with any H, Casimir functionals are preserved during the dynamics. In order to derive the Casimir invariants of the ETG model, it is convenient to introduce the variables

$$\mathcal{P}=\frac{p}{\sqrt{r}}+\sqrt{r}x, \quad \lambda=\phi-\nabla^2\phi - x, \qquad (8)$$

which correspond to a "translated" pressure and to a variable analogous to the potential vorticity of the Charney–Hasegawa–Mima equation,[35,36] respectively. In terms of these variables the model equations read

$$\frac{\partial}{\partial t}\lambda = -[\mathcal{L}^{-1}(\lambda+x),\lambda]+[\mathcal{P},\sqrt{r}x], \qquad (9)$$

$$\frac{\partial}{\partial t}\mathcal{P}=[\mathcal{P},\mathcal{L}^{-1}(\lambda+x)], \qquad (10)$$

whereas the Hamiltonian and the bracket become

$$H(\lambda,\mathcal{P})=\frac{1}{2}\int d^2x((\lambda+x)\mathcal{L}^{-1}(\lambda+x)-\mathcal{P}^2+2\sqrt{r}\mathcal{P}x), \qquad (11)$$

$$\{F,G\}=-\int d^2x(\lambda[F_\lambda,G_\lambda]+\mathcal{P}([F_\lambda,G_\mathcal{P}]+[F_\mathcal{P},G_\lambda])), \qquad (12)$$

where the bracket is seen to be identical to that for reduced MHD as first given in Ref. 32.

Applying Eq. (7) we deduce that the equations determining the Casimir invariants for our system are

$$[C_\lambda,\lambda]+[C_\mathcal{P},\mathcal{P}]=0, \qquad [C_\lambda,\mathcal{P}]=0. \qquad (13)$$

By solving Eq. (13) we see that the system admits two independent infinite families of Casimirs:

$$C_1=\int d^2x\mathcal{H}(\mathcal{P}), \quad C_2=\int d^2x\lambda\mathcal{F}(\mathcal{P}), \qquad (14)$$

with $\mathcal{H}$ and $\mathcal{F}$ arbitrary functions. The dynamics described by the inviscid ETG model is then subject to an infinite number of constraints imposed by the conservation of the Casimir invariants Eq. (14). For instance, as a consequence of the conservation of C_2, integrals of the potential vorticity λ over regions bounded by contour lines of $\mathcal{P}$ will be conserved during the dynamics (see Ref. 37).

Notice that the constant of motion $\mathcal{I}$ found in Ref. 30 is given by

$$\mathcal{I}=\int d^2x\left(\frac{p^2}{r}+2(\phi-\nabla^2\phi)p\right) = 2\sqrt{r}\int d^2x\lambda\mathcal{P}+\int d^2x\mathcal{P}^2-2r\int d^2xx\left(\phi-\nabla^2\phi-\frac{x}{2}\right), \qquad (15)$$

which is a linear combination of two particular Casimirs of Eq. (14) with the realization that the time derivative of

$$-2r\int d^2xx\left(\phi-\nabla^2\phi-\frac{x}{2}\right) \qquad (16)$$

vanishes if $\partial\phi/\partial x$ and $\partial\phi/\partial y$ vanish or are periodic at the boundaries. From this fact one can then identify the presence of a further constant of motion (which is not a Casimir), corresponding to

$$\int d^2xx(\phi-\nabla^2\phi). \qquad (17)$$

From Noether's theorem one infers that this constant of motion reflects the invariance of the Hamiltonian with respect to translations along the y direction.

III. ENERGY STABILITY

The Hamiltonian formalism provides a systematic procedure for implementing the Energy-Casimir method for investigating stability of equilibria (see, e.g. Refs. 10, 38, and 39), a stability method that originated in plasma physics in Refs. 40 and 41 that has often been referred to as nonlinear stability. This method has been adopted in many works; for example, in the context of fluid models for plasmas in Refs. 42–45. It provides sufficient conditions for stability by taking the second variation of the free energy, the Hamiltonian plus Casimir invariants, and extracting conditions that are necessary for definiteness. Because the method is based on nonlinear constants of motion, the stability conditions obtained are stronger than conditions that emerge from dispersion relations, i.e., spectral stability conditions, that follow entirely from the linear equations of motion. Indeed this second variation stability, which we will refer to simply as *energy stability*, implies linear stability,[10,46] but the converse is not true. In some works energy stability is called formal stability when an additional convexity estimate is not provided. Usually these estimates are rather trivial and even when they are provided they are only a small part of a mathematically rigorous stability proof—for this reason we eschew this terminology.

For noncanonical Hamiltonian systems, equilibrium solutions can be found by solving the equations that result from extremizing the free energy functional. For our system the free energy functional is given by $F=H+C_1+C_2$ (which is not to be confused with the generic functional F of our Poisson brackets), with H given by Eq. (12) and $C_{1,2}$ by Eq. (14). For convenience we introduce the new variables according to the transformation

$$\Lambda = \lambda + x, \quad \bar{\mathcal{P}} = \mathcal{P} \tag{18}$$

and then drop the bar on $\bar{\mathcal{P}}$ in the following. In terms of these variables the Hamiltonian and the bracket become

$$H(\Lambda,\mathcal{P}) = \frac{1}{2}\int d^2x(\Lambda\mathcal{L}^{-1}\Lambda - \mathcal{P}^2 + 2\sqrt{r}\mathcal{P}x),$$

$$\{F,G\} = \int d^2x(x-\Lambda)[F_\Lambda,G_\Lambda] - \mathcal{P}([F_\Lambda,G_\mathcal{P}] + [F_\mathcal{P},G_\Lambda]).$$

The free energy functional is then explicitly given by

$$F(\Lambda,\mathcal{P}) = \frac{1}{2}\int d^2x(\Lambda\mathcal{L}^{-1}\Lambda - \mathcal{P}^2 + 2\sqrt{r}\mathcal{P}x) + \int d^2x\mathcal{H}(\mathcal{P}) + \int d^2x(\Lambda - x)\mathcal{F}(\mathcal{P}), \tag{19}$$

and the equilibrium equations, obtained from setting the first variation δF equal to zero, are

$$F_\Lambda = \mathcal{L}^{-1}\Lambda + \mathcal{F}(\mathcal{P}) = 0, \tag{20}$$

$$F_\mathcal{P} = -\mathcal{P} + \sqrt{r}x + \mathcal{H}'(\mathcal{P}) + (\Lambda - x)\mathcal{F}'(\mathcal{P}) = 0, \tag{21}$$

where the prime denotes derivative with respect to the argument of the function. Due to the presence of the arbitrary functions in the Casimirs, such equilibrium equations possess free functions. Specifying these selects from a class of equilibrium solutions. In particular, choosing $\mathcal{F}$ corresponds to fixing the relation between the equilibrium stream function $\phi_{eq}=\mathcal{L}^{-1}\Lambda_{eq}$ and the equilibrium translated pressure $\mathcal{P}_{eq}$.

As indicated above, an equilibrium solution of Eqs. (20) and (21) is energy stable (and therefore linearly) stable, if the second variation of F, evaluated at that equilibrium, has a definite sign. In terms of the variables Λ and $\mathcal{P}$, the second variation of F reads

$$\begin{aligned}\delta^2 F = \int d^2x[&(1-\mathcal{F}'(\mathcal{P}))|\mathcal{L}^{-1}\delta\Lambda|^2 \\ &+ (1-2\mathcal{F}'(\mathcal{P}))|\mathcal{L}^{-1}\nabla\delta\Lambda|^2 + \mathcal{F}'(\mathcal{P})(\delta\Lambda + \delta\mathcal{P})^2 \\ &- \mathcal{F}'(\mathcal{P})(\mathcal{L}^{-1}\nabla^2\delta\Lambda)^2 + (\mathcal{H}''(\mathcal{P}) - 1 \\ &+ (\Lambda - x)\mathcal{F}''(\mathcal{P}) - \mathcal{F}'(\mathcal{P}))|\delta\mathcal{P}|^2].\end{aligned} \tag{22}$$

From Eq. (22) one immediately obtains sufficient conditions for positive definiteness of δ^2F in the case of no flow:

$$\mathcal{F}(\mathcal{P}_{eq}) \equiv 0 \quad \text{and} \quad \mathcal{H}''(\mathcal{P}_{eq}) > 1. \tag{23}$$

From Eq. (20), $\mathcal{F}(\mathcal{P}_{eq})\equiv 0$ implies no flow, while $\mathcal{H}''(\mathcal{P}_{eq})>1$ gives a condition on the pressure profile. Generally speaking, the situation with flow, when $\mathcal{F}(\mathcal{P}_{eq})\not\equiv 0$, is expected to have NEMs.[10] However, this case is more complicated to analyze, with the Poincaré inequality often being of use, but we will not pursue it further here.

IV. DISCRETE NEGATIVE ENERGY MODES

In this section, we first review the theory of NEMs in the finite degree-of-freedom Hamiltonian context,[47] since it applies directly to finite systems with discrete spectra (e.g. Ref. 8). Subsequently, after carrying out a spectral stability analysis of the system linearized around homogeneous equilibria with no flow, we make use of the Hamiltonian formalism in order to detect the presence of NEMs among the stable modes of the linearized system. Finally we carry out the explicit transformation that casts the corresponding Hamiltonian into normal form.

A. Review of mode signature and normal forms for linear Hamiltonian systems

A real canonical Hamiltonian linear system with N degrees of freedom is generated by the canonical Poisson bracket

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}, \tag{24}$$

and a quadratic Hamiltonian

$$H_L = \tfrac{1}{2}A_{ij}z^i z^j, \tag{25}$$

where $z=(q_1,\ldots,q_N,p_1,\ldots,p_N)$, A_{ij} are the elements of a $2N\times 2N$ matrix with constant coefficients, and repeated sum notation is assumed with $i,j=1,2,\ldots,N$. The resulting equations of motion can then be compactly written as

$$\dot{z} = J_c A z, \tag{26}$$

where

$$J_c = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix} \tag{27}$$

is the $2N\times 2N$ canonical symplectic matrix (cosymplectic form). Assuming

$$z = \tilde{z}e^{i\omega t} + \tilde{z}^* e^{-i\omega t}, \tag{28}$$

with * indicating complex conjugate, Eq. (26) yields the eigenvalue problem

$$i\omega_\alpha z_\alpha = J_c A z_\alpha, \quad \alpha = 1,\ldots,N, \tag{29}$$

where we have dropped the tilde on the eigenvectors and have added an eigenvalue label α. In what follows we will assume distinct eigenvalues, precluding the existence of nontrivial Jordan form and possible secular growth in time. We also assume that the eigenvalues ω_α are real, which is the case of interest for detecting mode signature. Because our dynamical variables are real, the remaining N eigenvalues are given by $\omega_{-\alpha}=-\omega_\alpha$ and the corresponding eigenvectors are $z_\alpha = z_\alpha^*$. Defining $\Omega := J_c^{-1}$, the symplectic two-form, we construct the quantity

$$h(\alpha,\beta) := i\omega_\alpha z_\beta^T \Omega z_\alpha = z_\beta^T A z_\alpha, \tag{30}$$

where T denotes transpose.

It can be easily shown that the property $h(\alpha,\beta) - h(\beta,\alpha) = 0$ holds. Then, from this relation and the antisymmetry of Ω, it follows that

$$h(\alpha,\beta) = 0, \quad \text{if } \beta \neq -\alpha. \tag{31}$$

On the other hand,

$$h(-\alpha,\alpha) = z_{-\alpha}^T A z_\alpha = z_\alpha^{*T} A z_\alpha = i\omega_\alpha z_\alpha^{*T}\Omega z_\alpha, \tag{32}$$

is clearly the *energy* (Hamiltonian H_L) of the mode $(z_\alpha, \omega_\alpha; z_\alpha^*, -\omega_\alpha)$. Evidently, $z_\alpha^{*T}\Omega z_\alpha$ is a purely imaginary number, and a normalization constant for the eigenvectors can be chosen in such a way that

$$z_\alpha^{*T}\Omega z_\alpha = \pm 2i, \tag{33}$$

with the sign, an invariant, depending on the specific mode under consideration. Note that the left-hand side of Eq. (33) represents the Lagrange bracket (symplectic two-form) of z_α^* and z_α. If z_α is an eigenvector, associated with a positive eigenvalue ω_α, and

$$z_\alpha^{*T}\Omega z_\alpha = -2\,i, \tag{34}$$

then $(z_\alpha, \omega_\alpha; z_\alpha^*, -\omega_\alpha)$ corresponds to a *positive energy mode*, otherwise it is a *negative energy mode*. This can be easily seen by observing that, in the case of a PEM, the corresponding energy is given by

$$h(-\alpha,\alpha) = i\omega_\alpha z_\alpha^{*T}\Omega z_\alpha = 2\omega_\alpha > 0. \tag{35}$$

Note that, although here we carried out an analysis with canonical coordinates, Sylvester's theorem guarantees that the signature of a mode (i.e., whether it is a PEM or a NEM), does not depend on the choice of the coordinate system.

The distinction between positive and negative energy modes becomes even more transparent when we are reminded that, for stable modes there exists[8,12,47] a canonical transformation $T:(Q_1,\ldots,Q_N,P_1,\ldots,P_N) \to (q_1,\ldots,q_N,p_1,\ldots,p_N)$, that casts the quadratic Hamiltonian of Eq. (25) into the following *normal form*:

$$H_L = \frac{1}{2}\sum_{\alpha=1}^{N} \sigma_\alpha \omega_\alpha (P_\alpha^2 + Q_\alpha^2), \tag{36}$$

where ω_α represents the positive eigenvalues of the linearized system, whereas $\sigma_\alpha \in \{-1, 1\}$ is the signature of the mode.

If the system contains unstable modes, then they have a different normal form. However, if the Hamiltonian is restricted to the stable modes, then it can be written as the Hamiltonian for a system of N harmonic oscillators with different frequencies. The modes for which $\sigma_\alpha = -1$, which give a negative contribution to the total energy, correspond to the NEMs, while those corresponding to $\sigma_\alpha = 1$ are, of course, PEMs.

Once the eigenvalues and eigenvectors of the system are known, the procedure for constructing the map T is algorithmic. First one needs to select, among the $2N$ eigenvectors of the system, the N eigenvectors z_α that satisfy

$$z_\alpha^{*T}\Omega z_\alpha = -2\,i, \quad \alpha = 1,\ldots,N. \tag{37}$$

Then the $2N\times 2N$ matrix that defines the transformation is given by

$$T = \mathrm{col}(\mathrm{Re}\, z_1, \mathrm{Re}\, z_2 \ldots \mathrm{Re}\, z_N, \mathrm{Im}\, z_1, \mathrm{Im}\, z_2, \ldots, \mathrm{Im}\, z_N), \tag{38}$$

which is the matrix with columns given by Re z_1 etc. It can be shown that the transformation constructed in this way is canonical and indeed provides the desired diagonalization.

B. Mode signature and stability for the ETG model

Now consider a special case of the no-flow equilibria of Sec. III, the homogeneous equilibria:

$$\Lambda_{eq} = \mathcal{L}^{-1}\Lambda_{eq} = 0, \quad \mathcal{P}_{eq} = \alpha_{\mathcal{P}} x, \tag{39}$$

where $\alpha_{\mathcal{P}}$ is a constant. The equilibrium solution Eq. (39) corresponds to the choices

$$\mathcal{F}(\mathcal{P}) = 0, \quad \mathcal{H}(\mathcal{P}) = \frac{1}{2}\left(1 - \frac{\sqrt{r}}{\alpha_{\mathcal{P}}}\right)\mathcal{P}^2, \tag{40}$$

for the Casimir functions that appear in Eqs. (20) and (21).

Linearizing the model equations around this equilibrium gives the system

$$\dot{\tilde{\Lambda}} = -\frac{\partial}{\partial y}\mathcal{L}^{-1}\tilde{\Lambda} - \sqrt{r}\frac{\partial}{\partial y}\tilde{\mathcal{P}},$$

$$\dot{\tilde{\mathcal{P}}} = \alpha_{\mathcal{P}}\frac{\partial}{\partial y}\mathcal{L}^{-1}\tilde{\Lambda}.$$

Expanding the perturbations as Fourier series, as follows:

$$\Lambda = \tilde{\Lambda} = \sum_{\mathbf{k}=-\infty}^{+\infty} \tilde{\Lambda}_{\mathbf{k}}(t) \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}},$$
$$\mathcal{P} = \alpha_{\mathcal{P}} x + \tilde{\mathcal{P}} = \alpha_{\mathcal{P}} x + \sum_{\mathbf{k}=-\infty}^{+\infty} \tilde{\mathcal{P}}_{\mathbf{k}}(t)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{41}$$

yields the amplitude equations

$$\dot{\tilde{\Lambda}}_{\mathbf{k}} = i\frac{k_y}{1+k_\perp^2}\tilde{\Lambda}_{\mathbf{k}} + i\sqrt{r}k_y\tilde{\mathcal{P}}_{\mathbf{k}}, \tag{42}$$

$$\dot{\tilde{\mathcal{P}}}_{\mathbf{k}} = -i\alpha_{\mathcal{P}}\frac{k_y}{1+k_\perp^2}\tilde{\Lambda}_{\mathbf{k}}, \tag{43}$$

whence, the dispersion relation for modes of the form $\mathrm{e}^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}$ is obtained,

$$\omega^2 - \frac{k_y}{1+k_\perp^2}\omega + \alpha_{\mathcal{P}}\frac{\sqrt{r}k_y^2}{1+k_\perp^2} = 0. \tag{44}$$

This expression is in agreement with that obtained in Ref. 31. The eigenvalues correspond to a slow and a fast mode, and are given by

$$\omega_s^k = \frac{k}{2(1+k_\perp^2)}(1 - \sqrt{1 - 4\,(1+k_\perp^2)\alpha_{\mathcal{P}}\sqrt{r}}), \tag{45}$$

$$\omega_f^k = \frac{k}{2(1+k_\perp^2)}(1 + \sqrt{1 - 4\,(1+k_\perp^2)\alpha_{\mathcal{P}}\sqrt{r}}), \tag{46}$$

where we have set $k_y = k$. The corresponding eigenvectors are

$$\begin{aligned}\tilde{\Lambda}_{s,f}^k &= -\,\omega_{s,f}^k \frac{1+k_\perp^2}{\alpha_{\mathcal{P}} k}\tilde{\mathcal{P}}_{s,f}^k \\ &= -\frac{1}{2\alpha_{\mathcal{P}}}(1 \pm \sqrt{1 - 4\,(1+k_\perp^2)\alpha_{\mathcal{P}}\sqrt{r}})\tilde{\mathcal{P}}_{s,f}^k,\end{aligned} \tag{47}$$

for $k > 0$. The system also possesses the eigenvalues $\omega_{-s,-f}^k = -\omega_{s,f}^k$, whose eigenvectors are the complex conjugates of those of Eq. (47).

From Eqs. (45) and (46) we obtain a necessary and sufficient condition for spectral stability, viz.

$$1 - 4\,(1+k_\perp^2)\alpha_{\mathcal{P}}\sqrt{r} > 0 \Rightarrow \alpha_{\mathcal{P}} < \frac{1}{4(1+k_\perp^2)\sqrt{r}}. \tag{48}$$

Therefore, if the electron pressure or magnetic field gradients are such that $\alpha_{\mathcal{P}} < 0$, such equilibria are always spectrally stable $\forall k$. If $\alpha_{\mathcal{P}} > 0$, on the other hand, the equilibrium will be stable only for $\mathbf{k}$ such that $0 < \alpha_{\mathcal{P}} < 1/4(1+k_\perp^2)\sqrt{r}$ is satisfied. In other words, there will always be instability for sufficiently large $k_\perp$.

The linearized system of Eqs. (42) and (43) inherits a Hamiltonian formulation from the nonlinear system, one that can be written in the framework described in Sec. IV A. We can then take advantage of this fact in order to see whether NEMs are present in the system and to cast the Hamiltonian into its normal form.

By using the relation (see, e.g., Refs. 8, 34, and 41)

$$\frac{\delta F}{\delta \tilde{\Lambda}} = \sum_{k=-\infty}^{k=+\infty}\left(\frac{\delta F}{\delta \tilde{\Lambda}}\right)_k \mathrm{e}^{-iky} = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\frac{\partial \bar{F}}{\partial \tilde{\Lambda}_{-k}}\mathrm{e}^{-iky}, \tag{49}$$

where $F(\tilde{\Lambda}) = \bar{F}(\tilde{\Lambda}_k)$, it can be shown that the Hamiltonian structure of Eqs. (42) and (43), is given by the bracket

$$\begin{aligned}\{F,G\} = \sum_{k=1}^{+\infty}\frac{ik}{2\pi}\Bigg[&\left(\frac{\partial F}{\partial \tilde{\Lambda}_k}\frac{\partial G}{\partial \tilde{\Lambda}_{-k}} - \frac{\partial F}{\partial \tilde{\Lambda}_{-k}}\frac{\partial G}{\partial \tilde{\Lambda}_k}\right) \\ &- \alpha_{\mathcal{P}}^2\left(\frac{\partial F}{\partial \tilde{\Lambda}_k}\frac{\partial G}{\partial \tilde{\mathcal{P}}_{-k}} + \frac{\partial F}{\partial \tilde{\mathcal{P}}_k}\frac{\partial G}{\partial \tilde{\Lambda}_{-k}} - \frac{\partial F}{\partial \tilde{\mathcal{P}}_{-k}}\frac{\partial G}{\partial \tilde{\Lambda}_k} \right.\\ &\left. - \frac{\partial F}{\partial \tilde{\Lambda}_{-k}}\frac{\partial G}{\partial \tilde{\mathcal{P}}_k}\right)\Bigg].\end{aligned} \tag{50}$$

and the linear Hamiltonian, which is proportional to $\delta^2 F$,

$$H_L = \sum_{k=1}^{+\infty} H_L^k = 2\pi\sum_{k=1}^{+\infty}\left(\frac{|\tilde{\Lambda}_k|^2}{1+k_\perp^2} - \frac{\sqrt{r}}{\alpha_{\mathcal{P}}}|\tilde{\mathcal{P}}_k|^2\right), \tag{51}$$

where we have suppressed the sum on k_x (note that k_x only appears in the combination $k_\perp^2 = k_x^2 + k^2$). Although not canonical, this formulation, in principle is sufficient in order to detect the presence of NEMs for the equilibrium under consideration. Indeed, as already done for the four-field model of Ref. 20, we can make use of the property that NEMs can change their signature only if they become unstable through a "Kreĭn bifurcation,"[10] or if the corresponding eigenvalues go through zero frequency. It is then sufficient to identify a NEM in a particular limit, and we are then guaranteed that its signature does not change as long as one of the two above mentioned phenomena does not occur. Sylvester's theorem also guarantees that the signature is independent on the choice of the coordinates we make. If we fix a wave vector k, then we can first evaluate the energy associated with the corresponding mode in the $(\tilde{\Lambda}_k, \tilde{\Lambda}_{-k}, \tilde{\mathcal{P}}_k, \tilde{\mathcal{P}}_{-k})$ coordinates by inserting eigenvalues and eigenvectors associated to k in the expression for H_L^k. This results in

$$\begin{aligned}H_{L_{s,f}}^k = 2\pi&\left(\frac{1 - 4\,\sqrt{r}\alpha_{\mathcal{P}} - 2k_\perp^2\alpha_{\mathcal{P}}\sqrt{r} \pm \sqrt{1 - 4\,(1+k_\perp^2)\alpha_{\mathcal{P}}\sqrt{r}}}{2\alpha_{\mathcal{P}}^2}\right) \\ &\times|\tilde{\mathcal{P}}_{s,f}^k|^2,\end{aligned} \tag{52}$$

which is the energy of the slow and the fast modes of wave number k (summed over k_x). In order to identify PEMs and NEMs, it is sufficient to consider the limit $k_\perp \to 0$, which yields

$$H_{L_{s,f}}^{k,k_\perp=0} = 2\pi\left(\frac{1 - 4\,\sqrt{r}\alpha_{\mathcal{P}} \pm \sqrt{1 - 4\,\alpha_{\mathcal{P}}\sqrt{r}}}{2\alpha_{\mathcal{P}}^2}\right)|\tilde{\mathcal{P}}_{s,f}^k|^2. \tag{53}$$

We can then see that, for the fast mode [corresponding to the + sign in Eq. (53)], $H_{L_f}^{k,k_\perp=0}$ is positive, and therefore a PEM. Also, it will remain a PEM as parameters are varied in a continuous way, until the instability threshold is reached. For the slow mode, two cases exist for finite $\alpha_{\mathcal{P}}$. If $\alpha_{\mathcal{P}} < 0$, then $H_{L_s}^{k,k_\perp=0} > 0$ and, again, we have a PEM. If $0 < \alpha_{\mathcal{P}} < 1/(4\sqrt{r})$, on the other hand, the slow mode is a NEM.

The above mentioned instability, occurring at large $k_\perp$, for $\alpha_{\mathcal{P}} > 0$, indicates a Kreĭn bifurcation, which is one of the possible types of bifurcations occurring in Hamiltonian systems. These situations are illustrated in Fig. 1. When ω_s^k is real and negative, that is for $\alpha_{\mathcal{P}} < 0$, both the slow and the fast modes are PEMs and the two branches correspond to two dispersive waves with opposite sign. These waves correspond to inviscid drift waves modified by the presence of the ETG and magnetic field curvature. In the limit $r \to 0$, corresponding to vanishing ETG, the two modes degenerate into a Hasegawa-Mima drift wave. When ω_s^k is real and positive, i.e., for $0 < \alpha_{\mathcal{P}} < 1/4(1+k_\perp^2)\sqrt{r}$, both modes are still stable but the slow mode is now a NEM. Comparing the two plots of Fig. 1, one observes that ω_s^k went from negative to positive, i.e., it crossed through zero frequency, while changing from a PEM to a NEM. For the parameters chosen for the figure, the instability threshold, due to the presence of ETG,

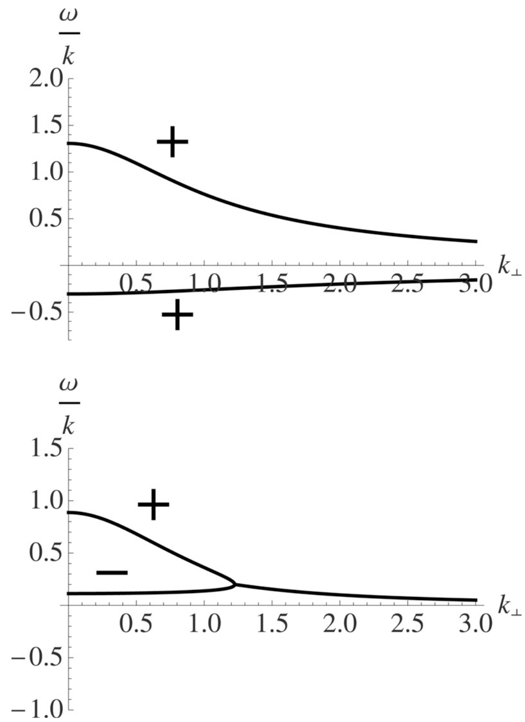

FIG. 1. Depiction of two possible mode signature situations for ETG modes, depending on the sign of $\alpha_{\mathcal{P}}$. The plot on the left refers to the case of negative $\alpha_{\mathcal{P}}$ ($\alpha_{\mathcal{P}}=-0.3$, $\sqrt{r}=0.2$). In this case both modes are PEMs and the system is energy stable. In the plot on the right, $\alpha_{\mathcal{P}}$ is positive ($\alpha_{\mathcal{P}}=0.5$, $\sqrt{r}=0.2$). For $0<k_\perp<1.22$, the equilibrium is stable, but the slow mode is a NEM. A Kreĭn bifurcation occurs at $k_\perp=1.22$.

occurs at $k_\perp \approx 1.22$. For perpendicular wave numbers above this value, the equilibrium is indeed unstable. The transition of the two real eigenvalues into a complex conjugate pair, occurring at $k_\perp \approx 1.22$, is an example of a Kreĭn bifurcation. Note that, as predicted by Kreĭn's theorem (see, e.g., Ref. 10), if a Kreĭn bifurcation between two eigenvalues occurs, one of them must be a NEM. This is indeed our case.

If we consider now the stability condition Eq. (23) for no-flow equilibria, together with Eq. (40), we find that the homogeneous equilibrium Eq. (39) is energy stable if $\mathcal{H}''(\mathcal{P}_{\rm eq})>1$, which is equivalent to $\alpha_{\mathcal{P}}<0$. Indeed, if this condition is satisfied, the equilibrium is stable, with no NEMs. If we now use the actual pressure p, as a variable, and consider the corresponding homogeneous equilibrium $p_{\rm eq}=a_p x=\sqrt{r}(\alpha_{\mathcal{P}}-\sqrt{r})x$, we can reformulate our results in the following way. If the equilibrium pressure gradient is such that $a_p<-r$ (i.e., $\alpha_{\mathcal{P}}<0$), then the system is energy stable for every $\mathbf{k}$. If, on the other hand, $-r<a_p<-r+1/4(1+k_\perp^2)$ (i.e., $0<\alpha_{\mathcal{P}}<1/(4(1+k_\perp^2)\sqrt{r})$ then the system becomes unstable through a Kreĭn bifurcation at a critical $k_\perp$, for fixed r. For $k_\perp$ below this critical value, the equilibrium is spectrally stable but not energy stable. Indeed, the slow mode in this case is a NEM, which makes the equilibrium fragile with respect to the addition of dissipation or nonlinearities.

C. Normal form for the Hamiltonian of the ETG model

The explicit knowledge of the eigenvalues of the linearized system, makes it possible to cast the Hamiltonian into its normal form, as discussed in Sec. IV A. First of all we point out that the transformation $(\tilde\Lambda_k,\tilde\Lambda_{-k},\tilde{\mathcal{P}}_k,\tilde{\mathcal{P}}_{-k}) \to (q_k^1,q_k^2,p_k^1,p_k^2)$, corresponding to

$$q_k^1=\sqrt{\frac{\pi}{k\alpha_{\mathcal{P}}^2}}(\tilde{\mathcal{P}}_k+\alpha_{\mathcal{P}}\tilde\Lambda_k+\tilde{\mathcal{P}}_{-k}+\alpha_{\mathcal{P}}\tilde\Lambda_{-k}),$$

$$p_k^1=-i\sqrt{\frac{\pi}{k\alpha_{\mathcal{P}}^2}}(\tilde{\mathcal{P}}_k+\alpha_{\mathcal{P}}\tilde\Lambda_k-\tilde{\mathcal{P}}_{-k}-\alpha_{\mathcal{P}}\tilde\Lambda_{-k}), \tag{54}$$

$$q_k^2=\sqrt{\frac{\pi}{k}}(\tilde\Lambda_k+\tilde\Lambda_{-k}), \quad p_k^2=i\sqrt{\frac{\pi}{k}}(\tilde\Lambda_k-\tilde\Lambda_{-k}),$$

puts the system into canonical Hamiltonian form. Indeed, in terms of the variables $(q_k^1,q_k^2,p_k^1,p_k^2)$, the bracket of Eq. (50) takes the canonical form

$$\{F,G\}=\sum_{k=1}^{\infty}\frac{\partial F}{\partial q_1^k}\frac{\partial G}{\partial p_1^k}-\frac{\partial F}{\partial p_1^k}\frac{\partial G}{\partial q_1^k}+\frac{\partial F}{\partial q_2^k}\frac{\partial G}{\partial p_2^k}-\frac{\partial F}{\partial p_2^k}\frac{\partial G}{\partial q_2^k}. \tag{55}$$

The Hamiltonian Eq. (51) in terms of the new variables, on the other hand, reads

$$H_L=\frac{1}{2}\sum_{k=1}^{\infty}\sum_{i,j=1}^{4}A_{ij}^k z_i^k z_j^k, \tag{56}$$

where

$$A^k=\begin{pmatrix} a & c & 0 & 0\\ c & b & 0 & 0\\ 0 & 0 & a & -c\\ 0 & 0 & -c & b\end{pmatrix}, \quad z^k=(q_1^k,q_2^k,p_1^k,p_2^k) \tag{57}$$

with $a=-\sqrt{r}\alpha_{\mathcal{P}}k$, $b=k(1/(1+k_\perp^2)-\sqrt{r}\alpha_{\mathcal{P}})$, and $c=\sqrt{r}|\alpha_{\mathcal{P}}|k$. We emphasize that the transformation Eq. (54) is constructed in such a way that the canonical variables q_i^k and p_i^k are *real*. We are thus in the framework depicted in Sec. IV A. For each k, the equations of motion are given by

$$z^k=J_c A^k z^k. \tag{58}$$

Upon writing the variables as

$$q_{1,2}^k=\tilde z_{1,2}^k e^{i\omega^k t}+\tilde z_{1,2}^{k*}e^{-i\omega^k t}, \quad p_{1,2}^k=\tilde z_{3,4}^k e^{i\omega^k t}+\tilde z_{3,4}^{k*}e^{-i\omega^k t}, \tag{59}$$

and dropping the tilde in the following, we obtain that the eigenvectors of the linearized system are

$$z_s^k=q_{1s}^k\begin{pmatrix}1\\ -B_-\\ -i\\ -iB_-\end{pmatrix}, \quad z_{-s}^k=q_{1s}^{k*}\begin{pmatrix}1\\ -B_-\\ i\\ iB_-\end{pmatrix}, \tag{60}$$

$$z_f^k = q_{1f}^k \begin{pmatrix} 1 \\ -B_+ \\ -i \\ -iB_+ \end{pmatrix}, \quad z_{-f}^k = q_{1f}^{k*} \begin{pmatrix} 1 \\ -B_+ \\ i \\ iB_+ \end{pmatrix}.$$

These are the eigenvectors corresponding to the eigenvalues ω_s^k, ω_{-s}^k, ω_f^k, and ω_{-f}^k, respectively. In Eq. (60) we introduced the quantities

$$B_\pm = \frac{b+a \pm \sqrt{(b+a)^2 - 4c^2}}{2c}, \tag{61}$$

whereas $q_{1_{s,f}}^k$ are complex coefficients. Following Sec. IV A, the Lagrange bracket for the slow mode reads

$$z_{-s}^{k^T} \Omega z_s^k = 2i(1 - B_-^2) q_{1s}^{k*} q_{1s}^k. \tag{62}$$

Direct calculation shows that

$$1 - B_-^2 = -\frac{\sqrt{(b+a)^2 - 4c^2}(\sqrt{(b+a)^2 - 4c^2} - (b+a))}{2c^2}$$
$$> 0. \tag{63}$$

To obtain the inequality in Eq. (63), we made use of the fact that, for stable modes, $\sqrt{(b+a)^2 - 4c^2} < b+a$.

The inequality Eq. (63) tells us that, for slow modes, we must pick up the plus sign in the general expression Eq. (33). Moreover, if we choose the normalization constants so that

$$q_{1s}^k = q_{1s}^{k*} = \frac{1}{D_-} \equiv \frac{1}{\sqrt{1 - B_-^2}}, \tag{64}$$

we obtain that the Lagrange bracket for slow modes becomes

$$z_{-s}^{k^T} \Omega z_s^k = 2i. \tag{65}$$

Of course,

$$z_s^{k^T} \Omega z_{-s}^k = -2\,i, \tag{66}$$

and consequently, according to the definition, given in Sec. IV A, we have a PEM, when $\omega_{-s}^k = -\omega_s^k > 0$, and a NEM when $\omega_{-s}^k = -\omega_s^k < 0$. This confirms the results we obtained in Sec. IV B with noncanonical variables.

Following the same procedure for the fast mode, we find

$$z_{-f}^{k^T} \Omega z_f^k = 2i(1 - B_+^2) q_{1f}^{k*} q_{1f}^k. \tag{67}$$

Given that $1 - B_+^2 < 0$, the Lagrange bracket for the fast mode becomes

$$z_{-f}^{k^T} \Omega z_f^k = -2\,i, \tag{68}$$

after having chosen the following normalization for the eigenvectors:

$$q_{1f}^k = q_{1f}^{k*} = \frac{1}{D_+} \equiv \frac{1}{\sqrt{B_+^2 - 1}}. \tag{69}$$

Because ω_f^k is always positive, according to the definition, Eq. (68) tells us that the fast mode, as expected, is always a PEM.

The transformation that casts the Hamiltonian Eq. (56) into normal form, following Sec. IV A, will be a real canonical transformation T^k that, for each k, maps a new set of coordinates $\bar{z}^k = (Q_1^k, Q_2^k, P_1^k, P_2^k)$, in terms of which the Hamiltonian is diagonal, into z^k.

After noticing that

$$z_{-s}^{k*^T} \Omega z_{-s}^k = -2\,i, \tag{70}$$

$$z_f^{k*^T} \Omega z_f^k = -2\,i, \tag{71}$$

the matrix associated with the application T^k is constructed in the following way:

$$T^k = \begin{pmatrix} \frac{1}{D_-} & \frac{1}{D_+} & 0 & 0 \\ -\frac{B_-}{D_-} & -\frac{B_+}{D_+} & 0 & 0 \\ 0 & 0 & \frac{1}{D_-} & -\frac{1}{D_+} \\ 0 & 0 & \frac{B_-}{D_-} & -\frac{B_+}{D_+} \end{pmatrix}. \tag{72}$$

Direct calculation shows that

$$T^{k^T} A^k T^k = \begin{pmatrix} -\omega_s^k & 0 & 0 & 0 \\ 0 & \omega_f^k & 0 & 0 \\ 0 & 0 & -\omega_s^k & 0 \\ 0 & 0 & 0 & \omega_f^k \end{pmatrix}. \tag{73}$$

Consequently, the Hamiltonian Eq. (56), restricted to the stable modes, can be finally written as

$$H_L' = \frac{1}{2} \sum_k{}' \sum_{i,j=1}^{4} (T^k \bar{z}^k)_i^T A_{ij}^k (T^k \bar{z}^k)_j$$
$$= \frac{1}{2} \sum_k{}' \omega_f^k (Q_2^{k^2} + P_2^{k^2}) - \omega_s^k (Q_1^{k^2} + P_1^{k^2}), \tag{74}$$

where the prime on the sum indicates that the latter includes only the stable modes. The expression Eq. (74) corresponds to the normal form for the Hamiltonian of the linearized ETG model for stable modes. It clearly shows how the corresponding energy can be decomposed into the sum of energies of harmonic oscillators which possess, as characteristic frequencies, those of the fast and slow modes. The harmonic oscillators associated with the fast modes always provide a positive contribution to the total energy. Those associated to the slow modes, on the other hand, give a negative contribution if $\omega_s^k > 0$, which translates into the condition on the equilibrium pressure gradient discussed in Sec. IV B.

V. A CASE WITH EQUILIBRIUM SHEAR FLOW

The analysis of Sec. IV, restricted to homogeneous equilibria, serves as a first step for understanding the modes present in a model and their stability. More realistic equilibria possess spatial dependence and the possibility of flows with shear. The analysis of systems linearized around such equilibria, however, is considerably more complicated than in the case of homogeneous equilibria, and the presence of a continuous spectrum is to be expected. Here we treat the problems of the identification of negative energy modes and the reduction to normal form for such an equilibrium with a shear flow.

To simplify matters we consider a reduced stable situation. The combined action of forcing and dissipation can maintain both density and pressure gradients. For example, in the context of rotating fluids Ekman drag and the input of vorticity by appropriate pumping can be used to select a particular ϕ (see e.g. Ref. 48). In a similar vein, here we follow[49] and envision a situation where the pressure profile is maintained relative to the density profile. With such a frozen pressure profile, where $\mathbf{v}\cdot\nabla p$ is maintained by the combined action of drive and dissipative processes, Eq. (2) is solved by the linear relation $p=\kappa\phi-rx$, with constant parameter κ. Then, the dynamics is determined by Eq. (1) alone, which becomes

$$\frac{\partial}{\partial t}(1-\nabla^2)\phi=[\phi,\nabla^2\phi]-(1+\kappa)\frac{\partial\phi}{\partial y}, \qquad (75)$$

and it is clear that our system still supports drift waves; since, with the above assumptions, we have a version of the Hasegawa–Mima equation. Note, no explicit assumption has been made on the size of r. By effecting the transformation $\phi\to\bar{\phi}+(1+\kappa)x$ and $\bar{y}=y+(1+\kappa)t$, the above equation becomes

$$\frac{\partial}{\partial t}(1-\nabla^2)\bar{\phi}=[\bar{\phi},\nabla^2\bar{\phi}]. \qquad (76)$$

Lastly, since unstable modes at very large wavenumbers are unphysical and, particularly, because we are interested in demonstrating how to handle the continuous spectrum we consider the part of the spectrum with $k\gg 1$, and obtain

$$\frac{\partial}{\partial t}\zeta+[\phi,\zeta]=0, \qquad (77)$$

where $\zeta=\nabla^2\phi$, and we dropped the overbars for simplicity. Thus we obtain a system homologous to Euler's equation for two-dimensional fluids.

With the above reduction the Hamiltonian of Eq. (5) reduces to

$$H[\zeta]=\frac{1}{2}\int d^2x|\nabla\phi|^2, \qquad (78)$$

where the integration here (and in all integrals below) is over $[-1,1]\times[0,2\pi]$, the noncanonical Poisson bracket of Eq. (6) reduces to

$$\{F,G\}=\int d^2x\zeta[F_\zeta,G_\zeta], \qquad (79)$$

and the equation of motion of Eq. (77) is generated as follows:

$$\frac{\partial\zeta}{\partial t}=\{\zeta,H\}=[\zeta,\phi]. \qquad (80)$$

We note that this kind of Hamiltonian reduction can be done in a systematic way by using Dirac constraint theory,[36] but we will not pursue this here.

The bracket of Eq. (79) is the standard bracket for fluid and plasma theories with two 'spatial' variables. It first appeared in the Vlasov–Poisson context in Ref. 50, for two-dimensional Euler equation in Refs. 33, 51, and 52, and a generalization with arbitrary interaction in Refs. 53. All of these systems are amenable to the kind of analysis presented below.

This system Eq. (80) conserves the quantities

$$C[\zeta]:=\int d^2x\mathcal{C}(\zeta),$$
$$P_x[\zeta]:=\int d^2xy\zeta, \quad P_y[\zeta]:=-\int d^2xx\zeta, \qquad (81)$$

where the first satisfies $\{F,C\}=0$ for all functionals F and is thus a family of Casimir invariants, while the second two, the relative momenta, are dynamical invariants and satisfy $\{P_x,H\}=\{P_y,H\}=0$.

For the shear flow equilibrium of interest here, the potential ϕ_{eq} and charge ζ_{eq} must satisfy

$$[\zeta_{\text{eq}},\phi_{\text{eq}}]=0, \quad \nabla^2\phi_{\text{eq}}=\zeta_{\text{eq}}, \qquad (82)$$

the case of interest here being $\zeta_{\text{eq}}(x)=v'_{\text{eq}}(x)=\phi''_{\text{eq}}(x)$, where prime denotes d/dx. We take v_{eq} to be monotonic for $x\in[-1,1]$ and assume stability, which can be guaranteed by the techniques of Ref. 54 for establishing necessary and sufficient conditions for stability of shear flow.

Setting $\phi=\phi_{\text{eq}}+\delta\phi$ and $\zeta=\zeta_{\text{eq}}+\delta\zeta$ and expanding Eq. (77) to first order, gives

$$\frac{\partial\delta\zeta}{\partial t}+v_{\text{eq}}\frac{\partial\delta\zeta}{\partial y}-v''_{\text{eq}}\frac{\partial\delta\phi}{\partial y}=0, \qquad (83)$$

with $\nabla^2\delta\phi=\delta\zeta$. This linear dynamics conserves the functional

$$H_L[\delta\zeta]=\frac{1}{2}\int d^2x\left[\frac{v_{\text{eq}}}{v''_{\text{eq}}}(\delta\zeta)^2-\delta\zeta\delta\phi\right], \qquad (84)$$

which physically corresponds to the total energy contained in a perturbation away from the equilibrium state and is the Hamiltonian for the linear dynamics. Thus with the linear bracket

$$\{F,G\}_L=\int d^2x\zeta_{\text{eq}}[F_{\delta\zeta},G_{\delta\zeta}], \qquad (85)$$

Eq. (83) has the form $\partial\delta\zeta/\partial t=\{\delta\zeta,H_L\}_L$.

As in Sec. IV, we consider a Fourier expansion $\delta\phi=\Sigma_k\phi_k(x,t)\exp(iky)$ and $\delta\zeta=\Sigma_k\zeta_k(x,t)\exp(iky)$, but unlike in Sec. IV we only expand in y, where k is the poloidal wavenumber, and obtain

$$\frac{\partial\zeta_k}{\partial t}+ikv_{\text{eq}}\zeta_k-ikv''_{\text{eq}}\phi_k=0, \qquad (86)$$

with

$$\phi_k(x,t)=\int_{-1}^{1}dx'\mathcal{K}_k(x,x')\zeta_k(x',t), \qquad (87)$$

where

$$\mathcal{K}_k(x,x') = \begin{cases} \sinh[k(x-1)]\sinh[k(x+1)]/k\sinh[2k] & \text{for } x > x', \\ \sinh[k(x'-1)]\sinh[k(x+1)]/k\sinh[2k] & \text{for } x \le x'. \end{cases} \tag{88}$$

The boundary conditions are $\phi_k(x=\pm 1,t)=0$.

Canonization of the bracket of Eq. (85) proceeds as in Sec. IV. With the Fourier expansion it becomes

$$\{F,G\}_L = \sum_{k=1}^{\infty} \int dxik\zeta'_{\rm eq}\left(\frac{\delta F}{\delta \zeta_k}\frac{\delta G}{\delta \zeta_{-k}} - \frac{\delta G}{\delta \zeta_k}\frac{\delta F}{\delta \zeta_{-k}}\right), \tag{89}$$

and by introducing the new variables $q_k(x,t) := \zeta_k(x,t)$ and $p_k(x,t) := -\zeta_{-k}(x,t)/(ik\zeta'_e)$, we obtain the canonical form

$$\{F,G\}_L = \sum_{k=1}^{\infty} \int dx\left(\frac{\delta F}{\delta q_k}\frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k}\frac{\delta F}{\delta p_k}\right), \tag{90}$$

where q_k and p_k are canonically conjugate variables.

The Hamiltonian for the linear dynamics is given by Eq. (84), which with the insertion of the Fourier expansion becomes

$$\begin{aligned} H_L &= \sum_{k=1}^{\infty} \int dx\left(\frac{v_{\rm eq}}{v''_{\rm eq}}\zeta_k - \phi_k\right)\zeta_{-k} \\ &= \sum_{k,k'=1}^{\infty} \int dx \int dx'\, \zeta_k(x)\mathcal{O}_{k,k'}(x|x')\zeta_{k'}(x'), \end{aligned} \tag{91}$$

where $\mathcal{O}_{k,k'}(x|x') := \delta_{k,-k'}(v_{\rm eq}\delta(x-x')/v''_{\rm eq} - \mathcal{K}_k(x,x'))$. From Eq. (91) it is clear that this Hamiltonian is not diagonal; i.e., it does not possess a form that is the infinite-dimensional generalization of a sum over oscillators. This remains true even after rewriting it in terms of the canonical variables (q_k,p_k). However, we will see that a coordinate change that uses a particular integral transform produces this diagonalization.

To diagonalize the Hamiltonian while maintaining the Hamiltonian form we introduce the following *mixed variable generating functional*, the essence of which is determined by an integral transform $\hat{G}$ described in Appendix A:

$$\mathcal{F}[q,P] = \sum_{k=1}^{\infty} \int dx P_k(x)\hat{G}[q_k](x). \tag{92}$$

The transformation to the new canonical variables $(\mathcal{Q}_k,\mathcal{P}_k)$ is given by the following:

$$\begin{aligned} p_k(x) &= \frac{\delta \mathcal{F}[q,\mathcal{P}]}{\delta q_k(x)} = \hat{G}^{\dagger}[\mathcal{P}_k](x), \\ \mathcal{Q}_k(\bar{x}) &= \frac{\delta \mathcal{F}[q,\mathcal{P}]}{\delta \mathcal{P}_k(\bar{x})} = \hat{G}[q_k](\bar{x}), \end{aligned} \tag{93}$$

where $\bar{x}$ is the independent variable for the transformed function.

Now we insert the transformations of Eq. (93) into Eq. (91) and show that this gives a diagonal form. Rewriting Eq. (91) as $H_L = \Sigma_{k=1}^{\infty}\int dx(-ikv_{\rm eq}q_kp_k + ikv''_{\rm eq}\psi_kp_k)$ and inserting $p_k = \hat{G}^{\dagger}[P_k]$ and $q_k = G[\mathcal{Q}_k]$ yields

$$H_L = -i\sum_{k=1}^{\infty} \int d\bar{x}kP_k(\hat{G}[v_{\rm eq}G[\mathcal{Q}_k]] - \hat{G}[v_{\rm eq}\phi_k]), \tag{94}$$

which upon making use of Eqs. (A15) and (A16) gives

$$\begin{aligned} H_L &= i\sum_{k=1}^{\infty} \int d\bar{x}\sigma_k(\bar{x})\omega_k(\bar{x})\mathcal{P}_k\mathcal{Q}_k \\ &= \frac{1}{2}\sum_{k=1}^{\infty} \int d\bar{x}\sigma_k(\bar{x})\omega_k(\bar{x})(P_k^2 + Q_k^2), \end{aligned} \tag{95}$$

where the transformation giving the second equality of Eq. (95) is elementary, $\omega_k(\bar{x}) := |kv_{\rm eq}(\bar{x})|$, and the signature is defined by $\sigma_k(\bar{x}) := \mathrm{sgn}(v''_{\rm eq}/v_{\rm eq})$.

Several comments can be made about the form of Eq. (95). Because of the integration over $\bar{x}$, there is a continuum of eigenvalues $\omega_k(\bar{x})$, and because of the presence of σ_k, this continuum may have positive or negative energy. The existence of the continuum is expected, since it is well-known that systems of the form of Eq. (86) possess a continuous spectrum, which can be shown rigorously (see, e.g., Refs. 53 and 55). Next, if $v''_{\rm eq} \neq 0$, i.e., Rayleigh's condition[56] is satisfied, then the system is stable and it is always possible to transform to the normal form of Eq. (95) with a sign definite (positive or negative) energy. However, when there is a point within the domain where $v''_{\rm eq}=0$, then the system may still be stable, yet the energy not sign definite. When this is the case there exist negative energy continua. This situation is analogous to that of Sec. IV and one anticipates Kreĭn-like possibilities for the transition to instability. Indeed such is the case, and the continuum generalization of Kreĭn's theorem has been thoroughly investigated in the context of the Vlasov equation in Ref. 55. The same general results apply here, but this is beyond the scope of the present work.

VI. CONCLUSIONS

By making use of the Hamiltonian formalism, we have analyzed the mode signature and the stability properties of an ETG fluid model. The families of Casimir invariants of the model were obtained, thereby showing that the dynamics of the model is subject to an infinite number of constraints. A general energy stability condition has been derived, according to which, the absence of equilibrium flow and a restriction on the pressure equilibrium profile imply stability. Subsequently, after reviewing the concept of mode signature in the Hamiltonian framework, we have explicitly determined the energies of stable modes about homogeneous equilibria. From the stability viewpoint, the dispersion relation gives us a spectral stability condition which, however, does not give us information about the stronger condition of energy stability. Indeed, our analysis shows that spectrally stable homogeneous equilibria can be of two types, depending on the value of the parameters. If $\alpha_{\mathcal{P}}<0$, equilibria are spectrally and energy stable (i.e., with no NEMs). If $0<\alpha_{\mathcal{P}}<1/(4(1+k_{\perp}^2)\sqrt{r})$, on the other hand, equilibria are still spectrally stable, but they are not energy stable. Indeed, the sign of the second variation of the free energy functional, in this case, is indefinite because of the presence of NEMs. Equilibria of the second type might then be prone to dissipation-induced or nonlinearity-induced instabilities. Finally, we analyzed a reduced model with shear flow and described an integral transform that allows the transformation to normal form when there is a continuous spectrum and in this way identified negative energy continua. As anticipated in Sec. I, one of the advantages of the Hamiltonian formalism for investigating stability and mode signature, is that it is very general and can be applied to any plasma model with a Hamiltonian structure.

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APPENDIX A: THE INTEGRAL TRANSFORM—COORDINATE CHANGE

In this appendix we examine a particular example of a class of integral transforms that generate diagonalizing canonical transformations for theories with Poisson brackets of the form of Eq. (79). Specific examples and related background material for this Hamiltonian approach can be found in Refs. 14, 53–55, and 57–64. Alternatively, one could approach diagonalization by considering the eigenvalue problem, as was done in Sec. IV, but here because of the continuous spectrum this entails a generalization of Van Kampen's treatment of plasma oscillations[65] or use of an approach based on hyperfunction theory (microlocal analysis).[21,22] We have found the Hamiltonian integral transform approach, based on a generalization of the Hilbert transform, to be a natural tool for handling continuous spectra. It has also proven useful, in much the same was as the Hilbert transform for analytic signal processing, for analyzing experimental data.[66,67]

1. Transform pair

The transform used here is defined by

$$G_k[\Lambda_k](x,t) := \int_{-1}^{1} d\bar{x}\mathcal{G}_k(x,\bar{x})\Lambda_k(\bar{x},t) = \zeta_k(x,t), \quad \text{(A1)}$$

where $\Lambda_k(\bar{x},t)$ is here being transformed into $\zeta_k(x,t)$, with t only acting as a parameter, and the kernel defined by

$$\mathcal{G}_k(x,\bar{x}) := \varepsilon_k(x)\delta(x-\bar{x}) + \mathcal{P}\frac{v''_{\rm eq}(x)\phi_k(x,\bar{x})}{v_{\rm eq}(x)-v_{\rm eq}(\bar{x})}, \quad \text{(A2)}$$

$$\varepsilon_k(x) := 1-\mathcal{P}\int_{-1}^{1} dx' \frac{v''_{\rm eq}(x')\phi_k(x',x)}{v_{\rm eq}(x')-v_{\rm eq}(x)}, \quad \text{(A3)}$$

where $\mathcal{P}$ denotes Cauchy principal value and $\phi_k(x,\bar{x})$ is a solution to a regular Fredholm equation

$$\phi_k(x,\bar{x}) = \mathcal{K}_k(x,\bar{x}) + \int_{-1}^{1} dx' \mathcal{F}_k(x,x';\bar{x})\phi_k(x',\bar{x}), \quad \text{(A4)}$$

with the kernel defined by

$$\mathcal{F}_k(x,x';\bar{x}) = \left[\frac{\mathcal{K}_k(x,x')-\mathcal{K}_k(x,\bar{x})}{v_{\rm eq}(x')-v_{\rm eq}(\bar{x})}\right]v''_{\rm eq}(x'), \quad \text{(A5)}$$

with $\mathcal{K}_k$ defined by Eq. (88).

Evidently from the above, the transform of Eq. (A1) is tailored to the particular $v_{\rm eq}(x)$ under consideration. Also, note that upon insertion of Eq. (A2) into Eq. (A1), the transform is seen to be the sum of a multiplicative piece plus a piece that is a generalization of the Hilbert transform (see e.g., Ref. 68). Lastly, note that the functions ε_k and $\phi_k(x,\bar{x})$ only need to be calculated once for a given $v_{\rm eq}$. Generally this must be done numerically, but solution techniques are readily available to calculate the Cauchy integral of Eq. (A3), and it is a simple matter to rapidly obtain a solution to the regular Fredholm problem Eq. (A5).

The inverse of Eq. (A1), subject to the conditions of monotonicity and no discrete spectrum, which we have imposed on $v_{\rm eq}$, is given by

$$\hat{G}_k[\zeta_k](\bar{x},t) := \int_{-1}^{1} dx\hat{\mathcal{G}}_k(\bar{x},x)\zeta_k(x,t), \quad \text{(A6)}$$

where

$$\hat{\mathcal{G}}_k(\bar{x},x) = \frac{1}{|\varepsilon_k(\bar{x})|^2}\left[\varepsilon_k(\bar{x})\delta(x-\bar{x}) + \mathcal{P}\frac{v''_{\rm eq}(\bar{x})\phi_k(x,\bar{x})}{v_{\rm eq}(x)-v_{\rm eq}(\bar{x})}\right], \quad \text{(A7)}$$

and where $|\varepsilon_k(x)|^2 := \varepsilon_k^2 + \hat{\varepsilon}_k^2$ with $\hat{\varepsilon}_k(x) := -\pi\phi_k(\bar{x},\bar{x})v''_{\rm eq}(x)/v'_{\rm eq}(x)$.

If $\hat{G}_k$ is to be the inverse of the transform G_k, then $\hat{G}_k[G_k[\Lambda_k]] \equiv \Lambda_k$, which follows if

$$\int_{-1}^{1} dx\hat{\mathcal{G}}_k(\bar{x}',x)\mathcal{G}_k(x,\bar{x}) = \delta(\bar{x}-\bar{x}'). \quad \text{(A8)}$$

This inverse relation can also be viewed as a completeness relation for the continuous spectrum. We verify Eq. (A8) below. In a similar fashion we have the reciprocal completeness relation,

$$\int_{-1}^{1} d\bar{x}\hat{\mathcal{G}}_k(\bar{x},x')\mathcal{G}_k(x,\bar{x}) = \delta(x-x'). \tag{A9}$$

2. Transform inverse

Now we show that the transform Eq. (A1) is the inverse of Eq. (A6). Under mild restriction on the profiles v_{eq}, many rigorous results can be obtained. We will not pursue these here, but instead direct the reader to Refs. 55 and 59 where corresponding proofs are given in the context of the Vlasov-Poisson equation—the situation in the present case is much the same.

Substituting the explicit forms of $\hat{\mathcal{G}}_k(\bar{x},x)$ and $\mathcal{G}_k(x,\bar{x}')$ into Eq. (A8), gives

$$\begin{aligned}\int_{-1}^{1} dx\hat{\mathcal{G}}_k(\bar{x}',x)\mathcal{G}_k(x,\bar{x}) &= \varepsilon_k(\bar{x})^2\delta(\bar{x}-\bar{x}') + \frac{v''_{\text{eq}}(\bar{x}')}{v_{\text{eq}}(\bar{x})-v_{\text{eq}}(\bar{x}')}[\varepsilon_k(\bar{x})\phi_k(\bar{x},\bar{x}') \\ &\quad - \varepsilon_k(\bar{x}')\phi_k(\bar{x}',\bar{x})] \\ &\quad + \frac{v''_{\text{eq}}(\bar{x}')}{|\varepsilon_k(\bar{x}')|^2}\mathcal{P}\int_{-1}^{1} dx\frac{v''_{\text{eq}}(x)\phi_k(x,\bar{x})\phi_k(x,\bar{x}')}{[v_{\text{eq}}(x)-v_{\text{eq}}(\bar{x})][v_{\text{eq}}(x)-v_{\text{eq}}(\bar{x}')]}.\end{aligned} \tag{A10}$$

To evaluate the integral of the final term of Eq. (A10) we use the relation,

$$\begin{aligned}v''_{\text{eq}}(\bar{x}')\mathcal{P}\int_{-1}^{1} dx\frac{v''_{\text{eq}}(x)\phi_k(x,\bar{x}')\phi_k(x,\bar{x})}{[v_{\text{eq}}(x)-v_{\text{eq}}(\bar{x}')][v_{\text{eq}}(x)-v_{\text{eq}}(\bar{x})]} &= \hat{\varepsilon}_k(\bar{x})^2\delta(\bar{x}-\bar{x}') + \frac{v''_{\text{eq}}(\bar{x}')}{v_{\text{eq}}(\bar{x})-v_{\text{eq}}(\bar{x}')}\mathcal{P} \\ &\quad \times\int_{-1}^{1} dx v''_{\text{eq}}(x)\phi_k(x,\bar{x}')\phi_k(x,\bar{x})\left[\frac{1}{v_{\text{eq}}(x)-v_{\text{eq}}(\bar{x})}\right. \\ &\quad \left. - \frac{1}{v_{\text{eq}}(x)-v_{\text{eq}}(\bar{x}')}\right],\end{aligned} \tag{A11}$$

which is a form of the Poincaré–Bertrand transposition formula (see, e.g., Ref. 69). Using Eq. (A11) and collecting together terms, Eq. (A10) becomes

$$\int_{-1}^{1} dx\hat{\mathcal{G}}_k(\bar{x}',x)\mathcal{G}_k(x,\bar{x}) = \delta(\bar{x}-\bar{x}') + \frac{v''_{\text{eq}}(\bar{x}')}{v_{\text{eq}}(\bar{x})-v_{\text{eq}}(\bar{x}')}\frac{\mathcal{J}_k(\bar{x},\bar{x}')}{|\varepsilon_k(\bar{x}')|^2}, \tag{A12}$$

where

$$\begin{aligned}\mathcal{J}_k(\bar{x},\bar{x}') &:= \varepsilon_k(\bar{x})\phi_k(\bar{x},\bar{x}') \\ &\quad + \mathcal{P}\int_{-1}^{1} dx\frac{v''_{\text{eq}}(x)\phi_k(x,\bar{x})\phi_k(x,\bar{x}')}{v_{\text{eq}}(x)-v_{\text{eq}}(\bar{x})} \\ &\quad - \varepsilon_k(\bar{x}')\phi_k(\bar{x}',\bar{x}) \\ &\quad - \mathcal{P}\int_{-1}^{1} dx\frac{v''_{\text{eq}}(x)\phi_k(x,\bar{x}')\phi_k(x,\bar{x})}{v_{\text{eq}}(x)-v_{\text{eq}}(\bar{x}')}.\end{aligned} \tag{A13}$$

Equation (A4) can be rewritten as

$$\begin{aligned}\phi_k(x,\bar{x}) &= \varepsilon_k(\bar{x})\mathcal{K}_k(x,\bar{x}) \\ &\quad + \mathcal{P}\int_{-1}^{1} dx'\mathcal{K}_k(x,x')\frac{v''_{\text{eq}}(x')\phi_k(x',\bar{x})}{v_{\text{eq}}(x')-v_{\text{eq}}(\bar{x})}.\end{aligned} \tag{A14}$$

Insertion of Eq. (A14) into the first and third terms of Eq. (A13) and into the second ϕ_k of the integrands of the second and fourth terms, reveals that $\mathcal{J}_k(\bar{x},\bar{x}')\equiv 0$. Hence, we have verified Eq. (A8) and finally that Eq. (A1) is the inverse of Eq. (A6).

3. Transform identities

Just as Fourier and Laplace transforms possess many useful identities, there are a variety of identities possessed by G_k and $\hat{G}_k$. We state two such transform identities used in Sec. V:

$$\begin{aligned}\hat{G}_k[v_{\text{eq}}\zeta_k](\bar{x},t) &= v_{\text{eq}}(\bar{x})\hat{G}_k[\zeta_k](\bar{x},t) \\ &\quad + \frac{v''_{\text{eq}}(\bar{x})}{|\varepsilon_k|^2(\bar{x})}\mathcal{P}\int_{-1}^{1} dx\zeta_k(x,t)\phi_k(x,\bar{x})\end{aligned} \tag{A15}$$

and

$$\hat{G}_k[v''_{\text{eq}}\phi_k](\bar{x},t) = \frac{v''_{\text{eq}}(\bar{x})}{|\varepsilon_k|^2(\bar{x})}\mathcal{P}\int_{-1}^{1} dx\zeta_k(x,t)\phi_k(x,\bar{x}), \tag{A16}$$

where Eq. (A16) is valid if ζ_k is related to ϕ_k according to Eq. (87). The validity of these identities can be demonstrated in a manner similar to our demonstration of Eq. (A8).

For the record we state two additional orthogonality-like identities,

$$\int_{-1}^{1} dx\hat{\mathcal{G}}_k(\bar{x},x)\hat{\mathcal{G}}_k(\bar{x}',x)v''_{\text{eq}}(x) = \frac{v''_{\text{eq}}(\bar{x})}{|\varepsilon_k|^2(\bar{x})}\delta(\bar{x}-\bar{x}')$$

and

$$\int_{-1}^{1} dx\left[\frac{v_{\text{eq}}(x)}{v'_{\text{eq}}(x)}\mathcal{G}_k(x,\bar{x}) - \phi_k(x,\bar{x})\right]\mathcal{G}_k(x,\bar{x}') = \frac{v_{\text{eq}}(\bar{x})}{v''_{\text{eq}}(\bar{x})}|\varepsilon_k|^2(\bar{x})\delta(\bar{x}-\bar{x}').$$

These last two identities are not used in this paper.

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