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ON KREIN-LIKE THEOREMS FOR NONCANONICAL HAMILTONIAN SYSTEMS WITH CONTINUOUS SPECTRA: APPLICATION TO VLASOV-POISSON

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The notions of spectral stability and the spectrum for the Vlasov-Poisson system linearized about homogeneous equilibria, $f_0(v)$, are reviewed. Structural stability is reviewed and applied to perturbations of the linearized Vlasov operator through perturbations of f_0. We prove that for each f_0 there is an arbitrarily small $\delta f_0'$ in $W^{1,1}(\mathbb{R})$ such that $f_0 + \delta f_0$ is unstable. When f_0 is perturbed by an area preserving rearrangement, f_0 will always be stable if the continuous spectrum is only of positive signature, where the signature of the continuous spectrum is defined as in Morrison and Pfirsch (1992) and Morrison (2000). If there is a signature change, then there is a rearrangement of f_0 that is unstable and arbitrarily close to f_0 with f_0' in $W^{1,1}$. This result is analogous to Krein's theorem for the continuous spectrum. We prove that if a discrete mode embedded in the continuous spectrum is surrounded by the opposite signature there is an infinitesimal perturbation in C^n norm that makes f_0 unstable. If f_0 is stable we prove that the signature of every discrete mode is the opposite of the continuum surrounding it.

1. Introduction

The perturbation of point spectra for classical vibration and quantum mechanical problems has a long history (Rayleigh, 1896; Rellich, 1969). The more difficult problem of assessing the structural stability of the continuous spectrum in scattering problems has also been widely investigated (Friedrichs, 1965; Kato, 1966). Because general linear Hamiltonian systems are not governed by Hermitian or symmetric operators, the spectrum need not be

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stable and a transition to instability is possible. For finite degree-of-freedom Hamiltonian systems, the situation is described by Krein's theorem (Kreĭn, 1950; Kreĭn and Jakubovič, 1980; Moser, 1958), which states that a necessary condition for a bifurcation to instability under perturbation is to have a collision between eigenvalues of opposite signature. The purpose of the present article is to investigate Krein-like phenomena in Hamiltonian systems with continuous spectra. Of interest are systems that describe continuous media that are Hamiltonian in terms of noncanonical Poisson brackets (see, e.g., Morrison, 1998, 2005).

Our study differs from that of Grillakis (1990), which considered canonical Hamiltonian systems with continuous spectra in a Hilbert space where the time evolution operator is self-adjoint. The effects of relatively compact perturbations on such a system were studied and it was proved that the existence of a negative energy mode in the continuous spectrum caused the system to be structurally unstable. It was also proved that such systems are otherwise structurally stable. In addition, our study differs from analyses of fluid theories concerning point spectra (MacKay and Saffman, 1986; Kueny and Morrison, 1995) and point and continuous spectra (Hirota and Fukumoto, 2008), the latter using hyperfunction theory.

A representative example of the kind of Hamiltonian system of interest is the Vlasov-Poisson equation (Morrison, 1980), which, when linearized about stable homogeneous equilibria, gives rise to a linear Hamiltonian system with pure continuous spectra that can be brought into action-angle form (Morrison and Pfirsch, 1992; Morrison, 2000, 1994; Morrison and Shadwick, 1994). A definition of signature was given in these works for the continuous spectrum. In the present article we concentrate on the Vlasov-Poisson equation, but the same structure is possessed by Euler's equation for the two-dimensional fluid, where signature for shear flow continuous spectra was defined (Balmforth and Morrison, 1998, 2002) as well as, a large class of systems (Morrison, 2003). Thus, modulo technicalities—the behavior treated here—is expected to cover a large class of systems.

In Section 2 we review the noncanonical Hamiltonian structure for a class of systems on a formal level that includes the Vlasov-Poisson equation as a special case. Linearization about equilibria, the concept of dynamical accessibility, and the linear

Hamiltonian operator T—the main subject of the remainder of the article—are defined. Next we sketch proofs in varying levels of detail pertaining to properties of this linear operator for various equilibria. In Section 3 we describe spectral stability in general terms and analyze the spectrum of T for the Vlasov case. The existence of a continuous component to the spectrum is demonstrated and Penrose plots are used to describe the point component. In Section 4 we describe structural stability and, in particular, consider the structural stability of T under perturbation of the equilibrium state. We show that any equilibrium is unstable under perturbation of an arbitrarily small function in $W^{1,1}$ In Section 5 we introduce the Krein-Moser theorem and restrict it to dynamically accessible perturbations. We prove that equilibria without signature changes are structurally stable and those with changes are structurally unstable. In Section 6 we define critical states of the linearized Vlasov equation that are structurally unstable under perturbations that are further restricted. We prove that a mode with the opposite signature of the continuum is structurally unstable and that the opposite combination cannot exist unless the system is already unstable. Finally, in Section 7, we conclude.

2. Noncanonical Hamiltonian Form

The class of equations of interest have a single dependent variable $\zeta(x, v, t)$, such that for each time t, $\zeta : \mathcal{D} \rightarrow \mathbb{R}$, where the particle phase space $\mathcal{D}$ is a two-dimensional domain with coordinates (x, v). The dynamics are assumed to be Hamiltonian in terms of a noncanonical Poisson bracket of the form

$$\{F, G\} = \int_{\mathcal{D}} dx dv \, \zeta \left[\frac{\delta F}{\delta \zeta}, \frac{\delta G}{\delta \zeta} \right], \qquad (1)$$

where $[f, g] := f_x g_v - f_v g_x$ is the usual Poisson bracket, the subscripts denote partial differentiation, and $\delta F / \delta \zeta$ denotes the functional derivative of a functional $F[\zeta]$. The equation of motion is generated from a Hamiltonian functional $\mathcal{H}[\zeta]$ as follows:

$$\zeta_t = \{\zeta, \mathcal{H}\} = -[\zeta, \mathcal{E}], \qquad (2)$$

where $\mathcal{E} := \delta\mathcal{H}/\delta\zeta$. The Poisson bracket (1) is noncanonical: it uses only a single noncanonical variable ζ, instead of the usual canonically conjugate pair; it possesses degeneracy reflected in the existence of Casimir invariants, $C = \int_{\mathcal{D}} dx\, dv\, \mathcal{C}(\zeta)$ that satisfy $\{F, C\} = 0$ for all functionals F; but it does satisfy the Lie-algebraic properties of usual Poisson brackets. (For further details see Morrison, 1982, 1998, 2003; Holm et al., 1985).

For the Vlasov-Poisson equation we assume $\mathcal{D} = X \times \mathbb{R}$, where $X \subset \mathbb{R}$ or $X = S$—the circle; the distinction will not be important. The dependent variable is the particle phase space density $f(x, v, t)$ and the Hamiltonian is given by

$$\mathcal{H}[f] = \frac{1}{2} \int_X dx \int_{\mathbb{R}} dv\, v^2 f + \frac{1}{2} \int_X dx\, |\phi_x|^2 , \qquad (3)$$

where ϕ is shorthand for the functional dependence on f obtained through a solution of Poisson's equation, $\phi_{xx} = 1 - \int_{\mathbb{R}} f\, dv$, for a positive charge species with a neutralizing background. Using $\delta\mathcal{H}/\delta f = \mathcal{E} = v^2/2 + \phi$, we obtain

$$f_t = \{f, \mathcal{H}\} = -[f, \mathcal{E}] = -v f_x + \phi_x f_v , \qquad (4)$$

where, as usual, the plasma frequency and Debye length have been used to nondimensionalize all variables.

This Hamiltonian form for the Vlasov-Poisson equation was first published in Morrison (1980). For a discussion of a general class of systems with this Hamiltonian form to which the ideas of the present analysis can be applied see Morrison (2003). In a sequence of papers (Morrison, 1987, 2000; Morrison and Pfirsch, 1992; Morrison and Shadwick, 1994, 2008; Shadwick and Morrison, 1994) various ramifications of the Hamiltonian form have been explored—notably, canonization and diagonalization of the linear dynamics to which we now turn.

Because of the noncanonical form, linearization requires expansion of the Poisson bracket as well as the Hamiltonian. Equilibria ζ_0 are obtained by extremization of a free energy functional, $\mathcal{F} = \mathcal{H} + C$, as was first done for Vlasov-like equilibria in Kruskal and Oberman (1958). Writing $\zeta = \zeta_0 + \zeta_1$ and expanding gives the

Hamiltonian form for the linear dynamics

$$\zeta_{1t} = \{\zeta_1, H_L\}_L, \tag{5}$$

where the linear Hamiltoinian, $H_L = \frac{1}{2}\int_{\mathcal{D}} dx\,dv\,\zeta_1 \mathcal{O}\,\zeta_1$, is the second variation of $\mathcal{F}$, a quadratic form in ζ_1 defined by the symmetric operator $\mathcal{O}$, and $\{F, G\}_L = \int_{\mathcal{D}} dx\,dv\,\zeta_0[F_1, G_1]$ with $F_1 := \delta F/\delta\zeta_1$. Thus the linear dynamics are governed by the time evolution operator $T \cdot := -\{\,\cdot\,, H_L\}_L = [\zeta_0, \mathcal{O}\cdot\,]$.

Linearizing the Vlasov-Poisson equation about an homogeneous equilibrium, $f_0(v)$, gives rise to the system,

$$f_{1t} = -v f_{1x} + \phi_{1x} f_0' \tag{6}$$

$$\phi_{1xx} = -\int_{\mathbb{R}} dv\,f_1 , \tag{7}$$

for the unknown $f_1(x, v, t)$. Here $f_0' := df_0/dv$. This is an infinite-dimensional linear Hamiltonian system generated by the Hamiltonian functional:

$$H_L[f_1] = -\frac{1}{2}\int_X dx \int_{\mathbb{R}} dv\,\frac{v}{f_0'}\,|f_1|^2 + \frac{1}{2}\int_X dx\,|\phi_{1x}|^2 . \tag{8}$$

We concentrate on systems where x is an ignorable coordinate and either Fourier expand or transform. For Vlasov-Poisson this gives the system

$$f_{kt} = -ikv f_k + \frac{if_0'}{k}\int_{\mathbb{R}} d\bar{v}\,f_k(\bar{v}, t) =: -T_k f_k, \tag{9}$$

where $f_k(v, t)$ is the Fourier dual to $f_1(x, v, t)$. Perturbation of the spectrum of the operator defined by Eq. (9) is the primary subject of this article. The operator T_k is a Hamiltonian operator generated by the Hamiltonian functional

$$H_L[f_k, f_{-k}] = \frac{1}{2}\sum_k \left(-\int_{\mathbb{R}} dv\,\frac{v}{f_0'}\,|f_k|^2 + |\phi_k|^2\right), \tag{10}$$

with the Poisson bracket

$$\{F, G\}_L = \sum_{k=1}^{\infty} ik \int_{\mathbb{R}} dv \, f_0' \left(\frac{\delta F}{\delta f_k} \frac{\delta G}{\delta f_{-k}} - \frac{\delta F}{\delta f_{-k}} \frac{\delta G}{\delta f_k} \right) . \qquad (11)$$

Observe from (11) that $k \in \mathbb{N}$ and thus f_k and f_{-k} are independent variables that are almost canonically conjugate. Thus the complete system is

$$f_{k_t} = - T_k f_k \qquad \text{and} \qquad f_{-k_t} = - T_{-k} f_{-k} , \qquad (12)$$

from which we conclude the spectrum is Hamiltonian.

Lemma 2.1 *If λ is an eigenvalue of the Vlasov equation linearized about the equilibrium $f_0'(v)$, then so are $-\lambda$ and $\bar{\lambda}$ (complex conjugate). Thus if $\lambda = \gamma + i\omega$, then eigenvalues occur in the pairs, $\pm\gamma$ and $\pm i\omega$, for purely real and imaginary cases, respectively, or quartets, $\lambda = \pm\gamma \pm i\omega$, for complex eigenvalues.*

Proof. That $-\lambda$ is an eigenvalue follows immediately from the symmetry $T_{-k} = -T_k$, and that $\bar{\lambda}$ is an eigenvalue follows from $T_k f_k = -\overline{(T_k f_k)}$. □

In Morrison and Pfirsch (1992), Morrison and Shadwick (1994, 2008), and Morrison (2000) it was shown how to scale f_k and f_{-k} to make them canonically conjugate variables. In order to do this requires the following definition of dynamical accessibility, a terminology introduced in Morrison and Pfirsch (1989, 1990).

Definition. A particle phase space function k is *dynamically accessible* from a particle phase space function h, if k is an area-preserving rearrangement of h; i.e., in coordinates $k(x, v) = h(X(x, v), V(x, v))$, where $[X, V] = 1$. A peturbation δh is *linearly dynamically accessible* from h if $\delta h = [G, h]$, where G is the infinitesimal generator of the canonical transformation $(x, v) \leftrightarrow (X, V)$.

Dynamically accessible perturbations come about by perturbing the particle orbits under the action of some Hamiltonian. Since electrostatic-charged particle dynamics is Hamiltonian, one can make the case that these are the only perturbations allowable within the confines of Vlasov-Poisson theory.

Given an equilibrium state f_0, linear dynamically accessible perturbations away from this equilibrium state satisfy $\delta f_0 = [G, f_0] = G_x f_0'$. Therefore assuming the initial condition for the linear dynamics is linearly dynamically accessible, we can define

$$q_k(v, t) = f_k \qquad \text{and} \qquad p_k(v, t) = -i f_{-k}/(k f_0') \tag{13}$$

without worrying about a singularity at the zeros of f_0' and $k = 0$. With the definitions of (13), the Poisson bracket of (11) achieves canonical from

$$\{F, G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} dv \left(\frac{\delta F}{\delta q_k} \frac{\delta G}{\delta p_k} - \frac{\delta F}{\delta p_k} \frac{\delta G}{\delta q_k} \right). \tag{14}$$

The full system has the new Hamiltonian $\bar{\mathcal{H}} = \mathcal{H} + U\mathcal{P}$ in a frame moving with speed U, where $\mathcal{P} = \int_{\mathcal{D}} dx\,dv\,vf$. Linearizing in this frame yields the linear Hamiltonian $\bar{H}_L = H_L + P_L$, from which we identify the linear momentum

$$P_L[f_k, f_{-k}] = \frac{1}{2} \sum_{k=1}^{\infty} \int_{\mathbb{R}} dv \frac{k}{f_0'} |f_k|^2, \tag{15}$$

which must be conserved by the linear dynamics. It is easy to show directly that this is the case.

Lemma 2.2 *The momentum P_L defined by (15) is a constant of motion, i.e., $\{P_L, H_L\} = 0$.*

Proof. This follows immediately from (12): $\int_{\mathbb{R}} dv\,(f_k T_{-k} + f_{-k} T_k) = 0$. □

Observe that like the Hamiltonian, H_L, the momentum P_L is conserved for each k, which in all respects appears only as a parameter in our system. Assuming the system size to be L yields $k = 2\pi n/L$ with $n \in \mathbb{N}$, and, thus, this parameter can be taken to be in $\mathbb{R}^+/\{0\}$. Alternatively, we could suppose $X = \mathbb{R}$, Fourier transform, and split the Fourier integral to obtain an expression similar to (11) with the sum replaced by an integral over positive values of k. For the present analysis we will not be concerned with issues of convergence for reconstructing the spatial variation

of $f_1(x, v, t)$, but will only consider $k \in \mathbb{R}^+/\{0\}$ to be a parameter in our operator. We will see in Section 3 that the operator T_k possesses a continuous component to its spectrum. However, we emphasize that this continuous spectrum of interest arises from the multiplicative nature of the velocity operator, i.e. the term vf_k of T_k, not from having an infinite spatial domain, as is the case for free particle or scattering states in quantum mechanics. In the remainder of the article, f will refer to either f_1 or f_k, which will be clear from context, and the dependence on k will be suppressed, e.g. in T_k, unless k dependence is being specifically addressed.

3. Spectral Stability

Now we consider properties of the evolution operator T defined by (9). We define spectral stability in general terms, record some properties of T, and describe the tools necessary to characterize the spectrum of T. We suppose f_k varies as $\exp(-i\omega t)$, where ω is the frequency and $i\omega$ is the eigenvalue. For convenience we also use $u := \omega/k$, where $k \in \mathbb{R}^+$. The system is spectrally stable if the spectrum of T is less than or equal to zero or the frequency is always in the closed lower half plane. Since the system is Hamiltonian, the question of stability reduces to deciding if the spectrum is confined to the imaginary axis.

Definition. The linearized dynamics of a Hamiltonian system around some equilibrium solution, with the phase space of solutions in some Banach space $\mathcal{B}$, is *spectrally stable* if the spectrum $\sigma(T)$ of the time evolution operator T is purely imaginary.

Spectral stability does not guarantee that the system is stable or that the equilibrium f_0 is linearly stable. (See, e.g., Morrison, 1998, for general discussion.) The solutions of a spectrally stable system are guaranteed to grow at most sub-exponentially, and one can construct a spectrally stable system with polynomial temporal growth for certain initial conditions. (See, e.g., Degond, 1986, for analysis of the Vlasov system.)

Spectral stability relies on functional analysis for its definition, since the spectrum of the operator T may depend on the choice of function space $\mathcal{B}$. The time evolution operators arising from the types of noncanonical Hamiltonian systems that are of

interest here generally contain a continuous spectrum (Morrison, 2003), and the effects of perturbations that we study can be categorized by properties of the continuous spectrum of these operators. In general for the operators in Morrison (2003), the operator T is the sum of a multiplication operator and an integral operator. In the Vlasov case, the multiplicative operator is $iv \cdot$ and the integral operator, is $f_0' \int dv \cdot$. As we will see, the multiplication operator causes the continuous spectrum to be composed of the entire imaginary axis except possibly for some discrete points.

Instability comes from the point spectrum. In particular, the linearized Vlasov Poisson equation is not spectrally stable when the time evolution operator has a spectrum that includes a point away from the imaginary axis, with the necessary counterparts implied by Lemma 2.2. For the operator T this will always be a discrete mode; i.e. an eigenmode associated with an eigenvalue in the point spectrum.

Theorem 3.1 *The one-dimensional linearized Vlasov-Poisson system with homogeneous equilibrium f_0 is spectrally unstable if for some $k \in \mathbb{R}^+$ and u in the upper half plane, the plasma dispersion relation*

$$\varepsilon(k, u) := 1 - k^{-2} \int_{\mathbb{R}} dv \frac{f_0'}{v - u} = 0 .$$

Otherwise it is spectrally stable.

Proof. The details of this proof are given in plasma textbooks. It follows directly from (6) and (7), and the assumption $f_1 \sim \exp(ikx - i\omega t)$. □

Using the Nyquist method that relies on the argument principle of complex analysis, Penrose (1960) was able to relate the vanishing of $\varepsilon(k, u)$ to the winding number of the closed curve determined by the real and imaginary parts of ε as u runs along the real axis. Such closed curves are called Penrose plots. The crucial quantity is the integral part of ε as u approaches the real axis from above:

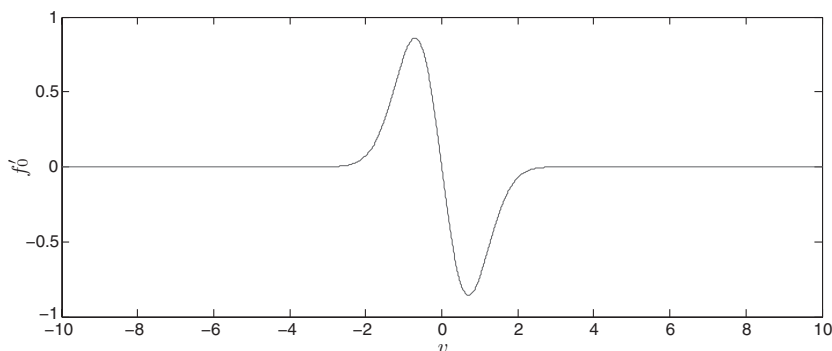$$\lim_{u \to 0^+} \frac{1}{\pi} \int_{\mathbb{R}} dv \frac{f_0'}{v - u} = H[f_0'](u) - if_0'(u) ,$$

FIGURE 1 f_0' for a Maxwellian distribution.

where $H[f_0']$ denotes the Hilbert transform, $H[f_0'] = \frac{1}{\pi}\fint dv\, f_0'/(v-u)$, where $\fint := PV \int_{\mathbb{R}}$ indicates the Cauchy principle value. (See King, 2009, for an in-depth treatment of Hilbert transforms.) The graph of the real line under this mapping is the essence of the Penrose plot, and so we will refer to these closed curves as Penrose plots as well. When necessary to avoid ambiguity, we will refer to the former as ε-plots.

For example, Figure 1 shows the derivative of the distribution function, f_0', for the case of a Maxwellian distribution, and Figure 2 shows the contour $H[f_0'] - if_0'(u)$ that emerges from the origin in the complex plane at $u = -\infty$, descends, and then wraps around to return to the origin at $u = \infty$. From this figure it is evident that the winding number of the $\varepsilon(k, u)$-plot is zero for any fixed $k \in \mathbb{R}$, and as a result there are no unstable modes.

Making use of the argument principle as described, Penrose obtained the following criterion:

Theorem 3.2 *The linearized Vlasov-Poisson system with homogeneous equilibrium f_0 is spectrally unstable if there exists a point u such that*

$$f_0'(u) = 0 \qquad \text{and} \qquad \fint dv\, \frac{f_0'(v)}{v-u} > 0\,,$$

with f_0' traversing zero at u. Otherwise it is spectrally stable.

Penrose plots can be used to visually determine spectral stability. As described, the Maxwellian distribution $f_0 = e^{-v^2}$ is
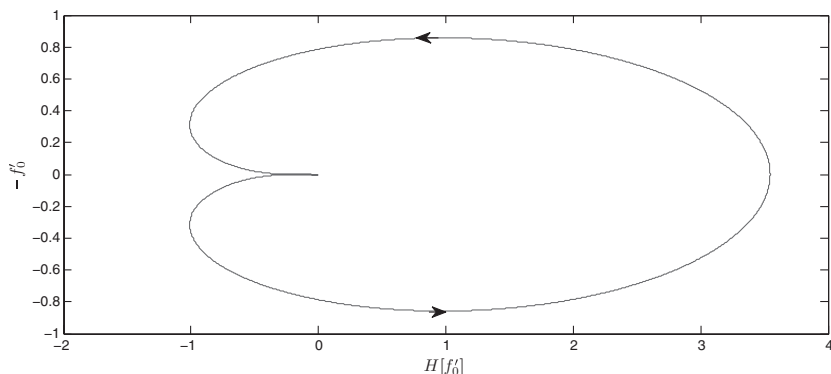
FIGURE 2 Stable Penrose plot for a Maxwellian distribution.

stable, as the resulting ε-plot does not encircle the origin. However, it is not difficult to construct unstable distribution functions. The superposition of two displaced Maxwellian distributions, $f_0 = e^{-(v+c)^2} + e^{-(v-c)^2}$, is such a case. As c increases the distribution goes from stable to unstable. Figures 3 and 4 demonstrate how the transition from stability to instability is manifested in a Penrose plot. The two examples are $c = 3/4$ and $c = 1$. (Note, the normalization of f_0 only affects the overall scale of the Penrose plots and so is ignored for convenience.) It is evident from Figure 4 that for some $k \in \mathbb{R}$ the ε-plot (which is a displacement of the curve shown by multiplying by $-k^{-2}$ and adding unity) will encircle the origin, and thus will be unstable for such k-values.
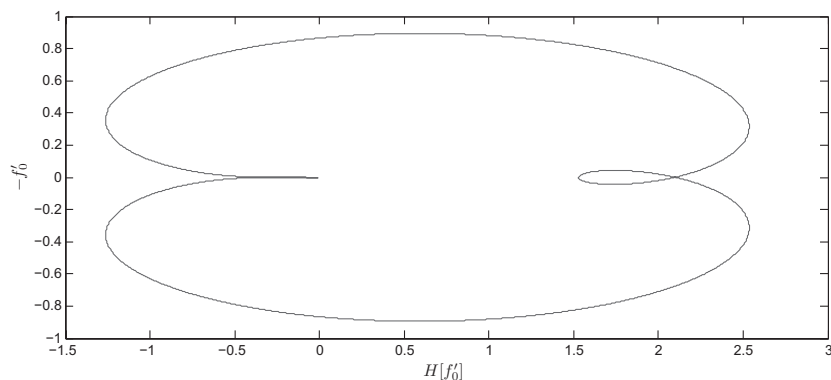

FIGURE 3 Penrose plot for a stable superposition of Maxwellian distributions.
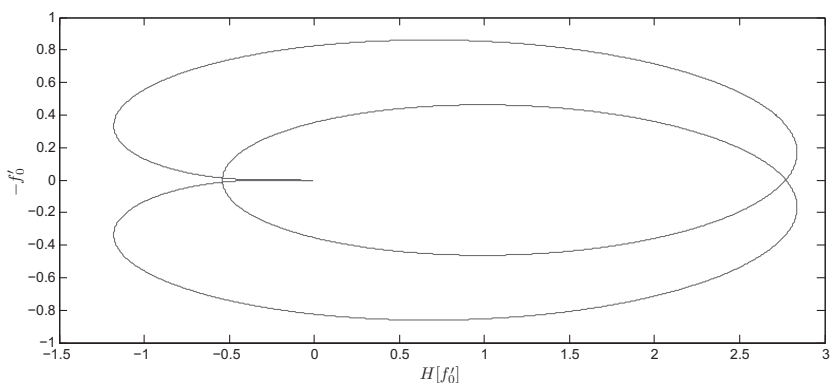
FIGURE 4 The unstable Penrose plot corresponding to two separated Maxwell distributions.

We now are positioned to completely determine the spectrum. For convenience we set $k = 1$ when it does not affect the essence of our arguments, and consider the operator $T\colon f \mapsto ivf - if_0' \int f$ in the space $W^{1,1}(\mathbb{R})$, but we also discuss the space $L^1(\mathbb{R})$. The space $W^{1,1}(\mathbb{R})$ is the Sobolev space containing the closure of functions under the norm $\|f\|_{1,1} = \|f\|_1 + \|f'\|_1$. Thus it contains all functions that are in $L^1(\mathbb{R})$ whose weak derivatives are also in $L^1(\mathbb{R})$. First we establish the expected facts that T is densely defined and closed.

In $W^{1,1}$ the operator T is the sum of the multiplication operator and a bounded operator—it is densely defined and closed because the multiplication operator is densely defined and closed in these spaces, where

$$D_1(T) := \{f | vf \in W^{1,1}(\mathbb{R})\}.$$

Theorem 3.3 *The operator* $T\colon W^{1,1}(\mathbb{R}) \to W^{1,1}(\mathbb{R})$ *with domain* $D_1(T)$ *is both (i) densely defined and (ii) closable.*

Proof. (i) The set of all smooth functions with compact support, $C_c^\infty(\mathbb{R})$ is a subset of D_1. This set is dense in $W^{1,1}(\mathbb{R})$ so D_1 is dense and T is densely defined. (ii) The operator T is closable if the operator v is closable because T and v differ by a bounded operator. The multiplication operator v is closed if for each sequence

$f_n \subset W^{1,1}(\mathbb{R})$ that converges to 0 either vf_n converges to 0 or vf_n does not converge. Suppose vf_n converges. At each point f_n converges to 0. Therefore vf_n converges to 0 at each point, so vf_n converges to 0 if it converges.　　　　　　　　　　　　　　□

Therefore some domain D exists such that the graph (D, TD) is closed.

In determining the spectrum of the operator T, denoted $\sigma(T)$, we split the spectrum into point, residual, and continuous components as follows.

Definition. For $\lambda \in \sigma(T)$ the resolvent of T is $R(T, \lambda) = (T - \lambda I)^{-1}$, where I is the identity operator. We say λ is (i) in the point spectrum, $\sigma_p(T)$, if $T - \lambda I$ fails to be injective. (ii) In the residual spectrum, $\sigma_r(T)$, if $R(T, \lambda)$ exists but is not densely defined. (iii) In the continuous spectrum, $\sigma_c(T)$, if $R(T, \lambda)$ exists and is densely defined but unbounded.

Using this definition we characterize the spectrum of the operator T.

Theorem 3.4 *The component $\sigma_p(T)$ consists of all points $\lambda = iu \in \mathbb{C}$ where $1 - k^{-2} \int_{\mathbb{R}} dv\, f_0'/(v - u) = 0$, $\sigma_c(T)$ consists of all $\lambda = iu$ with $u \in \mathbb{R} \setminus (-i\sigma_p(T) \cap \mathbb{R})$, and $\sigma_r(T)$ contains all points $\lambda = iu$ in the complement of $\sigma_p(T) \cup \sigma_c(T)$ that satisfy $f_0'(u) = 0$.*

Proof. By the Penrose criterion we can identify all the points in the point spectrum. If $1 - k^{-2} \int_{\mathbb{R}} dv\, f_0'/(v - u) = 0$ then $iu = \lambda \in \sigma_p(T)$. Because the system is Hamiltonian these modes will occur for the linearized Vlasov-Poisson system in quartets (two for T_k and two for T_{-k}), as follows from Lemma 2.2. It is possible for there to be discrete modes with real frequencies, and these will occur in pairs. If for real u the map $u \mapsto \varepsilon$ passes through the origin then there will be an embedded mode.

For convenience we drop the wavenumber subscript k on f_k and add the subscript n to identify f_n as an element of a sequence of functions that converges to zero with, for each n, support contained in an interval of length $2\epsilon(n)$ surrounding the point u and zero average value. Let $u \in \mathbb{R}$ and choose the sequence $\{f_n\}$ so

that $\epsilon(n) \to 0$. Then for each n

$$
\begin{aligned}
\|R(T, iu)\| &\geq \frac{\|f_n\|_{1,1}}{\|(v-u)f_n\|_{1,1}} \\
&\geq \frac{\|f_n\|_{1,1}}{\|v-u\|_{W^{1,1}(u-\epsilon, u+\epsilon)}\|f_n\|_{1,1}} \\
&= \frac{1}{\|v-u\|_{W^{1,1}(u-\epsilon, u+\epsilon)}}.
\end{aligned}
$$

In the expression, $W^{1,1}(u - \epsilon, u + \epsilon)$ refers to the integral of $|f| + |f'|$ over the interval $(u-\epsilon, u+\epsilon)$. Therefore the resolvent is an unbounded operator and $iu = \lambda$ is in the spectrum. If the frequency u has an imaginary component $i\gamma$ then $\|R(T, iu)\| < 1/\gamma$, so unless $iu = \lambda$ is part of the point spectrum it is part of the resolvent set.

The residual spectrum of T is contained in the point spectrum of T^*. The dual of $W^{1,1}$ is the space $W^{-1,1}$ defined by pairs $(g, h) \in W^{-1,1}$ with $\|(g, h)\|_{-1,1} < \infty$ (Adams, 2003). The operator $T^*(g, h) = i(vg - h + \int(gf_0' - hf_0'')\,dv, -vh)$ is the adjoint of T. If we search for a member $iu = \lambda$ of the point spectrum we get two equations, one of which is $(v - u)h = 0$. This forces $h = 0$ because h cannot be a δ-function in $W^{-1,1}$. The other equation is then $(v - u)g + \int gf_0'\,dv = 0$, which can only be true if the integral is zero or if $(v - u)g$ is a constant. This $g = \frac{1}{v-u}$ and the resulting equation for u is the same equation as that for the frequency of the point modes of T. If the integral is zero then $g = \delta(v - u)$ is a solution when $f_0'(u) = 0$. Therefore the residual spectrum contains the points $\lambda = iu$ satisfying $f_0'(u) = 0$. $\qquad\square$

This characterization of the spectrum fails in Banach spaces with less regularity than $W^{1,1}$, such as L^p spaces, because the Dirac δ is not contained in the dual space. In this case the residual spectrum vanishes because $\sigma_p(T^*) = \sigma_p(T)$. This calculation is nearly identical to that of Degond (1986), who characterizes the residual spectrum slightly differently than we do. In any event, the result is that the Penrose criterion determines whether T is spectrally stable. If the winding number of the ε-plot is positive, then there is spectral instability, and if it is zero there is spectral stability.

4.   Structural Stability

Spectral stability characterizes the linear dynamics of a nonlinear Hamiltonian system in a neighborhood of an equilibrium. The main question now is to determine when a spectrally stable system can be made spectrally unstable with a small perturbation. When this is impossible for our choice of allowed perturbations, we say the equilibrium is structurally stable, and when there is an infinitesimal perturbation that makes the system spectrally unstable we say that the equilibrium is spectrally unstable. We can make this more precise by stating it in terms of operators on a Banach space.

Definition. Consider an equilibrium solution of a Hamiltonian system and the corresponding time evolution operator T for the linearized dynamics, with a phase space some Banach space $\mathcal{B}$. Suppose that T is spectrally stable. Consider perturbations δT of T and define a norm on the space of such perturbations. Then we say that the equilibrium is *structurally stable* under this norm if there is some $\delta > 0$ such that for every $\|\delta T\| < \delta$ the operator $T + \delta T$ is spectrally stable. Otherwise the system is *structurally unstable.*

Because we are dealing with physical systems it makes sense to have physical motivation for the choice of norm on the space of perturbations. In this article we are interested in perturbations of the Vlasov equation through changes in the equilibrium. This choice is motivated by the Hamiltonian structure of the equations and Krein's theorem for finite-dimensional systems. In general the space of possible perturbations is quite large, but perturbations of equilibria give rise to operators in certain Banach spaces and motivate the definition of norm. Even in the case of unbounded perturbations such a norm may exist (see Kato, 1966, for instance).

Consider a stable equilibrium function f_0. We will consider perturbations of the equilibrium function and the resulting perturbation of the time evolution operator. Suppose that the time evolution operator of the perturbed system is $T + \delta T$. In the function space that we will consider, these perturbations are bounded operators and their size can be measured by the norm $\|\delta T\|$. This norm will be proportional to the norm of $\|\delta f_0'\|$, where δf_0 is the perturbation of the equilibrium.

Definition. Consider the formulation of the linearized Vlasov-Poisson equation in the Banach space $W^{1,1}(\mathbb{R})$ with a spectrally stable homogeneous equilibrium function f_0. Let $T_{f_0 + \delta f_0}$ be the time evolution operator corresponding to the linearized dynamics around the distribution function $f_0 + \delta f_0$. If there exists some δ depending only on f_0 such that $T_{f_0 + \delta f_0}$ is spectrally stable whenever $\|\delta T_{\delta f_0}\| = \| T_{f_0} - T_{f_0 + \delta f_0}\| < \delta$, then the equilibrium f_0 is structurally stable under perturbations of f_0.

The aim of this work is to characterize the structural stability of the linearized Vlasov-Poisson equation. We will prove that if the perturbation function is some homogeneous δf_0 and the norm is $W^{1,1}$ (and L^1 as a consequence) every equilibrium distribution function is structurally unstable to an infinitesimal perturbation in this space. This fact will force us to consider more restricted sets of perturbations.

4.1. Winding Number

We need to compute the winding number of Penrose plots and the change in winding number under a perturbation, both in this section and in the rest of the article. We use the fact that one way to compute the winding number is to draw a ray from the origin to infinity and to count the number of intersections with the contour accounting for orientation.

Lemma 4.1 *Consider an equilibrium distribution function f_0'. The winding number of the Penrose ε-plot around the origin is equal to $\sum_u \text{sgn}(f_0''(u))$ for all $u \in \mathbb{R}^-$, satisfying $f_0'(u) = 0$.*

To calculate the winding number of the Penrose ε-plot using this lemma one counts the number of zeros of f_0' on the negative real line and adds them with a positive sign if f_0'' is positive, a Penrose crossing from the upper half plane to the lower half plane; a negative sign if f_0'' is negative, a crossing from the lower half plane to the upper half plane; and zero if u is not a crossing of the x-axis, a tangency. This lemma comes from the following equivalent characterization of the winding number from differential topology (Guillemin and Pollack, 1974).

Definition. If X is a compact, oriented, l-dimensional manifold and $f: X \to \mathbb{R}^{l+1}$ is a smooth map, the winding number of f around any point $z \in \mathbb{R}^{l+1} - f(X)$ is the degree of the direction map $u: X \to S^l$ given by $u(x) = \frac{f(x)-z}{|f(x)-z|}$.

In our case the compact manifold is the real line plus the point at ∞ and $l = 1$. The degree of u is the intersection number of u with any point on the circle taken with a plus sign if the differential preserves orientation and a minus sign if it reverses it. The lemma is just a specialization of this definition to the negative x-direction on the circle. If more than one derivative of f_0 vanishes at a zero of f_0' there is a standard procedure for calculating the winding number by determining if there is a sign change in f_0' at the zero.

4.2. *Structural Instability of General f_0*

In a large class of function spaces it is possible to create infinitesimal perturbations that make any equilibrium distribution function unstable. This can happen in any space where the Hilbert transform is an unbounded operator. In these spaces there will be an infinitesimal δf_0 such that $H[\delta f_0']$ is order one at a zero of f_0'. Such a perturbation can turn any point where $f_0' = 0$ into a point where $H[f_0' + \delta f_0'] > 0$ as well. Because $\delta f_0'$ is small and $H[\delta f_0']$ is not small only within a small region, the only effect on the Penrose plot will be to move the location of the zero. Thus, such a perturbation will increase the winding number and cause instability.

We will explicitly demonstrate this for the Banach space $W^{1,1}(\mathbb{R})$ and, by extension, the Banach space $L^1 \cap C_0$. This will imply that any distribution function is infinitesimally close to instability when the problem is set in one of these spaces, implying the structural instability of every distribution function.

Suppose we perturb f_0 by a function δf_0. The resulting perturbation to the operator T is the operator mapping f to $\delta f_0' \int dv f$. In the space $W^{1,1}$ this is a bounded operator and thus we take the norm of the perturbing operator to be $\|\delta f_0'\|_{1,1}$. Now we introduce a class of perturbations that can be made infinitesimal, but have Hilbert transform of order unity.
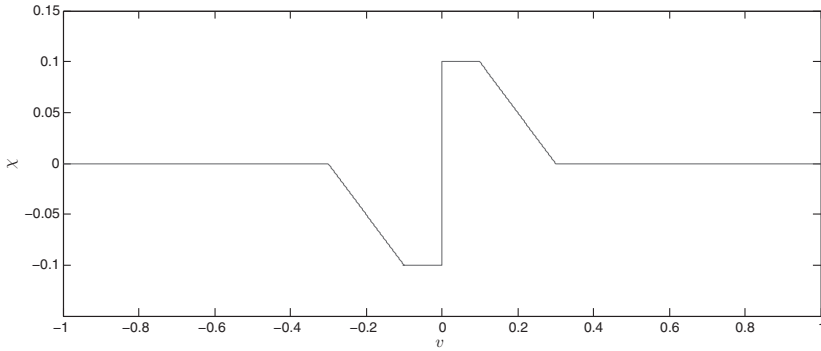
FIGURE 5 The perturbation χ for $\epsilon = e^{-10}$, $h = d = .1$.

Consider the function $\chi(v, h, d, \epsilon)$ defined by

$$
\chi = \begin{cases}
hv/\epsilon & |v| < \epsilon \\
h \operatorname{sgn}(v) & \epsilon < |v| < d + \epsilon \\
h + d/2 + \epsilon/2 - v/2 & 2h + d + \epsilon > v > d + \epsilon \\
-h - d/2 - \epsilon/2 - v/2 & 2h + d + \epsilon > -v > d + \epsilon \\
0 & |v| > 2h + d + \epsilon
\end{cases}.
$$

Figures 5 and 6 show the graph of χ and its Hilbert transform, $H[\chi]$, respectively.

Lemma 4.2 *If we choose $d = h$ and $\epsilon = e^{-(1/h)}$, then for any $\delta, \gamma > 0$ we can choose an h such that $\|\chi\|_{1,1} < \delta$ and $\fint dv\, \chi/v > 1 - O(h)$, and $|\fint dv\, \chi/(u - v)| < |\gamma/u|$ for $|u| > |2h + d + \epsilon|$.*
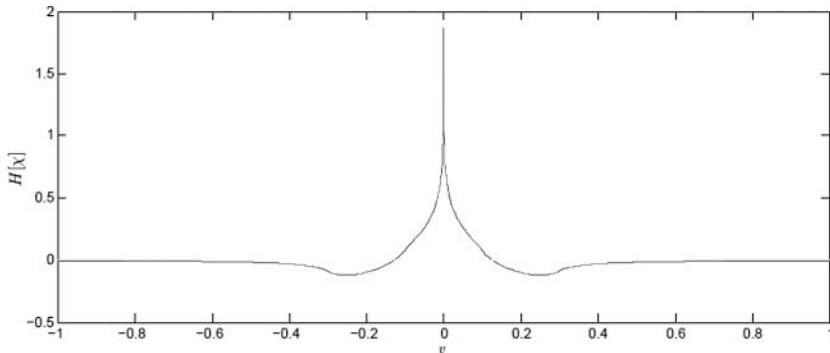

FIGURE 6 The Hilbert transform of χ.

Proof. In the space $W^{1,1}$ the function χ has norm $2h^2 + 2hd + h\epsilon + 4h$, which is less than any δ for small enough h. We can compute the value of the Hilbert transform of this function at a given point u by calculating the principal values:

$$\fint dv \, \frac{\chi}{v-u} = \frac{hu}{\epsilon} \log\left(\frac{|u-\epsilon|}{|u+\epsilon|}\right) + h \log\left(\frac{|d+\epsilon-u||u+d+\epsilon|}{|\epsilon+u||\epsilon-u|}\right)$$
$$+ \frac{1}{2}(d+\epsilon+2h-u)\log\left(\frac{|d+\epsilon+2h-u|}{|d+\epsilon-u|}\right)$$
$$+ \frac{1}{2}(d+\epsilon+2h+u)\log\left(\frac{|d+\epsilon+2h+u|}{|d+\epsilon+u|}\right). \quad (16)$$

We analyze the asymptotics of this function as h, d, and ϵ go to zero, with the desiderata that (i) the norm of χ goes to zero, (ii) the maximum of the Hilbert transform of χ is $O(1)$, and (iii) there is a band of vanishing width around the origin outside of which the Hilbert transform can be made arbitrarily close to zero.

Note that (16) can be written as a linear combination of translates of the function $x \log x$:

$$\fint dv \, \frac{\chi}{v-u} = \frac{h}{\epsilon}((u-\epsilon)\log(|u-\epsilon|) - (u+\epsilon)\log(|u+\epsilon|))$$
$$- \frac{1}{2}(d+u+\epsilon)\log(|d+u+\epsilon|)$$
$$- \frac{1}{2}(d-u+\epsilon)\log(|d-u+\epsilon|)$$
$$+ \frac{1}{2}(d+u+\epsilon+2h)\log(|d+u+\epsilon+2h|)$$
$$+ \frac{1}{2}(d-u+\epsilon+2h)\log(|d-u+\epsilon+2h|). \quad (17)$$

The function $x \log x$ has a local minimum for positive x at $x = 1/e$. This is the point at which the function is most negative. It has zeros at $x = 0$ and $x = 1$. For values of u, d, ϵ, h close to zero all of the arguments of the log functions are less than $1/e$. Therefore, for $|u| < d + \epsilon + 2h$ the $x \log x$ terms are all monotonically

decreasing functions of the argument x. Of the terms of (17), $\frac{h}{\epsilon}((u-\epsilon)\log(|u-\epsilon|)-(u+\epsilon)\log(|u+\epsilon|))$ has by far the largest coefficient as long as ϵ is much smaller than h. We choose $h = d$ and $\epsilon = 0(e^{-1/h})$. Then the terms that do not involve ϵ are all smaller than $(6h + \epsilon)\log(6h + \epsilon)$. With these choices χ satisfies

$$
\begin{aligned}
\chi(0) &= 2 - (h + e^{-1/h})\log(|h + e^{-1/h}|) \\
&\quad + (3h + e^{-1/h})\log(|3h + e^{-1/h}|) \\
&= 2 + O(h\log h)\,.
\end{aligned}
$$

Consider the pair of functions $-(u+c)\log(|u+c|)+(u-c)\log(|u-c|)$. The derivative with respect to u is $-\log(|u+c|)+\log(|u-c|)$. This is zero for $u = 0$ and for $u > 0$ it is always negative and the pair is always decreasing, and for small values of h the pair is guaranteed to be positive. Suppose that $u > \epsilon$. Then we can bound the term with the h/ϵ coefficient:

$$
\begin{aligned}
&\frac{h}{\epsilon}\left|(u-\epsilon)\log(|u-\epsilon|) - (u+\epsilon)\log(|u+\epsilon|)\right| \\
&= \left|\frac{h}{\epsilon}(u-\epsilon)\log\frac{|u-\epsilon|}{|u+\epsilon|} - 2\epsilon\log(|u+\epsilon|)\right| \\
&= \frac{h}{\epsilon}\left|(u-\epsilon)\log\frac{1-\frac{\epsilon}{u}}{1+\frac{\epsilon}{u}} - 2\epsilon\log(|u+\epsilon|)\right| \\
&< \frac{h}{\epsilon}\left|(u-\epsilon)\log(e^{-\epsilon/u})\right| + 2\left|h\log(|u+\epsilon|)\right| \\
&= \frac{h(u-\epsilon)}{u} + 2\left|h\log(|u+\epsilon|)\right|\,.
\end{aligned}
$$

For $u >> \epsilon$, for example if $u = O(h^2)$, this term is $O(h\log h)$. Therefore, for $|u| > h^2$ we have $\chi = O(h\log h)$, which can be made arbitrarily small. When $|u| > 3h+\epsilon$ the function χ decreases at least as fast as $O(1/u)$. With these choices of h, d, and ϵ, the norm of χ is $O(h)$, which proves the Lemma. $\qquad\square$

Now we state the theorem that any equilibrium is strucutrally unstable in both the spaces $W^{1,1}$ and $L^1 \cap C_0$. In order to prove this theorem we will make use of a result from Morse theory Hirsch

(1976). A Morse function is a function that has no degenerate critical points.

Lemma 4.3 *Let M be a smooth manifold. The set of Morse functions is open and dense in the space $C^r(M, \mathbb{R})$.*

Therefore if f_0 is C^2 there is an infinitesimal perturbation f_1 such that $f_0 + f_1$ is a Morse function. Because the winding number is stable under homotopy there is an f_1 such that all the zeros of $f_0 + f_1$ are non-degenerate and the winding number of the Penrose plot is the same as that of f_0. Therefore we will assume that f_0 is a Morse function. A consequence of this assumption is that all the zeros f_0' are isolated.

Theorem 4.4 *A stable equilibrium distribution $f_0 \in C^2$ is structurally unstable under perturbations of the equilibrium in the Banach spaces $W^{1,1}$ and $L^1 \cap C_0$.*

Proof. If f_0 is stable then the Penrose ϵ-plot of f_0' has a winding number of zero. Because the point at ∞ corresponds to a crossing where f_0' goes from negative to positive there exists a point u_0 with $f_0'(u_0) = 0$ that is an isolated zero, $H[f_0'](u_0) < 0$, and $f_0''(u_0) < 0$. Let $F = \sup |f_0''|$. Choose h to always be smaller than the distance from u_0 to the nearest 0 of f_0'. Then if $\epsilon = O(e^{-1/h})$ and $d = h$ the support of $\chi(u - u_0)$ will contain only one zero of f_0'. For h small enough the slope of χ at u_0 will be greater than F so that the function $f_0' + \chi$ will be positive for u in the set (u_0, u^+) for some u^+ in the support of χ. Similarly $f_0' + \chi$ will be negative for u in the set (u^-, u_0) for some u^- in the support of χ. Because χ has compact support the function $f_0' + \chi$ is positive in a neighborhood outside the support of χ so that the intermediate value theorem guarantees one additional zero of the function $f_0' + \chi$ for $u > u_0$ and also for $u < u_0$. Choose χ so that this Hilbert transform of $f_0' + \chi$ is positive at the point u_0 and h is small enough that it is negative before the next zero of $f_0' + \chi$ on either side of u_0. Then the winding number of $f_0' + \chi$ is positive because an additional positive crossing has been added on the negative real line.

Because the norm of χ is $O(h)$ in both $W^{1,1}$ and L^1 the distribution f_0 is unstable to an arbitrarily small perturbation and is therefore structurally unstable. □
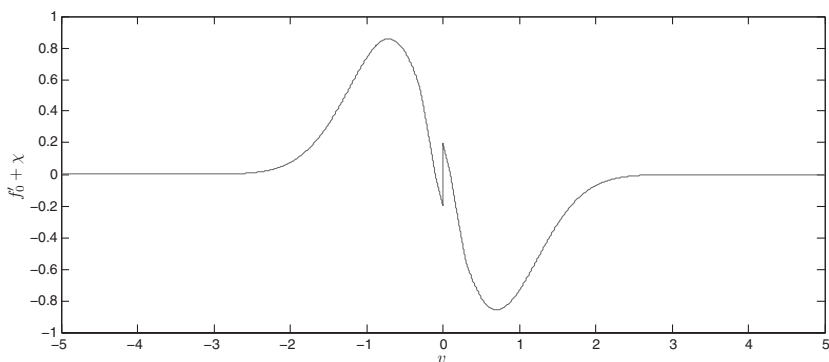
FIGURE 7 $f_0' + \chi$ for a Maxwellian distribution.

Thus we emphasize that we can always construct a perturbation that makes our linearized Vlasov-Poisson system unstable. For the special case of the Maxwellian distribution, Figure 7 shows the perturbed derivative of the distribution function and Figure 8 shows the Penrose plot of the unstable perturbed system. Observe the two crossings created by the perturbation on the positive axis as well as the negative crossing arising from the unboundedness of the perturbation.

In some sense Theorem 4.4 represents a failure of our class of perturbations to produce any interesting structure for the Vlasov equation. Indeed signature appears to play no role in delineating bifurcation to instability. In order to derive a nontrivial result we develop a new theory analogous to the finite-dimensional
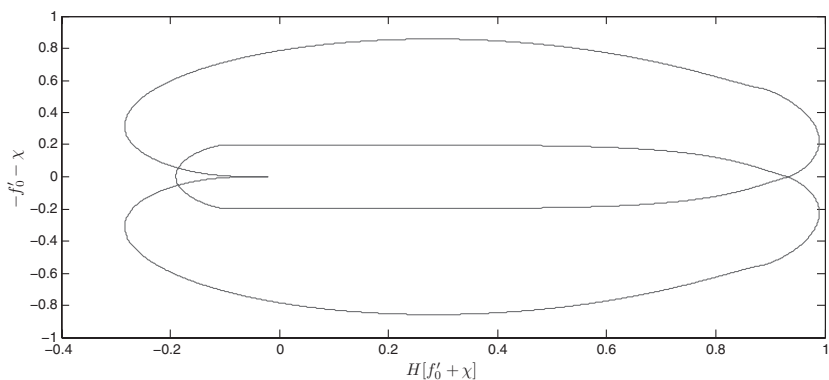

FIGURE 8 Penrose plot for perturbed Maxwellian.

Hamiltonian perturbation theory developed by Krein and Moser. This new theory involves a restriction to dynamically accessible perturbations of the equilibrium state. This is natural since the noncanonical Hamiltonian structure can be viewed as the union of canonical Hamiltonian motions (on symplectic leaves) labelled by the equilibrium state—to compare with traditional finite-dimensional theory requires restriction to the given canonical Hamiltonian motion under consideration.

5. The Kreĭn-Moser Theorem

For linear finite-dimensional Hamiltonian systems, Hamilton's equations are a set of first order linear ordinary differential equations (ODEs). If the Hamiltonian is time-independent, then the behavior of solutions is characterized by the eigenfrequencies. If all the eigenfrequencies are on the real axis and nondegenerate, then the system will be stable. If there are degenerate eigenvalues the system will be stable as long as the time evolution operator does not have any nontrivial Jordan blocks, but there will be secular growth if it does. Any complex eigenfrequencies will lead to instability. The Hamiltonian of a linear finite-dimensional Hamiltonian system is a quadratic form in the canonical variables. If we consider perturbations of the coefficients of the quadratic form it is trivial to define a notion of small perturbations, as the resulting perturbation of the Hamiltonian will be a bounded operator. Krein and Moser independently proved a theorem characterizing the structural stability of these systems in terms of a signature, a quantity that amounts to the sign of the energy evaluated on the eigenvector of a mode. The original theorem described the stability in terms of a quantity called the Kreĭn signature, which is equivalent to the sign of the energy (Sturrock, 1958, 1960; MacKay, 1986; Morrison and Kotschenreuther, 1990). It is of historical interest to note that the fact that bifurcations to instability occur through collisions of modes of opposite sign was observed by Sturrock (1958, 1960) in the plasma physics literature.

Theorem 5.1 (Kreĭn-Moser) *Let H define a stable linear finite-dimensional Hamiltonian system. Then H is structurally stable if all the eigenfrequencies are nondegenerate. If there are any degeneracies, H*

is structurally stable if the asssociated eigenmodes have energy of the same sign. Otherwise H is structurally unstable.

This Kreĭn-Moser theorem gives a clear picture of the behavior of these systems under small perturbations. The eigenfrequencies move around, but remain confined to the real line unless there is a collision between a positive energy and negative energy mode, in which case they may leave the axis. This theorem was first proved by Kreĭn in the early 1950s and was later rediscovered by Moser in the late 1950s. Our goal is to place the perturbation theory of infinite-dimensional Hamiltonian systems in the language of the finite-dimensional theory.

The appropriate definition of signature for the continuous spectrum of the Vlasov-Poisson equation was introduced in Morrison and Pfirsch (1992) and Morrison (2000) (see also Morrison and Shadwick, 2008), where an integral transform was also introduced for constructing a canonical transformation to action-angle variables for the infinite-dimensional system. The transformation is a generalization of the Hilbert transform and it can be used to show that the linearized Vlasov-Poisson equation is equivalent to the system with the following Hamiltonian functional:

$$H_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} du\, \sigma_k(u)\omega_k(u)J_k(u, t) , \qquad (18)$$

where $\omega_k(u) = |ku|$ and $\sigma_k(u) = -\text{sgn}(kuf_0'(u))$ is the analog of the Kreĭn signature corresponding to the mode labeled by $u \in \mathbb{R}$. (Note, the transformation can always be carried out in a frame where $f_0'(0) = 0$. Because the Hamiltonian does not transform as a scalar for frame shifts, which are time-dependent transformations, signature is frame dependent. The Hamiltonian in a shifted frame is obtained by adding a constant times the momentum P_L of (15) to H_L. Later we will see that Hamiltonians that can be made sign definite in some frame are structurally stable in a sense to be defined.)

Definition 1. Suppose $f_0'(0) = 0$. Then the signature of the point $u \in \mathbb{R}$ is $-\text{sgn}(uf_0'(u))$.

Figure 9 illustrates the signature for a bi-Maxwellian distribution function.
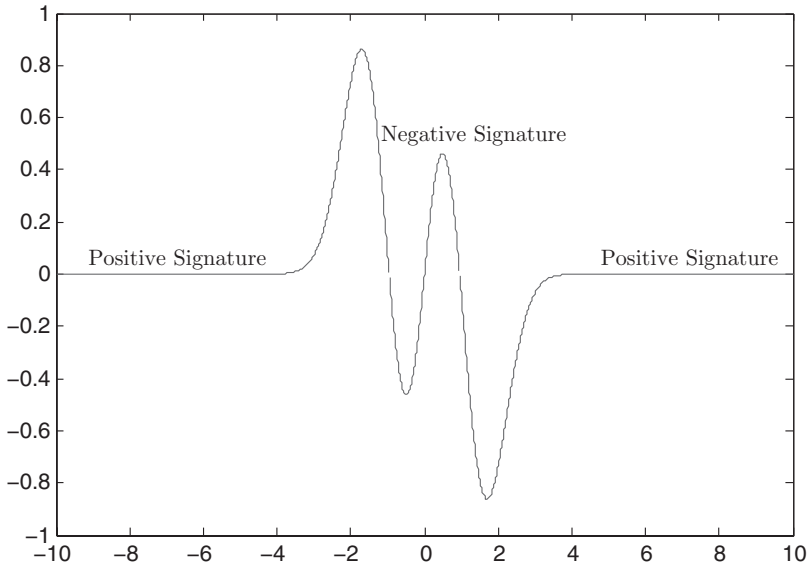
FIGURE 9 Signature for a bi-Maxwellian distribution function.

5.1. *Dynamical Accessibility and Structural Stability*

Now we discuss the effect of restricting to dynamically accessible perturbations on the structural stability of f_0. In this work we only study perturbations of f_0 that preserve homogeneity. Because dynamically accessible perturbations are area-preserving rearrangements of f_0, it is impossible to construct a dynamically accessible perturbation for the Vlasov equation in a finite spatial domain that preserves homogeneity.

To see this we write a rearrangement as $(x, v) \leftrightarrow (X, V)$, where V is a function of v alone. Because $[X, V] = 1$ and $V(v)$ is not a function of x, we have $V' \partial X/\partial x = 1$, or $X = x/V'$. If the spatial domain is finite, this map is not a diffeomorphism unless $V' = 1$. In the infinite spatial domain case, this is not a problem and these rearrangments exist. First we note that a rearrangement cannot change the critical points of f_0.

Lemma 5.2 *Let* (X, V) *be an area preserving diffeomorphism, and let* V *be homogeneous. Then the critical points of* $f_0(V)$ *are the points* $V^{-1}(v_c)$, *where* v_c *is a critical point of* $f_0(v)$.

Proof. By the chain rule $df_0(V(v))/dv = V(v)' f_0'(V(v))$. The function $V' \neq 0$ because (X, V) must be a diffeomorphism.

Therefore the critical points occur when $f_0'(V) = 0$ or at points $v = V^{-1}(v_c)$. $\qquad\Box$

Consider the perturbation χ that was constructed earlier. If v_c is a nondegenerate critical point of f_0 such that $f_0''(v_c) < 0$, then we want to prove that there is a rearrangement V such that $f_0(V) = f_0(v) + \int_{-\infty}^{v} \chi(v' - v_c)\,dv'$ or that $df_0(V)/dv = f_0'(v) + \chi(v - v_c)$. Such a rearrangement can be constructed as long as the parameters defining χ, the numbers h, d, ϵ, are chosen such that $f_0'(v) + \chi(v - v_c)$ has the same critical points as $f_0'(v)$. Using Morse theory it is possible to construct a V so that $f_0(V) = f_0(v) + \int \chi + O((v - v_c)^3)$, where $O((v - v_c)^3)$ has compact support and is smaller than $f_0(v) - f_0''(v_c)(v - v_c)^2/2$.

Theorem 5.3 *Let v_c be a nondegenerate critical point of f_0 with $f_0'(v_c) < 0$. Then there exists a rearrangement V such that $f_0(V) = f_0(v) + \int \chi + O((v - v_c)^3)$, where O is defined as above.*

We omit the proof but it is a simple application of the Morse lemma. In order to apply the Morse lemma f_0 must be C^2. This is not restrictive for practical applications where typically f_0 is smooth. The rearrangement of f_0 can also be made to be smooth if desired.

Using this result we prove a Kreĭn-like theorem for dynamically accessible perturbations in the $W^{1,1}$ norm.

Theorem 5.4 *Let f_0 be a stable equilibrium distribution function for the Vlasov equation on an infinite spatial domain. Then f_0 is structurally stable under dynamically accessible perturbations in $W^{1,1}$, if there is only one solution of $f_0'(v) = 0$. If there are multiple solutions, f_0 is structurally unstable and the unstable modes come from the zeros of f_0' that satisfy $f_0''(v) < 0$.*

Proof. Suppose that f_0' has only one zero on the real line. Because f_0 is an equilibrium this zero will have $f_0'' > 0$. Because a dynamically accessible perturbation can never increase the number of critical points, it will be impossible to change the winding number of the Penrose plot to a positive number. Therefore f_0 is structurally stable.

Suppose that $f_0' = 0$ has more than one solution on the real axis. Using the previous theorem perturb f_0' by $\chi(v - v_c)$ in a neighborhood of a critical point v_c with $f_0''(v_c) < 0$. This will increase the winding number to 1 since it will add a positively

oriented crossing on the negative real axis for the correct choices of h, d, and ϵ in the definition of χ. The norm of χ can be made as small as necessary and therefore f_0 is structurally unstable. Since no other critical points with $f_0'' < 0$ can be created the only critical points that lead to instabilities are the ones that already exist having $f_0'' < 0$.   □

The implication of this result is that in a Banach space where the Hilbert transform is an unbounded operator the dynamical accessibility condition makes it so that a change in the Kreĭn signature of the continuous spectrum is a necessary and sufficient condition for structural instability. The bifurcations do not occur at all points where the signature changes, however. Only those that represent valleys of the distribution can give birth to unstable modes.

6. Kreĭn Bifurcations in the Vlasov Equation

We identify two critical states for the Penrose plots that correspond to the transition to instability. In these states the system may be structurally unstable under infinitesimal perturbations of f_0' in the C^n norm for all n. The first critical state corresponds to the existence of an embedded mode in the continuous spectrum. If the equilibrium is stable, then such an embedded mode corresponds to a tangency of the Penrose plot to the real axis at the origin. If the system is perturbed so that the tangency becomes a pair of transverse intersections, then the winding number of the Penrose plot would jump to 1 and the system would be unstable. Considering a parametrized small perturbation, we see that the value of k for the unstable mode will correspond to some value of $k \neq 0$ for which the embedded mode exists. Figures 10 and 11 illustrate a critical Penrose plot for a bifurcation at $k \neq 0$. We explore this bifurcation in Section 6.1.

Another critical state occurs when $H[f_0'] = 0$ at a point where f_0' transversely intersects the real axis. If the Hilbert transform of f_0' is perturbed, there will be a crossing with a negative $H[f_0']$, and the winding number will be positive for some k. This mode enters through $k = 0$ because the smaller the perturbation of $H[f_0']$ the smaller k must be for T_k to be unstable. Figure 12 is a critical Penrose plot corresponding to the bi-Maxwellian
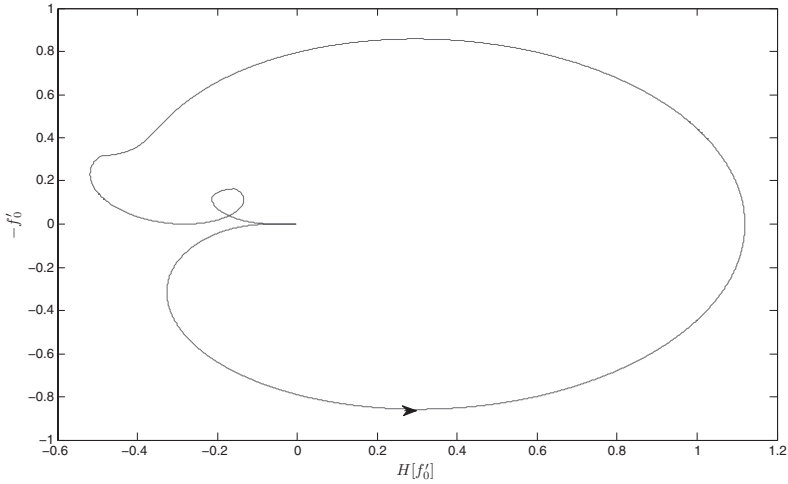
FIGURE 10 Critical Penrose plot for a $k \neq 0$ bifurcation.

distribution with the maximum stable separation. We explore this kind of bifurcation in Section 6.3.

6.1. Bifurcation at $k \neq 0$

The linearized Vlasov equation can support neutral plasma modes embedded within the continuous spectrum. The condition for
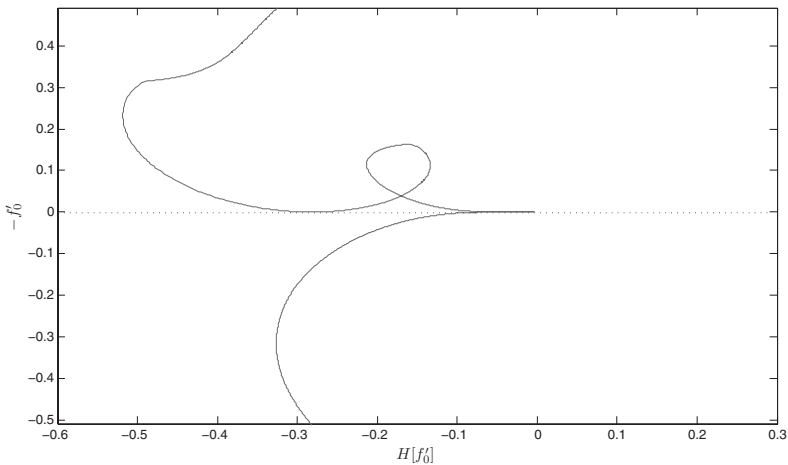

FIGURE 11 Close up of a critical Penrose plot for a $k \neq 0$ bifurcation.
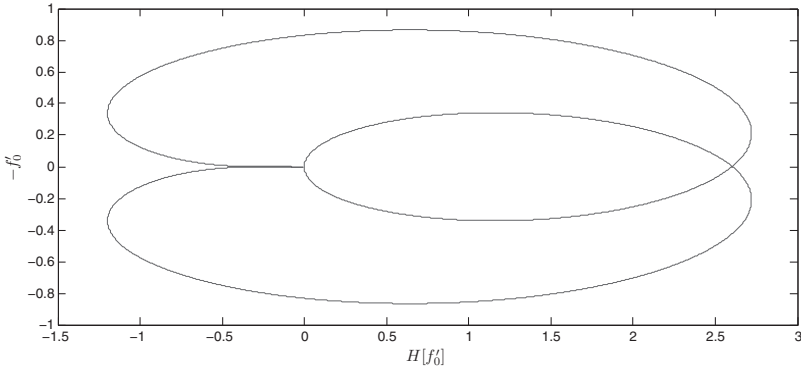
FIGURE 12 Critical Penrose plot for a bi-Maxwellian distribution function.

existence of a point mode is the vanishing of the plasma disper-
sion relation on the real axis,

$$\varepsilon(u) = 1 - H[f_0'] + if_0' = 0. \tag{19}$$

If the spatial domain is unbounded the point modes will be ana-
logues of the momentum eigenstate solutions of the Schrodinger
equation and have infinite energy. Any violation of the Penrose
criterion will guarantee the existence of zeros of the plasma dis-
persion function on the real axis because k can take any value in
this case.

   If the plasma dispersion relation vanishes at some u and
$f_0''(u) = 0$, there is an embedded mode in the continuous spec-
trum. The signature of the continuous spectrum will not change
signs at the frequency of the mode, and we will extend the def-
inition of signature to the point u even though $f_0'(u) = 0$. The
signature of an embedded mode is given by sgn$(u\partial\epsilon_R/\partial u)$ (see
Morrison and Pfirsch, 1992; Shadwick and Morrison, 1994). The
signature of the continuous spectrum is $-$sgn(uf_0'). These signa-
tures are the same if the value of f_0' in a neighborhood of its zero
is the same sign as $H[f_0'']$.

   We will prove that if f_0 is stable and mildly regular, it is impos-
sible for there to be a discrete mode embedded in the continuous
spectrum with signature that is the same as the signature of the
continuous spectrum surrounding it. The proof has a simple con-
ceptual outline. Suppose that there exists a discrete mode with the
same signature as the continuum. Then there exists some point u

satisfying $f_0'(u) = 0$, $-\mathrm{sgn}(f_0') = \mathrm{sgn}(\partial \epsilon_R / \partial u)$ in a neighborhood of u, and $H[f_0'](u) = 0$. Perturbations of f_0' centered around this point will give the Penrose plot a negative winding number, contradicting the analyticity of the plasma dispersion function in the upper half plane. We need f_0' to be Hölder continuous so that the Penrose plot is continuous and for the plasma dispersion function to converge uniformly to its values on the real line.

Lemma 6.1 *Let g be a function defined on the real line such that g is Hölder and let $h = H[f]$. Then the functions g_z, h_z that are the solutions of the Laplace equation in the upper half plane satisfying $f_z = f$ and $g_z = g$ on the real line converge uniformly to f and g.*

Proof. Because g can be defined as a bounded and continuous function on $\mathbb{R} \cup \{\infty\}$ and the g_z are analytic, the g_z must converge uniformly to g. The same properties hold for h and h_z must converge to $h = H[g]$. □

Lemma 6.2 *Let f_0' be the derivative of an equilibrium distribution function and let f_0' be sufficiently regular such that the assumptions of the previous lemma are true. Then the Penrose plot that is associated with f_0' cannot have a negative winding number.*

Proof. The Penrose plot associated with f_0' is the image of the real line under the map $\epsilon(u) = 1 - H[f_0'] + if_0'$. This is naturally defined as an analytic function if u is in the upper half plane. By the argument principle the image of $\mathbb{R} + it$ under this map has a non-negative winding number. Both the real and imaginary parts of this map converge uniformly to their values on the real line. Therefore the Penrose plot is a homotopy of these contours, making it possible to parametrize the contours by some t such that the distance from the Penrose plot to the contour produced by the image of $\mathbb{R} + i\delta$ is always less than some $\eta(t)$ that goes to 0. If the winding number of the Penrose plot were negative, there would be some t for which the winding number was negative because the winding number is a stable property under homotopy, contradicting the analyticity of the map. □

Theorem 6.3 *Let f_0' and f_0'' be Hölder continuous. If f_0 is stable there are no discrete modes with signature the same as the signature of the continuum.*

Proof. Because f_0 is stable the winding number of the Penrose plot is equal to 0. Assume that there is a discrete mode with the same signature as the contiuum surrounding it. Then there exists a point u with $f_0'(u) = 0$, $f_0''(u) = 0$, and $\mathrm{sgn}(f_0'(u + \delta)\, dH[f_0'](u)/du) = 1$. Then we search for a function g such that the Penrose plot of $f_0' + g$ has a negative winding number. If such a function exists it will contradict Lemma 6.1. Because f_0'' is Hölder $\partial \epsilon_R / \partial u$ is bounded away from zero in a neighborhood of the point. Suppose that in this neighborhood there is only one zero of f_0'. Then define g such that g has one sign, is smooth and has compact support, and such that the $|\partial H[g]/\partial u| < |f_0''|$ in this neighborhood. Then for small enough g the function $f_0' + g$ will have two zeros in a neighborhood of the point. Then both of the crossings will correspond to crossings of negative orientation and the resulting winding number will be -1, a contradiction. $\square$

Corollary 6.4 *If f_0 is stable it is impossible for there to be a point where $f_0' = 0$, $f_0'' < 0$, and $H[f_0'] > 0$.*

If f_0 is unstable the winding number is positive. In this case it may be possible for modes with the same signature as the continuum to exist. It is possible for a positive energy mode to be embedded in a section of negative signature and a negative energy mode to be embedded in a section of positive signature. This situation is structurally unstable under perturbations that are bounded by the C^n norm and remains so even when a linear dynamical accessibility constraint is enforced.

Theorem 6.5 *Let f_0' be the derivative of an equilibrium distribution function with a discrete mode embedded in the continuous spectrum. Then there exists an infinitesimal function with compact support in the C^n norm for each n such that $f_0' + \delta f'$ is unstable.*

Proof. Suppose that $H[f_0'']$ is nonzero in a neighborhood of the embedded mode. Define a dynamically accessible perturbation $\delta f = h f_0'$. Then assume that $f_0''' \neq 0$ at the mode. If we define h such that it does not vanish at the mode we find that $\delta f'' = h'' f_0' + h' f_0'' + h f_0'''$ and therefore we can choose h such that the discrete mode becomes a crossing. This can be done with h infinitesimal and smooth. The resulting perturbation will have an infinitesimal effect on f_0'. The new crossings will cause a violation

of the Penrose criterion, and therefore the system with the embedded mode is structurally unstable. □

This is an analog of Kreĭn's theorem for the Vlasov equation for the case where there is a discrete mode. As a result of this we see that all discrete modes are either unstable or structurally unstable.

6.2.   Little-Big Man Theorem

Consider a linearized equilibrium that supports three discrete modes. The signature of each mode depends on the reference frame. There is a result that applies to a number of Hamiltonian systems, the three-wave problem in particular (Coppi et al., 1969; Kueny and Morrison, 1995), that gives a condition on the signature of the modes and their frequencies in some reference frame such that no frame shift can cause all the modes to have the same signature. In a shifted frame, the Hamiltonian changes so the frequencies in the action-angle form are doppler shifted. Sometimes such shifts can render the Hamiltonian sign definite. A result for finite systems, which we call the little-big man theorem, indicates that this cannot happen when the mode of different signature has frequency with largest absolute value. A related result exists for the point spectrum of the Vlasov equation.

Theorem 6.6 *Let f_0' be the derivative of an equilibrium distribution function that has three discrete modes (elements of the point spectrum) with real frequencies. Consider a reference frame where all the modes have positive frequency. Then represent the energies of the three modes as a triplet $(\pm \pm \pm)$ where the plus and minus signs correspond to the signature of each mode, with the first mode being the one with the lowest phase velocity (ω / k) and the last one with the highest phase velocity. Then, if the triplet is of the form $(+ - +)$ or $(- + -)$ there is no reference frame in which all the modes have the same signature. If the triplet has any other form, then there is a reference frame in which all the modes have the same signature.*

Proof. The formula for the energy of an embedded mode is $\mathrm{sgn}(\omega \, \partial \epsilon_R / \partial \omega)$ (Shadwick and Morrison, 1994). If there are three embedded modes in a frame where the frequencies are all

positive, the triplet is

$$\left(\operatorname{sgn}\frac{\partial \epsilon_R}{\partial \omega}\bigg|_{\omega_1}, \operatorname{sgn}\frac{\partial \epsilon_R}{\partial \omega}\bigg|_{\omega_2}, \operatorname{sgn}\frac{\partial \epsilon_R}{\partial \omega}\bigg|_{\omega_3}\right).$$

If this is $(+-+)$ then as we shift frames the possible triplets are $(0-+), (--+), (-0+), (-++), (-+0), (-+-)$. All of these are indefinite. The other possibile initially indefinite triplet is $(--+)$. However if we shift the two $-$ modes to negative frequency the triplet becomes $(+++)$. All other examples are either definite or reduce to one of these two. □

A few observations are in order. First, frame shifts do not change the structure of Penrose plots, but only induce reparameterizations. Next, Theorem 6.6 differs from its finite-dimensional counterpart in that no restriction on the wave numbers is involved, a necessary part of the three-wave problem. Lastly, we are not addressing nonlinear stability here, as in the finite dimensional case, but should a frame exist in which the energy is definite, this is an important first step in a rigorous proof of nonlinear stability.

6.3.   Bifurcation at $k = 0$

Assume that there are no embedded modes and that f_0 is stable, but that there is a point that has $f_0' = 0$ and $H[f_0'] = 0$. This is the critical state for a bifurcation at $k = 0$. This can be destabilized in the same way as the critical state for $k \neq 0$. There will be a perturbation that makes $H[f_0'] < 0$ without changing f_0' at that point. Therefore the Penrose plot becomes unstable and the equilibrium is structurally unstable.

Theorem 6.7 *Suppose that f_0' is a stable equilibrium distribution function that has a zero at u of both f_0' and $H[f_0']$. Then f_0' is structurally unstable under perturbations bounded by the C^n norm for all n.*

Proof. Let δh be symmetric about the point u, be smooth with compact support and have its first n derivatives less than some ϵ. Then let $\delta f_0' = -H[\delta h]$. The resulting perturbation to $H[f_0']$ is h. If h is positive at u, then by the symmetry of h $f_0' + \delta f_0'$ has a zero at u and $H[f_0'] + h$ is positive there. Thus the Penrose plot has a

positive winding number and is unstable. Therefore f_0' is structurally unstable. □

The previous two sections demonstrated that when the Penrose plot is critical, no amount of regularity is sufficient to prevent f_0 from being structurally unstable. However, when the Penrose plot is not critical all that is required is that a small perturbation only change the Penrose plot by a small amount in addition to a condition to prevent perturbations near $v = \infty$. Suppose we arbitrarily restrict the support of the perturbations so that $|v| < v_{max}$. Then if we increase the required regularity such that if $\sup(H[\delta f_0'])$ is bounded there will be some δ such that for all $\delta f_0'$ with $\|\delta f_0'\| < \delta$ the distribution $f_0 + \delta f_0$ is structurally stable. This restriction can be motivated physically by restricting the particles in the distribution function to be travelling slower than the speed of light.

7. Conclusion

We have considered perturbations of the linearized Vlasov-Poisson equation through changes in the equilibrium function. The effect of these perturbations on the spectral stability of the equations is determined by the class of allowable perturbations and the signature of the contiuous and point spectra. Every equilibrium can be made unstable by adding an arbitrarily small function from the space $W^{1,1}$. If we rearrange f_0 then only when the signature of f_0 changes sign can an arbitrarily small perturbation destabilize it. When f_0 is stable discrete modes always have the opposite signature of the spectrum surrounding them. This is the result of Theorem 6.3. The equilibria are structurally unstable under C^n small perturbations for all n. The signature of the spectrum and the signature of the discrete mode can never be the same.

This generalization of Krein's theorem is more complicated than the finite-dimensional original. However the basic ideas of Krein's theorem are still important in the infinite-dimensional case. When the perturbations are more restricted than just belonging to $W^{1,1}$ the structural stability is determined by the signature of the spectrum. Just as in Krein's theorem there must be a positive signature interacting with a negative signature to produce structural instability.

This article was devoted primarily to the Vlasov equation, but other noncanonical Hamiltonian systems admit to a similar treatment, e.g., the 2D Euler equation with shear flow equilibria, and we hope to chronicle such cases in future publications.

References

Adams, R. A. (2003). *Sobolev Spaces.* Oxford, UK: Elsevier Science.

Balmforth, N. J., Morrison, P. J. (1998). A necessary and sufficient instability condition for inviscid shear flow. *Studies in Appl. Math.* 102:309–344.

Balmforth, N. J., Morrison, P. J. (2002). Hamiltonian description of shear flow. In J. Norbury and I. Roulstone, eds., *Large-Scale Atmosphere-Ocean Dynamics II*, (pp. 117–142). Cambridge, UK: Cambridge University Press.

Coppi, B., Rosenbluth, M. N., and Sudan, R. N. (1969). Nonlinear interactions of positive and negative energy modes in rarefied plasmas (I). *Ann. Phys.* 55:207–247.

Degond, P. (1986). Spectral theory of the linearized Vlasov-Poisson equation. *Trans. Am. Math. Soc.* 294:435–453.

Friedrichs, K. O. (1965). *Perturbation of Spectra in Hilbert Space.* Providence, RI: American Mathematical Society.

Grillakis, M. (1990). Analysis of the linearization around a critical point of an infinite dimensional Hamiltonian system. *Comm. Pure. Appl. Math.* 43:299–333.

Guillemin, V., Pollack, A. (1974). *Differential Topology.* Englewood Cliffs, New Jersey, USA: Prentice Hall.

Hirota, M., Fukumoto, Y. (2008). Energy of hydrodynamic and magnetohydrodynamic waves with point and continuous spectra. *J. Math. Phys* 49:083101-1–27.

Hirsch, M. (1976). *Differential Topology.* New York: Springer-Verlag.

Holm, D. D., Marsden, J. E., Ratiu, R., Weinstein, A. (1985). Nonlinear stability of fluid and plasma equilibria. *Phys. Rep.* 123:1–116.

Kato, T. (1966). *Perturbation Theory for Linear Operators.* Berlin: Springer-Verlag.

King, F. W. (2009). *Hilbert Transforms.* Cambridge, UK: Cambridge University Press.

Kreĭn, M. G. (1950). A generalization of some investigations on linear differential equations with periodic coefficients. *Dokl. Akad. Nauk SSSR*, 73A:445–448.

Kreĭn, M. G., Jakubovič, V. A. (1980). *Four Papers on Ordinary Differential Equations.* Providence, RI: American Mathematical Society.

Kruskal, M. D., Oberman, C. (1958). On the stability of plasma in static equilibrium. *Phys. Fluids* 1:275–280.

Kueny, C. S., Morrison, P. J. (1995). Nonlinear instability and chaos in plasma wave-wave interactions. I. Introduction. *Phys. Plasmas* 2:1926–1940.

MacKay, R. S. (1986). Stability of equilibria of hamiltonian systems. In *Nonlinear Phenomena and Chaos* (pp. 254–270).

MacKay, R. S., Saffman, P. G. (1986). Stability of water waves. *Proc. R. Soc. Lond. A* 406:115–125.

Morrison, P. J. (1980). The Maxwell-Vlasov equations as a continuous Hamiltonian system. *Phys. Lett.* 80:383–386.

Morrison, P. J. (1982). Poisson brackets for fluids and plasmas. In M. Tabor and Y. Treve, eds., *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems* (vol. 88, pp. 13–46). New York: Am. Inst. Phys.

Morrison, P. J. (1987). Variational principle and stability of nonmonotonic Vlasov-Poisson equilibria. *Zeitschrift f. Naturforschung* 42a:115–123.

Morrison, P. J. (1994). The energy of perturbations of Vlasov plasmas. *Phys. Plasmas* 1:1447–1451.

Morrison, P. J. (1998). Hamiltonian description of the ideal fluid. *Rev. Mod. Phys.* 70(2):467–521.

Morrison, P. J. (2000). Hamiltonian description of Vlasov dynamics: Action-angle variables for the continuous spectrum. *Trans. Theory Stat. Phys.* 29:397–414.

Morrison, P. J. (2003). Hamiltonian description of fluid and plasma systems with continuous spectra. In O. U. Velasco Fuentes, J. Sheinbaum, and J. Ochoa, eds., *Nonlinear Processes in Geophysical Fluid Dynamics* (pp. 53–69). Dordrecht: Kluwer.

Morrison, P. J. (2005). Hamiltonian and action principle formulations of plasma physics. *Phys. Plasmas* 12:058102-1–058102-13.

Morrison, P. J., Kotschenreuther, M. (1990). The free energy principle, negative energy modes, and stability. In V. G. Baryakhtar, V. M. Chernousenko, N. S. Erokhin, A. B. Sitenko, V. E. Zakharov, eds., *Nonlinear World: IV International Workshop on Nonlinear and Turbulent Processes in Physics* (vol. 2, pp. 910–932). Singapore: World Scientific.

Morrison, P. J., Pfirsch, D. (1989). Free energy expressions for Vlasov-Maxwell equilibria. *Phys. Rev.* 40A:3898–3910.

Morrison, P. J., Pfirsch, D. (1990). The free energy of Maxwell-Vlasov equilibria. *Phys. Fluids* 2B:1105–1113.

Morrison, P. J., Pfirsch, D. (1992) Dielectric energy versus plasma energy, and action-angle variables for the Vlasov equation. *Phys. Fluids* 4B:3038–3057.

Morrison, P. J., Shadwick, B. (1994). Canonization and diagonalization of an infinite dimensional noncanonical Hamiltonian system: Linear Vlasov theory. *Acta Phys. Pol.* 85:759–769.

Morrison, P. J., Shadwick, B. (2008). On the fluctuation spectrum of plasma. *Comm. Nonlinear Sci. Num. Simulations* 13:130–140.

Moser, J. (1958). New aspects in the theory of stability of Hamiltonian systems. *Comm. Pure Appl. Math.* 11:81–114.

Penrose, O. (1960). Electrostatic instabilities of a uniform non-maxwellian plasma. *Phys. Fluids* 3:258–265.

Rayleigh, J. W. S. (1896). *The Theory of Sound.* London: Macmillan.

Rellich, F. (1969). *Perturbation Theory of Eigenvalue Problems.* New York: Gordon and Breach Scientific Publishers.

Shadwick, B., Morrison, P. J. (1994). On neutral plasma oscillations. *Phys. Lett.* 184A:277–282.

Sturrock, J. A. (1958). Kinematics of growing waves. *Phys. Rev.* 112(5):1488–1503.

Sturrock, P. A. (1960). Action-transfer and frequency-shift relations in the nonlinear theory of waves and oscillations. *Ann. Phys.* 9:422–434.