

Variational coordinate transformation in plasma physics

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It is well-known from scaling arguments that action-based field theories do not possess localized solutions in spaces of more than one dimension. The same scaling argument, modified to account for external forces, is applied to magnetic plasma confinement in an axisymmetric torus. It yields an integral solvability condition of some interest. © 2009 American Institute of Physics.

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I. INTRODUCTION

In field theory the word “soliton” is sometimes used to denote any nonlinear solution to the field equations that is spatially localized, in the sense of vanishing in the limit of infinite distance from some origin. In 1964 Derrick¹ pointed out that a scalar field whose field equations possessed an action principle does not possess localized solutions, in a space of more than one dimension; such fields therefore cannot support solitons. Here we apply modified versions of Derrick’s argument to three variational equations of magnetized plasma physics: the Sinh–Poisson (SP) equation, the Spitzer problem for parallel transport, and the Grad–Shafranov (GS) equation. Our results uncover significant properties of the solutions to these equations. In one case the revealed feature is already well known, although the analytical method seems interesting even in this case. For the other two equations, we obtain families of integral relations that all solutions must satisfy. At least some of the integral relations obtained appear to be new.

Derrick’s method can be summarized as follows. Beginning with a variational principle, one uses a trial function that differs from the exact solution only by a transformation of the independent variables. The transformation is characterized by a parameter, λ , such that $\lambda=1$ corresponds to the identity transformation. By requiring the variational quantity to be extremal at this value, one is able to extract useful information about exact solution, although it remains unknown.

Derrick’s transformation was a simple scale transformation, $x \rightarrow \lambda x$. In this work we find it necessary to consider more general transformations. Furthermore the variational principle in Derrick’s case lacked an intrinsic scale length; in one of the problems we consider, a characteristic scale becomes important. Thus our results depend upon generalizing Derrick’s technique.

The bearing of Derrick’s analysis on plasma confinement has been noted previously by Faddeev *et al.*²

II. PARALLEL TRANSPORT THEORY

A. Variational principle

The analysis of plasma transport parallel to the magnetic field requires solution of the equation³

$$C^\ell(\hat{f}) = v_\parallel f_M \mathcal{A}(v/v_t), \quad (1)$$

where C^ℓ denotes a linearized collision operator,

$$\hat{f}(\mathbf{v}) \equiv \frac{f_1}{f_M}$$

is a normalized expression of the first order perturbation f_1 to the distribution function, f_M is a Maxwellian distribution with density n and thermal speed $v_t = \sqrt{2T/m}$ corresponding to temperature T , $v_\parallel = \mathbf{b} \cdot \mathbf{v}$ is the velocity component in the direction of the magnetic field $\mathbf{B} = b\mathbf{B}$, and $\mathcal{A}(v/v_t)$ is a function of the normalized velocity specifying the thermodynamic forces. The standard form is

$$\mathcal{A} = \nabla_\parallel \log n - \frac{e}{T} \nabla_\parallel \phi + \left(\frac{v^2}{v_t^2} - \frac{5}{2} \right) \nabla_\parallel \log T.$$

A characteristic feature of Eq. (1)—often called the “Spitzer problem”—is that the unknown function \hat{f} appears inside the collision operator. This is in general a complicated integro-differential operator, not easily inverted. Therefore the Spitzer problem is usually solved by means of a variational principle,⁴ the simplest version of which is expressed by

$$\delta S = 0, \quad S[\hat{f}, \hat{f}] = 2\mathcal{P}[\hat{f}] - \Phi[\hat{f}, \hat{f}]. \quad (2)$$

Here we have introduced the linear form

$$\mathcal{P}[\hat{f}] = \int d^3v \hat{f} v_\parallel f_M \mathcal{A}(v/v_t)$$

and the bilinear form

$$\Phi[\hat{f}, \hat{f}] = \int d^3v \hat{f} C^\ell(\hat{f}).$$

Note that the latter is symmetric in the sense that, for arbitrary perturbations g and h ,

$$\Phi[\hat{g}, \hat{h}] = \Phi[\hat{h}, \hat{g}]. \quad (3)$$

This symmetry reflects the self-adjointness of the collision operator. It can be assumed to hold for a large family of operators—essentially any operator that displays both Onsager symmetry and Boltzmann entropy increase. Note

also that we reserve the symbol \hat{f} for the exact solution to the Spitzer problem.

It is clear from the definitions that

$$\mathcal{P}[\hat{f}] = \Phi[\hat{f}, \hat{f}]. \quad (4)$$

Both of these forms measure the rate of entropy production due to parallel transport. The variational principle (2) follows easily from Eq. (3).

B. Scaled trial function

Here we follow Derrick, introducing the trial function

$$\hat{g}(\mathbf{v}; \lambda) = \hat{f}(\lambda \mathbf{v}).$$

What distinguishes the present problem is the existence of a natural velocity scale, provided by the thermal speed v_t . It is clear that varying the variational parameter λ will have properties in common with variation of the temperature. To express this relation we make the temperature and λ dependence of both forms explicit, using the notations

$$\mathcal{P}[\hat{g}] \rightarrow \mathcal{P}[\hat{g}; \lambda, T]$$

and

$$\Phi[\hat{g}, \hat{g}] \rightarrow \Phi[\hat{g}; \lambda, T].$$

Then, in particular, Eq. (4) takes the form

$$\mathcal{P}[\hat{f}; 1, T] = \Phi[\hat{f}; 1, T], \quad (5)$$

which must hold uniformly in T .

Next, beginning with the definition

$$\mathcal{P}[\hat{g}; \lambda, T] = \int d^3v \hat{f}(\lambda \mathbf{v}) v_{\parallel} f_M \mathcal{A}(v/v_t),$$

we change the integration variable

$$\mathbf{v} \rightarrow \bar{\mathbf{v}} = \lambda \mathbf{v}$$

to find that

$$\mathcal{P}[\hat{g}; \lambda, T] = \lambda^{-1} \mathcal{P}[\hat{f}; 1, \lambda^2 T] \quad (6)$$

since

$$\lambda v_t \propto \sqrt{\lambda^2 T}.$$

In other words scaling the velocity variable is equivalent, aside from the overall factor of λ^{-1} , to a scaling of the temperature.

This equivalence characterizes the existence of a natural velocity scale. We therefore expect a similar property to describe the bilinear form Φ , and write

$$\Phi[\hat{g}; \lambda, T] = \lambda^k \Phi[\hat{f}; 1, \lambda^2 T] \quad (7)$$

for some exponent k . We will show presently that in the case of Coulomb collisions, $k=3$. The present argument is more general, assuming only the existence (and uniqueness) of the velocity scale v_t , the self-adjointness property (3), and sufficient homogeneity of the collision operator to allow a single scaling law.

Note that the temperature dependence appearing in Eqs. (6) and (7) is that entering through the velocity scale v_t . Thus for example the gradient scale-length $\nabla_{\parallel} \log T$ is not included. Notice also that it makes no sense to equate the right-hand sides of Eqs. (6) and (7); these quantities are equal only at $\lambda=1$. To derive a nontrivial result we must turn to the variational property.

C. Variational result

When the functional S is evaluated on \hat{g} it becomes

$$S_{\lambda} \equiv 2\mathcal{P}[\hat{g}; \lambda, T] - \Phi[\hat{g}; \lambda, T].$$

Since

$$\hat{g}|_{\lambda=1} = \hat{f}$$

the variational principle (2) requires

$$\left. \frac{\partial S_{\lambda}}{\partial \lambda} \right|_{\lambda=1} = 0. \quad (8)$$

Now, according to Eq. (6),

$$\frac{\mathcal{P}}{\lambda} = 2T \frac{\mathcal{P}}{T} - \frac{\mathcal{P}}{\lambda^2}$$

while Eq. (7) provides

$$\frac{\Phi}{\lambda} = 2\lambda^{k+1} T \frac{\Phi}{T} + k\lambda^{k-1} \Phi.$$

After setting $\lambda=1$, as prescribed by Eq. (8), we can use Eq. (5) to obtain the differential equation

$$2T \frac{\Phi}{T} = (2+k)\Phi, \quad (9)$$

in which Φ is the exact entropy production. Thus we have shown that any collision operator that is self-adjoint, and that satisfies a scaling law of the form (7), must yield an entropy production that depends on temperature according to

$$\Phi(T) = T^{1+k/2} \Phi_0, \quad (10)$$

where Φ_0 is independent of T .

It should be emphasized that the result (10) is not a simple consequence of the assumed scaling (7), or of the temperature scaling of Eq. (1). In fact Eq. (10) cannot be derived without the variational principle; it depends implicitly on the (not otherwise specified) form of the exact solution \hat{f} .

D. Coulomb collisions

The form of the entropy production for Coulomb scattering is well known. We consider the simplest case of like-particle collisions, appropriate for ion collisions, but suppress species subscripts. Then we have

$$\Phi = \frac{\gamma}{4} \int d^3v d^3v' f_M f'_M \left(\frac{\hat{f}}{v_\alpha} - \frac{\hat{f}'}{v'_\alpha} \right) U_{\alpha\beta} \left(\frac{\hat{f}}{v_\beta} - \frac{\hat{f}'}{v'_\beta} \right).$$

Here γ is a constant depending on charge and mass,

$$U_\alpha = \frac{u^2 \delta_\alpha - u_\alpha u}{u^3}, \quad u_\alpha \equiv v_\alpha - v', \quad (11)$$

and we use primes to denote evaluation on v' ; for example,

$$f'_M \equiv \frac{n}{\pi^{3/2} v'^3} \exp(-v'^2/v_t^2).$$

To verify Eq. (7) for this case, we replace $\hat{f}(v)$ by $\hat{g} = \hat{f}(\lambda v)$, and then make the usual change in integration variables,

$$v \rightarrow \bar{v} = \lambda v, \quad v' \rightarrow \bar{v}' = \lambda v'.$$

The Maxwellian factors transform according to

$$\begin{aligned} d^3v d^3v' f_M(v; T) f'_M(v'; T) \\ \rightarrow d^3\bar{v} d^3\bar{v}' f_M(\bar{v}; \lambda^2 T) f'_M(\bar{v}'; \lambda^2 T). \end{aligned}$$

Each factor involving derivatives of \hat{g} contributes a factor λ , as does the tensor $U_{\alpha\beta}$. Thus we obtain

$$\Phi[g; \lambda, T] = \lambda^3 \Phi[f; 1, \lambda^2 T] \quad (12)$$

as anticipated in Eq. (7). It follows from Eq. (10) that Coulomb collisions are characterized by the entropy production rate

$$\Phi \propto T^{5/2}. \quad (13)$$

This prediction is confirmed by the standard analysis.

Of course the standard analysis of the Spitzer problem supplies much more information than Eq. (13); by use of appropriate trial functions, one can obtain a reasonable approximation to the complete transport matrix.⁵ What is interesting about the present analysis is its demonstration that the temperature dependence of transport follows entirely from the self-adjointness of the collision operator and the scaling law (12). No other features of the collisional process matter.

III. SINH-POISSON EQUATION

A. Formulation

The SP equation given by

$$\nabla^2 \psi + \kappa^2 \sinh(\psi) = 0 \quad (14)$$

governs, among other things, the electrostatic potential of a fully ionized plasma. It is a nonlinear, second-order partial differential equation with few known solutions.^{6,7} Because the SP equation is not readily solvable and the solutions are of considerable physical interest, it is worthwhile to extract as much information as possible regarding the behavior of the exact solution. Here we derive an exact integral relation, which provides a nontrivial constraint on all solutions to the SP equation.

B. Variational principle

The SP equation is the Euler-Lagrange equation of the following action integral:

$$A[F] = \int_R \left[\frac{1}{2} (\nabla F)^2 - \kappa^2 \cosh(F) \right] d^2x. \quad (15)$$

We assume the region R is compact, since the SP equation has no nontrivial solutions on infinite space. In this regard, our procedure departs from Derrick. Let ψ denote the exact solution to the SP equation (for given boundary data). In order to derive an integral identity from the action integral, we employ the following variation:

$$\psi + \delta\psi = \psi \circ f_\epsilon,$$

where f_ϵ is an element of a one-parameter family of coordinate transformations with the following properties:

- (i) The region ∂R is unaffected by all transformations in the set.
- (ii) The set contains a neighborhood of the identity transformation.

With this variation, the action integral can be considered a function of the single variable ϵ rather than a functional. We can then write

$$A(\epsilon) = A[\psi \circ f_\epsilon].$$

Let $f_{\epsilon=0}$ denote the identity transformation. Because ψ must be the extremum solution to the action functional, it is clear that

$$\left. \left(\frac{dA}{d\epsilon} \right) \right|_{\epsilon=0} = 0. \quad (16)$$

C. Example

Let R be given by

$$R = \{(x, y) : x_{\min} < x < x_{\max}, x_{\min} < y < x_{\max}\}.$$

So R is a square box in upper right quadrant of the xy -plane. We choose a change-of-variables of the form

$$X(x) = \epsilon x^2 + bx + c,$$

where ϵ, b and c are free parameters. From the requirement $X(x_{\min}) = x_{\min}$ and $X(x_{\max}) = x_{\max}$, we can solve for b and c in terms of ϵ . This yields

$$X_\epsilon(x) = \epsilon x^2 + [1 - \epsilon(x_{\min} + x_{\max})]x + \epsilon x_{\min} x_{\max} \quad (17)$$

and similarly we have $Y_\epsilon(y)$ with the same functional form. In accordance with our notation of the previous section, our family of transformations takes the form

$$f_\epsilon(x, y) = \{\epsilon x^2 + [1 - \epsilon(x_{\min} + x_{\max})]x + \epsilon x_{\min} x_{\max},$$

$$\epsilon y^2 + [1 - \epsilon(x_{\min} + x_{\max})]y + \epsilon x_{\min} x_{\max}\},$$

where $\epsilon=0$ clearly corresponds to the identity transformation. The action now takes the form

$$A_\epsilon = \int_R \left[\frac{1}{2} \{ \nabla \psi [X_\epsilon(x), Y_\epsilon(y)]^2 - \kappa^2 \cosh\{\psi [X_\epsilon(x), Y_\epsilon(y)]\} \right] dX_\epsilon dY_\epsilon.$$

Using the change-of-variables theorem and taking advantage of the fact that R is left unchanged under all of the mappings f_ϵ , we have

$$A_\epsilon = \int_R \left\{ \frac{1}{2} \left[\left(\frac{\partial \psi}{\partial X_\epsilon} \right)^2 \frac{dX_\epsilon}{dx} \frac{dy}{dY_\epsilon} + \left(\frac{\partial \psi}{\partial Y_\epsilon} \right)^2 \frac{dY_\epsilon}{dy} \frac{dx}{dX_\epsilon} \right] - \kappa^2 \cosh(\psi) \left(\frac{dx}{dX_\epsilon} \frac{dy}{dY_\epsilon} \right) \right\} dX_\epsilon dY_\epsilon.$$

The change-of-variables has placed all of the ϵ dependence in the integrand, allowing us to differentiate with respect to ϵ . In accordance with Eq. (16), we get

$$\begin{aligned} \frac{dA_\epsilon}{d\epsilon} = \int_R \frac{1}{2} \left[\left(\frac{\partial \psi}{\partial X_\epsilon} \right)^2 \frac{d}{d\epsilon} \left(\frac{dX_\epsilon}{dx} \frac{dy}{dY_\epsilon} \right) + \left(\frac{\partial \psi}{\partial Y_\epsilon} \right)^2 \frac{d}{d\epsilon} \left(\frac{dY_\epsilon}{dy} \frac{dx}{dX_\epsilon} \right) - \kappa^2 \cosh(\psi) \frac{d}{d\epsilon} \left(\frac{dx}{dX_\epsilon} \frac{dy}{dY_\epsilon} \right) \right] dX_\epsilon dY_\epsilon. \end{aligned} \quad (18)$$

We can compute all of these derivatives using the implicit function theorem. To avoid ambiguity, we will temporarily switch to functional notation, so the names of the variables are distinct from the names of the functions. Define $F: \mathbb{R}^3 \rightarrow \mathbb{R}$ as

$$F(X, x, \epsilon) = \epsilon x^2 + [1 - \epsilon(x_{\min} + x_{\max})]x + \epsilon x_{\min} x_{\max} - X.$$

From the equation $F=0$ which follows from Eq. (17), we can implicitly define x as a function g of X and ϵ . That is,

$$h(X, \epsilon) \equiv F[X, x = g(X, \epsilon), \epsilon] = 0.$$

We then have $Dh=0$ which implies

$$\partial_1 g = - \frac{1}{2} \frac{F}{F}.$$

Computing the partial derivatives and transforming back to variable notation, we obtain

$$\left(\frac{\partial x}{\partial X} \right) = \frac{1}{2\epsilon + [1 - \epsilon(x_{\max} + x_{\min})]},$$

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left(\frac{\partial x}{\partial X} \right) = 3 - [x_{\max} + x_{\min}].$$

All necessary remaining derivatives with respect to ϵ may be computed similarly. Inserting these expressions into Eq. (18) and simplifying, we have

$$\int_R \left\{ (x-y) \left[\left(\frac{\partial \psi}{\partial x} \right)^2 - \left(\frac{\partial \psi}{\partial y} \right)^2 \right] + 2xy\kappa^2 \cosh(\psi) \right\} dx dy = 0. \quad (19)$$

We can derive a similar relation for the case of a disk of radius ρ centered at the origin. We treat the disk as an annulus with inner radius r_{\min} and let r_{\min} tend to zero in the final result. The change of variables is similar to the one employed above,

$$R_\epsilon(r) = \epsilon r^2 + [1 - \epsilon(r_{\min} + \rho)]r + \epsilon r_{\min} \rho,$$

$$\Theta(\theta) = \theta.$$

We again use the change-of-variables theorem to place all ϵ dependence in the integrand. The calculation then proceeds in the same manner as the case above. Omitting the details of the calculation, we obtain

$$\int \left\{ \frac{1}{2} \left[r^2 \left(\frac{\partial \psi}{\partial r} \right)^2 - \left(\frac{\partial \psi}{\partial \theta} \right)^2 \right] - [2r\rho - 3r^2]\kappa^2 \cosh(\psi) \right\} dr d\theta = 0. \quad (20)$$

These relations hold for arbitrary boundary data and thus can be simplified with further assumptions regarding ψ on the boundary. For example, if the boundary conditions do not depend on θ , then Eq. (20) simplifies to

$$\int \left[\frac{r^2}{2} \left(\frac{\partial \psi}{\partial r} \right)^2 - [2r\rho - 3r^2]\kappa^2 \cosh(\psi) \right] dr d\theta = 0.$$

Notice that Eqs. (19) and (20) contain different informations; you cannot derive one from the other using a change of variables.

IV. GRAD-SHAFRANOV EQUATION

Equilibrium of a magnetically confined, axisymmetric plasma is described by the GS equation.^{8,9} This equation is to be solved for the poloidal magnetic flux $\psi(x)$ in some bounded, toroidal region \mathcal{V} ; it takes the form

$$\Delta^* \psi = -I(\psi)I'(\psi) - R^2 P'(\psi), \quad (21)$$

where

$$\Delta^* f \equiv R^2 \nabla \cdot \left(\frac{\nabla f}{R^2} \right)$$

and the coordinate R measures distance from the symmetry axis. The (toroidal) boundary surface $\partial\mathcal{V}=S$ is assumed to be a flux surface—that is, ψ is constant on the boundary,

$$\psi(x) = \psi_b, \quad \text{for } x \in S. \quad (22)$$

The functions $I(\psi)$, measuring the toroidal field, and $P(\psi)$, measuring the plasma pressure, are presumed to be known. Both sides of Eq. (21) measure the toroidal current density.

The GS equation possesses the variational principle¹⁰

$$\delta A = 0, \quad (23)$$

where

$$A[f] = \int_{\mathcal{V}} d^3x \left(\frac{|\nabla f|^2}{2R^2} - \frac{I^2}{2R^2} - P \right). \quad (24)$$

Here f is a trial function; we reserve the symbol ψ for the actual solution to GS. The integral is taken over the region \mathcal{V} . Note that, in view of Eq. (22), the flux at the boundary is not varied: $f(\mathbf{x}) = \psi(\mathbf{x})$ for \mathbf{x} in S . It is convenient to introduce the potential energy

$$U(f, R) = \frac{I(f)^2}{2R^2} + P(f). \quad (25)$$

Then the variational quantity A is seen to be analogous to mechanical action,

$$A[f] = \int_{\mathcal{V}} d^3x \left[\frac{|\nabla f|^2}{2R^2} - U(f, R) \right], \quad (26)$$

since the integrand can be identified with the classical Lagrangian—the difference between kinetic and potential energies. We make this identification explicit by introducing the kinetic energy functional

$$T[f] = \int_{\mathcal{V}} d^3x \frac{|\nabla f|^2}{2R^2}$$

and two potential energy functionals,

$$U_I[f] = \int_{\mathcal{V}} d^3x \frac{I(f)^2}{2R^2},$$

$$U_P[f] = \int_{\mathcal{V}} d^3x P(f).$$

Then

$$A[f] = T[f] - U_I[f] - U_P[f]. \quad (27)$$

An alternative version is more physical; since the poloidal (B_P) and toroidal (B_T) magnetic field components are given by

$$\mathbf{B}_P = \nabla \zeta \times \nabla \psi, \quad \mathbf{B}_T = I \nabla \zeta,$$

where ζ is the toroidal (azimuthal) angle for cylindrical coordinates (R, ζ, z) , we can write

$$A = \frac{1}{2} \int_{\mathcal{V}} d^3x (B_P^2 - B_T^2 - 2P). \quad (28)$$

A. Geometry and formulation

We suppose that the toroidal chamber has a rectangular cross section. The inner (outer) radius is denoted by R_i (R_o); the two horizontal walls are located at $z = \pm z_m$. Thus the action can be expressed as (suppressing a factor of 4π)

$$A[f] = \int_{R_i}^{R_o} dR \int_0^{z_m} dz \int \left[\frac{|\nabla f|^2}{2R^2} - U(f, R) \right]. \quad (29)$$

Now consider a coordinate transformation

$$\mathbf{r} \equiv (R, z) \rightarrow \bar{\mathbf{r}} = (\bar{R}, \bar{z})$$

that depends continuously on a parameter ϵ , in some neighborhood of $\epsilon=0$. We require the function $\bar{\mathbf{r}}(\mathbf{r}, \epsilon)$ to satisfy

$$\bar{\mathbf{r}}(\mathbf{r}, 0) = \mathbf{r}, \quad \text{for all } \mathbf{r}, \quad (30)$$

$$\bar{\mathbf{r}}(\mathbf{r}, \epsilon) = \mathbf{r}, \quad \text{for } \mathbf{r} \in S \text{ and } \epsilon \text{ sufficiently small.} \quad (31)$$

We use the transformed coordinates to define a trial function

$$f(R, z) = \psi(\bar{R}, \bar{z}), \quad (32)$$

where ψ is the exact solution. Then the action becomes a function of the parameter

$$A_\epsilon \equiv A[f_\epsilon],$$

which evidently satisfies

$$\left. \frac{dA_\epsilon}{d\epsilon} \right|_{\epsilon=0} = 0. \quad (33)$$

This relation provides an integral relation for the fields and pressure, for each choice of the coordinate transformation $\bar{\mathbf{r}}$. We next make this relation more explicit.¹¹

For simplicity we consider transformations that are uncoupled in the sense that \bar{R} depends only on R and ϵ , while \bar{z} depends only on z and ϵ . Then, after transforming integration variables, the action is expressed by Eq. (27), with

$$T_\epsilon = \int_{R_i}^{R_o} d\bar{R} \bar{R} \int_0^{z_m} d\bar{z} \frac{1}{2\bar{R}^2} [\xi_1(\bar{R}, \bar{z}, \epsilon) \psi_R^2 + \xi_2(\bar{R}, \bar{z}, \epsilon) \psi_z^2], \quad (34)$$

$$U_{I\epsilon} = \int_{R_i}^{R_o} d\bar{R} \bar{R} \int_0^{z_m} d\bar{z} \xi_3(\bar{R}, \bar{z}, \epsilon) \frac{I^2[\psi(\bar{R}, \bar{z})]}{2\bar{R}^2}, \quad (35)$$

$$U_{P\epsilon} = \int_{R_i}^{R_o} d\bar{R} \bar{R} \int_0^{z_m} d\bar{z} \xi_4(\bar{R}, \bar{z}, \epsilon) P[\psi(\bar{R}, \bar{z})]. \quad (36)$$

Here we have introduced the functions

$$\xi_1(\bar{R}, \bar{z}, \epsilon) = \frac{\bar{R} \bar{R}_R}{R \bar{z}_z}, \quad (37)$$

$$\xi_2(\bar{R}, \bar{z}, \epsilon) = \frac{\bar{R} \bar{z}_z}{R \bar{R}_R}, \quad (38)$$

$$\xi_3(\bar{R}, \bar{z}, \epsilon) = \frac{\bar{R}}{R \bar{R}_R \bar{z}_z}, \quad (39)$$

$$\xi_4(\bar{R}, \bar{z}, \epsilon) = \frac{R}{\bar{R} \bar{R}_R \bar{z}_z}. \quad (40)$$

Here we emphasize that all quantities must be evaluated in terms of the integration variables \bar{R} and \bar{z} .

It is now straightforward to compute the derivatives with respect to ϵ . Using the abbreviation

$$F' \equiv \left. \frac{\partial F}{\partial \epsilon} \right|_{\epsilon=0}$$

we have, suppressing bars,

$$T' = \int_{R_i}^{R_o} dRR \int_0^{z_m} dz \frac{1}{2R^2} [\xi_1'(R, z) \psi_R^2 + \xi_2'(R, z) \psi_z^2], \quad (41)$$

$$U_I' = \int_{R_i}^{R_o} dRR \int_0^{z_m} dz \xi_3'(R, z) \frac{I^2[\psi(R, z)]}{2R^2}, \quad (42)$$

$$U_P' = \int_{R_i}^{R_o} dRR \int_0^{z_m} dz \xi_4'(R, z) P[\psi(R, z)]. \quad (43)$$

Then Eq. (33) provides the integral identity

$$T' = U_I' + U_P'. \quad (44)$$

All that remains is the calculation of the ξ_i' .

B. Example

We consider the transformation

$$\bar{R}(R, \epsilon) = R + \left(R - \frac{R^2 + R_o R_i}{R_+} \right), \quad \bar{z}(z, \epsilon) = z,$$

where

$$R_{\pm} \equiv R_o \pm R_i.$$

The inverse function is

$$R(\bar{R}, \epsilon) = \frac{R_+}{2} (1 + \epsilon - \sqrt{X}),$$

where

$$X \equiv 1 - \frac{2}{R_+} (2\bar{R} - R_+) + \epsilon^2 \frac{R^2}{R_+^2}.$$

It is not hard to verify that this transformation satisfies Eqs. (30) and (31). The derivative is¹²

$$R' \equiv \left. \frac{\partial R}{\partial \epsilon} \right|_{\epsilon=0} = \frac{1}{R_+} [\bar{R}(\bar{R} - R_+) + R_i R_o]. \quad (45)$$

We use this result to compute

$$\xi_1' = 2 - 3 \frac{\bar{R}}{R_+} - \frac{R_i R_o}{R_+ \bar{R}} = -\xi_4', \quad (46)$$

$$\xi_2' = \frac{\bar{R}}{R_+} - \frac{R_i R_o}{R_+ \bar{R}} = \xi_3'. \quad (47)$$

The integral identity in this case is therefore

$$\begin{aligned} \int_{R_i}^{R_o} \frac{dR}{2R} \int_0^{z_m} dz \left(2 - 3 \frac{R}{R_+} - \frac{R_i R_o}{R_+ R} \right) [\psi_R^2 + P(\psi)] \\ = \int_{R_i}^{R_o} \frac{dR}{2R} \int_0^{z_m} dz \left(\frac{R_i R_o}{R_+ R} - \frac{R}{R_+} \right) [\psi_z^2 - I^2(\psi)]. \end{aligned} \quad (48)$$

V. SUMMARY

Variational principles are commonly used in plasma physics to find approximate solutions to equations for which general, exact solutions are not known. The variational method is very powerful in this context. This work shows that variational methods, based on the insight of Derrick, can also be used to obtain exact properties of the solutions. Such properties have intrinsic interest and also could be used to benchmark and validate numerical or other approximate solutions.

Derrick's key idea was to construct a trial function that differs from the exact solution only by a coordinate transformation, which is connected to the identity transformation; information about the solution is obtained by requiring that the variational quantity be stationary at the identity. In this way, the variational problem is reduced to one involving a single parameter rather than a space of functions. Derrick's use of a simple scaling transformation is applicable only for equations that hold in unbounded domains, and that lack intrinsic scale lengths. To extend the method to problems that involve boundaries or intrinsic scales, one must generalize this procedure, allowing in particular for more complicated coordinate transformations.

We have applied the generalized variational method to three basic equations of plasma physics, obtaining in each case conditions that every exact solution must satisfy. Some of our results appear to be new. In any case the method seems worth displaying because of its potentially broad applicability.

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