Symmetry analysis of the Grad–Shafranov equation

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Lie's technique of computing symmetries of differential equations is applied to a specific case of the Grad–Shafranov equation. The case considered contains the majority of exact solutions from literature. The full symmetry group is computed and new group-invariant solutions are obtained from these symmetries. The basic results and methods behind this technique are given to allow the reader who is unfamiliar with the subject to use the results given in this paper. Several plots of the level sets or flux surfaces of the new solutions are given. © *2009 American Institute of Physics*. [doi:10.1063/1.3267211]

I. INTRODUCTION

A symmetry group of a partial differential equation (PDE) is set of transformations that allow one to generate a whole family of solutions if a single solution is known. The transformation acts on both the independent and dependent variables and, in general, will not preserve the boundary geometry or boundary data of the original solution.

The Grad–Shafranov (GS) equation is given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial z^2}-\frac{1}{r}\frac{\partial u}{\partial r}+r^2F+G=0, \tag{1}$$

where F and G are functions of u. It can be derived from the ideal magnetohydrodynamic equations under the assumptions of static equilibrium and azimuthal symmetry.[1,2] The dependent variable u is the flux of the poloidal magnetic field. The function F is proportional to the pressure gradient dP/du while G measures the axial current density. Equation (1) is central to nearly all experiments in magnetic confinement physics. The solutions to Eq. (1) can be used to construct the magnetic field and pressure profile for an equilibrium plasma configuration with toroidal symmetry. Given a particular set of functions $F(u)$, $G(u)$, the corresponding solution to Eq. (1) provides an equilibrium configuration which is possible in principle. Any given level set $u=$const can be taken as the plasma boundary.

There are presently available very few analytic solutions to the GS equation; one usually must resort to numerical analysis. One exception is the case where F and G are constant for all u. Solov'ev,[3] Herrnegger,[4] and Maschke[5] all found families of analytic solutions with closed flux surfaces. For these reasons, in this paper we will assume that F and G are constant. Using any of the four symmetry transformation subgroups given in this paper, a whole family of solutions can be obtained from these known solutions. Thus the flux surfaces can be continuously deformed into new and possibly more interesting or more realistic configurations. In effect, each solution has been expanded to have more free parameters. From these solutions, we can also generate new solutions. In the case where the solution represents poloidal flux of a plasma, it must be noted that these new solutions will have a different plasma boundary which is a distortion of the original.

In addition, the symmetry group can be used to construct new independent and dependent variables such that Eq. (1) is transformed into an ordinary differential equation (ODE). If the ODE can be solved, a new solution to Eq. (1) is obtained. Solutions obtained in this way have special group-invariant properties which will be discussed below. An outline of the theory and methods used in applying symmetries of PDEs is given in Sec. II. The symmetry group of Eq. (1) is given in Sec. III. Special group-invariant solutions to Eq. (1) for several of subgroups of the full symmetry group have been computed and are given in Sec. IV.

II. OUTLINE OF METHOD

Let $\Delta(r,z;u;\partial u/\partial r,\dots)$ be a differential equation and let $X\times\mathbb{U}$ be the space of independent and dependent variables. Let $\mathbb{G}$ be a be a Lie group, $\Psi:\mathbb{G}\times(X\times\mathbb{U})\to X\times\mathbb{U}$ be a group action, and denote $\Psi[g,(r,z;u)]\equiv(\tilde{r},\tilde{z};\tilde{u})$, where $g\in G$. G is a *symmetry group* of the differential equation Δ if $\mathbb{G}$ maps solutions of Δ to new solutions under the group action Ψ. Simply stated, $\mathbb{G}$ maps the solution manifold into itself. If $u=f(r,z)$ is any solution to Eq. (1) then the function defined via $\tilde{u}=\tilde{u}\{r(\tilde{r},\tilde{z}),z(\tilde{r},\tilde{z});u=f[r(\tilde{r},\tilde{z}),z(\tilde{r},\tilde{z})]\}\equiv\tilde{f}(\tilde{r},\tilde{z})$ satisfies

$$\frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2}+\frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2}-\frac{1}{\tilde{r}}\frac{\partial \tilde{u}}{\partial \tilde{r}}+\tilde{r}^2F+G=0. \tag{2}$$

As an example, it will be shown below that

$$(\tilde{r},\tilde{z};\tilde{u})=\Bigg(re^{-\epsilon/2},ze^{-\epsilon/2};u+\frac{Gr^2}{2}\log(r)[1-e^{-\epsilon}]+\frac{Gr^2}{2}(e^{-\epsilon}-1)+\frac{Fr^4}{8}(1-e^{-2\epsilon})+\epsilon e^{-\epsilon}\frac{Gr^2}{4}\Bigg) \tag{3}$$

is a symmetry transformation of Eq. (1) for all ϵ. That is, if

$u(r,z)$ is a solution, then $\tilde{u}(\tilde{r},\tilde{z})$ is also a solution. The well-known Solov'ev solution to Eq. (1) is given by

$$u=\left[\tfrac{1}{2}(bR^2+r^2)z^2+\tfrac{1}{8}(a-1)(r^2-R^2)^2\right], \tag{4}$$

where

$$F=-a, \quad G=-bR^2,$$

and a, b, and R are free parameters. The level sets of this solution are closed nested surfaces which all circle the curves $r=R$ and $z=0$. We make the transformation given above and get a new one parameter family of solutions. The result is

$$\begin{aligned}\tilde{u}=&\left[\frac{1}{2}(bR^2+\tilde{r}^2e^{\epsilon})\tilde{z}^2e^{\epsilon}+\frac{1}{8}(a-1)(\tilde{r}^2e^{\epsilon}-R^2)^2\right]\\&+\frac{G\tilde{r}^2e^{\epsilon}}{2}\log(\tilde{r}e^{\epsilon/2})[1-e^{-\epsilon}]+\frac{G\tilde{r}^2e^{\epsilon}}{2}(e^{-\epsilon}-1)\\&+\frac{F\tilde{r}^4e^{2\epsilon}}{8}(1-e^{-2\epsilon})+\epsilon e^{-\epsilon}\frac{G\tilde{r}^2e^{\epsilon}}{4},\end{aligned}$$

which has been verified using symbolic software to satisfy Eq. (2). The solution has now been extended to have four free parameters.

Any symmetry transformation can be expanded in a power series in ϵ,

$$\tilde{r}=r+\epsilon\xi_r(r,z;u)+O(\epsilon^2), \tag{5}$$

$$\tilde{z}=z+\epsilon\xi_z(r,z;u)+O(\epsilon^2), \tag{6}$$

$$\tilde{u}=u+\epsilon\eta_u(r,z;u)+O(\epsilon^2). \tag{7}$$

The *infinitesimal generator* corresponding to this transformation is given by $\boldsymbol{v}=\xi_r(r,z;u)(\partial/\partial r)+\xi_z(r,z;u)(\partial/\partial z)+\eta_u(r,z;u)(\partial/\partial u)$. In transformation (3) we get

$$\tilde{r}=r-\frac{r}{2}\epsilon+O(\epsilon^2),$$

$$\tilde{z}=z-\frac{z}{2}\epsilon+O(\epsilon^2),$$

$$\tilde{u}=u+\epsilon\left(\frac{Gr^2}{2}\log(r)+\frac{Fr^4}{4}-\frac{Gr^2}{4}\right)+O(\epsilon^2).$$

Thus we see that the infinitesimal generator is given by

$$\boldsymbol{v}=-\frac{r}{2}\frac{\partial}{\partial r}-\frac{z}{2}\frac{\partial}{\partial z}+\left(\frac{Gr^2}{2}\log(r)+\frac{Fr^4}{4}-\frac{Gr^2}{4}\right)\frac{\partial}{\partial u}.$$

The full transformation group can be given by its infinitesimal generators $\{\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_n\}$. That is, any transformation can be written as $g=\exp(\epsilon_1\boldsymbol{v}_1)\exp(\epsilon_2\boldsymbol{v}_2),\ldots,\exp(\epsilon_n\boldsymbol{v}_n)$. In order to make the relationship between a transformation and its infinitesimal generator explicit, we use the conventional notation $\Psi[\epsilon,(r,z;u)]\equiv\exp(\epsilon\boldsymbol{v})(r,z;u)=(\tilde{r},\tilde{z};\tilde{u})$. One can also start with a given generator and construct the corresponding transformation group. Given an infinitesimal generator

$$\boldsymbol{v}=\xi_r(r,z;u)\frac{\partial}{\partial r}+\xi_z(r,z;u)\frac{\partial}{\partial z}+\eta_u(r,z;u)\frac{\partial}{\partial u},$$

the corresponding family of transformations is found by solving the following system of ODEs:

$$\frac{d\tilde{r}}{d\epsilon}(\epsilon)=\xi_r[\tilde{r}(\epsilon),\tilde{z}(\epsilon);\tilde{u}(\epsilon)], \quad \tilde{r}(0)=r,$$

$$\frac{d\tilde{z}}{d\epsilon}(\epsilon)=\xi_z[\tilde{r}(\epsilon),\tilde{z}(\epsilon);\tilde{u}(\epsilon)], \quad \tilde{z}(0)=z,$$

$$\frac{d\tilde{u}}{d\epsilon}(\epsilon)=\eta_u[\tilde{r}(\epsilon),\tilde{z}(\epsilon);\tilde{u}(\epsilon)], \quad \tilde{u}(0)=u.$$

III. SYMMETRY GROUP

A. Algorithm for generators

There is an algorithm for generating the full set of infinitesimal generators for a system of PDEs.[6] This algorithm involves solving an overdetermined system of linear PDEs and there are many software packages available[7] to aid in this laborious task. The infinitesimal generators in this paper were found using GEM for Maple.[8]

The complete set of infinitesimal generators is given by the span of the following vector fields:

$$\begin{aligned}\boldsymbol{v}_1=&2zr\frac{\partial}{\partial r}+(z^2-r^2)\frac{\partial}{\partial z}\\&+\left(uz-\frac{3}{2}Gzr^2\log(r)-\frac{7}{8}Fzr^4\right)\frac{\partial}{\partial u},\end{aligned}$$

$$\boldsymbol{v}_2=r\frac{\partial}{\partial r}+z\frac{\partial}{\partial z}+\left(2u-\frac{Fr^4}{4}\right)\frac{\partial}{\partial u},$$

$$\boldsymbol{v}_3=r\frac{\partial}{\partial r}+z\frac{\partial}{\partial z}+\left(\frac{Gr^2}{2}-\frac{Fr^4}{2}-Gr^2\log(r)\right)\frac{\partial}{\partial u},$$

$$\boldsymbol{v}_4=\frac{\partial}{\partial z},$$

$$\boldsymbol{v}_5=\alpha(r,z)\frac{\partial}{\partial u},$$

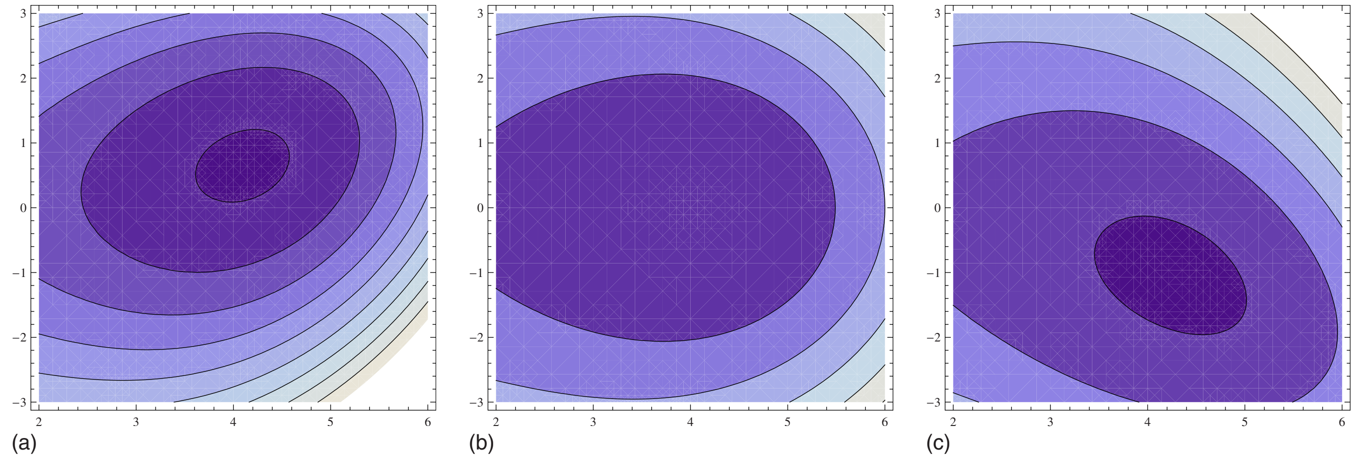

FIG. 1. (Color online) Level sets for Solov'ev solution transformed under the action of G_1.

where $\alpha(r,z)$ satisfies the GS equation for the case $F=0$ and $G=0$ (the homogeneous case). The span of these vector fields forms a five-dimensional Lie algebra.

B. Symmetry transformations

The one parameter group G_i corresponding to $\boldsymbol{v}_i$ transforms the point $(r,z;u)$ to $(\tilde{r},\tilde{z};\tilde{u})=\exp(\epsilon\boldsymbol{v}_i)(r,z;u)$. The transformations are found to be

$$G_1:(\tilde{r},\tilde{z};\tilde{u}) = \left(\frac{r}{\beta_-},\frac{z-\epsilon_1(z^2+r^2)}{\beta_-};\right.$$
$$\frac{1}{\sqrt{\beta_-}}\left[u+\frac{Gr^2}{2}(1-\beta_-^{3/2})\right.$$
$$\left.\left.+\frac{Gr^2}{6}\left(\frac{2+3\log(\beta_-)}{\beta_-^{3/2}}-2\right)+\frac{Fr^4}{8}(1-\beta_-^{-7/2})\right]\right),$$

$$G_2:(\tilde{r},\tilde{z};\tilde{u}) = \left(re^{\epsilon_2},ze^{\epsilon_2};u-\frac{Fr^4}{8}\sinh(\epsilon_2)\right),$$

$$G_3:(\tilde{r},\tilde{z};\tilde{u}) = \left(re^{-\epsilon_3/2},ze^{-\epsilon_3/2};u+\frac{Gr^2}{2}\log(r)[1-e^{-\epsilon_3}]\right.$$
$$\left.+\frac{Gr^2}{2}(e^{-\epsilon_3}-1)+\frac{Fr^4}{8}(1-e^{-2\epsilon_3})+\epsilon_3 e^{-\epsilon_3}\frac{Gr^2}{4}\right),$$

$$G_4:(\tilde{r},\tilde{z};\tilde{u}) = (r,z+\epsilon_4;u),$$

$$G_5:(\tilde{r},\tilde{z};\tilde{u}) = [r,z;u+\epsilon_5\alpha(r,z)],$$

where $\beta_\pm=(1\pm z\epsilon_1)^2+r^2\epsilon_1^2$. It is worth noting that $\boldsymbol{v}_1$ is a conformal mapping of the independent variables. The flux surface configurations corresponding to groups G_1 and G_2 are displayed in Figs. 1 and 2, respectively.

If $u=f(r,z)$ is any solution to the GS equation then the following functions $u^{(i)}=\tilde{f}_i(r,z)$ are also solutions:

$$u^{(1)} = \frac{1}{\sqrt{\beta_-}}\left[f\left(\frac{r}{\beta_+},\frac{z+\epsilon_1(z^2+r^2)}{\beta_+}\right)+\frac{Gr^2}{2\beta_+^2}(1-\beta_-^{3/2})\right.$$
$$\left.+\frac{Gr^2}{6\beta_+^2}\left(\frac{2+3\log(\beta_-)}{\beta_-^{3/2}}-2\right)+\frac{Fr^4}{8\beta_+^4}(1-\beta_-^{-7/2})\right],$$

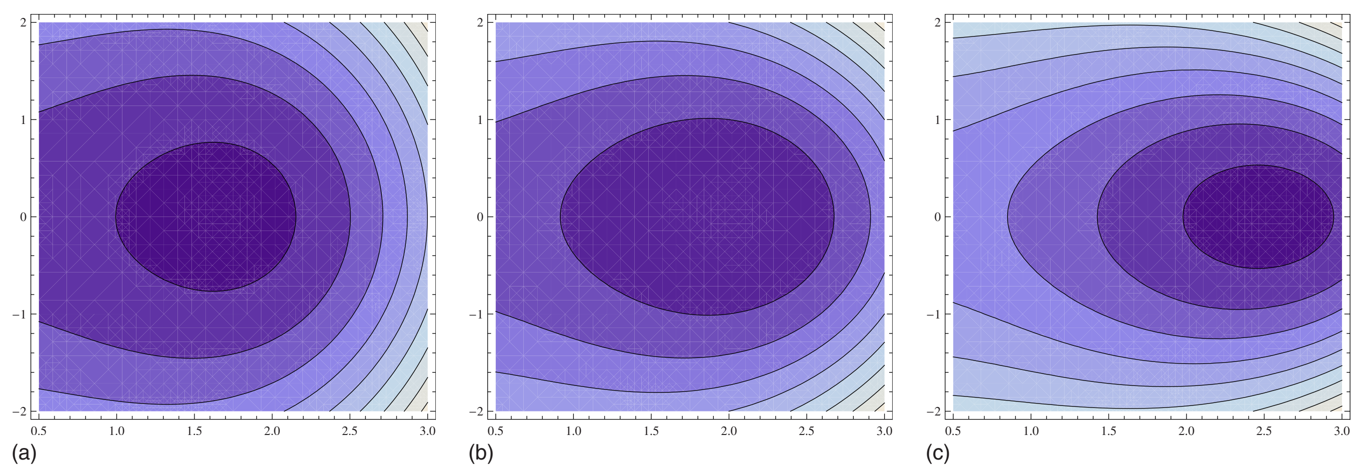

FIG. 2. (Color online) Level sets for Solov'ev solution transformed under the action of G_2.

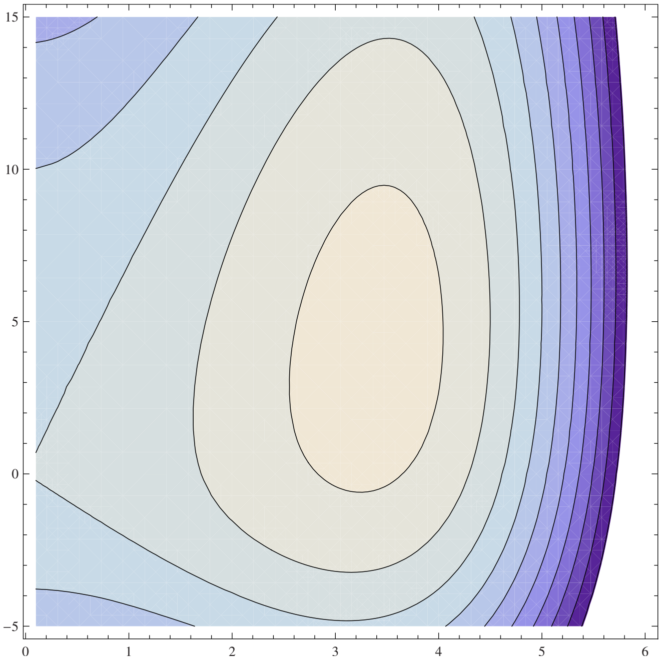

FIG. 3. (Color online) Flux surfaces for invariant solution corresponding to $U^{(1)}$.

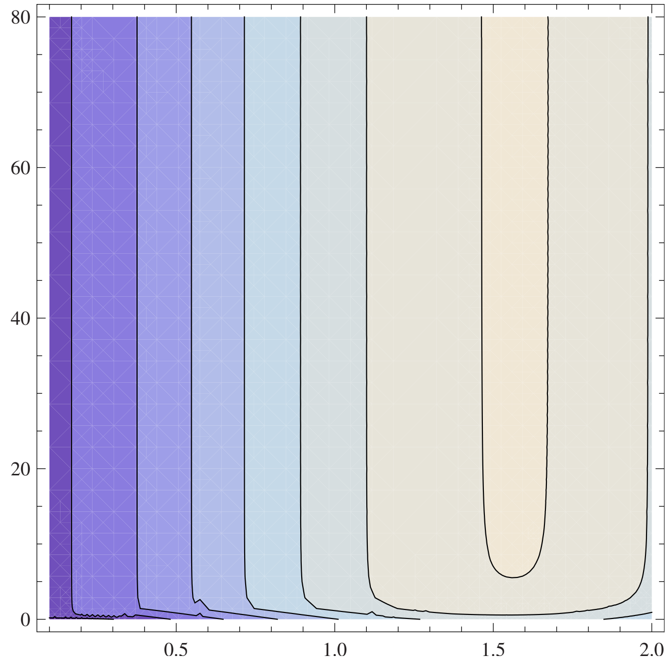

FIG. 4. (Color online) Flux surfaces for invariant solution corresponding to $U^{(2)}$.

$$u^{(2)} = f(re^{-\epsilon_2}, ze^{-\epsilon_2}) - \frac{Fr^4 e^{-4\epsilon_2}}{8}\sinh(\epsilon_2),$$

$$u^{(3)} = f(re^{\epsilon_3/2}, ze^{\epsilon_3/2}) + \frac{Gr^2 e^{\epsilon_3}}{2}\log(re^{\epsilon_3/2})[1 - e^{-\epsilon_3}] + \frac{Gr^2 e^{\epsilon_3}}{2}(e^{-\epsilon_3} - 1) + \frac{Fr^4 e^{2\epsilon_3}}{8}(1 - e^{-2\epsilon_3}) + \epsilon_3 e^{-\epsilon_3}\frac{Gr^2 e^{\epsilon_3}}{4},$$

$$u^{(4)} = f(r, z - \epsilon_4),$$

$$u^{(5)} = f(r,z) + \epsilon_5 \alpha(r,z).$$

It is worth emphasizing that these transformations can also be composed. For example, if $f(r,z)$ is a solution to Eq. (1), then

$$g(r,z) = f(r, z - \epsilon_4) + \epsilon_5 \alpha(r,z)$$

is also a solution.

IV. GROUP INVARIANT SOLUTIONS

H-invariant solutions are particular solutions to Eq. (1) which are invariant under a particular subgroup $H \subset \mathbb{G}$. These solutions are constructed by choosing new dependent and independent variables that are separately invariant under the subgroup H having dimension s. In general, such a transformation will reduce the number of dependent variables by the dimension s.[6] Because the GS equation has only two independent variables, we can only consider the one parameter subgroups $H_i = \{\exp[\epsilon(\boldsymbol{v}_i)] : \epsilon \in \mathbb{R}\}$. The result is to transform the GS equation into an *ODE* (single independent variable). We will find this equation to be solvable in closed form; thus we find a new family of exact analytic solutions to the GS equation. The vector fields $\boldsymbol{v}_4$ and $\boldsymbol{v}_5$ give only trivial symmetries so we omit them from consideration here.

We begin by constructing the solution to the GS equation invariant under H_1. There are two independent functions invariant under the vector field $\boldsymbol{v}_1$,

$$F_1(r,z;U) = \frac{r^2 + z^2}{r},$$

$$F_2(r,z;U) = \frac{1}{\sqrt{r}}\left[U - \frac{r^2 G}{3} + \frac{Fr^4}{8} + \frac{Gr^2}{2}\log(r)\right].$$

These are easily found by solving $\boldsymbol{v}_1(F_i) = 0$, which gives the first order PDE,

$$2zr\frac{\partial F_i}{\partial r} + (z^2 - r^2)\frac{\partial F_i}{\partial z} + \left(Uz - \frac{3}{2}Gzr^2\log(r) - \frac{7}{8}Fzr^4\right)\frac{\partial F_i}{\partial U} = 0.$$

We choose new independent variables to be $y = F_1(r,z)$ and r and the new dependent variable to be

$$v = F_2(r,z;U) = \frac{1}{\sqrt{r}}\left[U - \frac{r^2 G}{3} + \frac{Fr^4}{8} + \frac{Gr^2}{2}\log(r)\right]. \tag{8}$$

In Eq. (8), v ultimately depends only on r and z since U is a function of these variables. Indeed, U will be our H-invariant solution to Eq. (1). If we assume that v depends on r and z only through the variable y [i.e., $v(r,z) = v[y(r,z)]$], use the chain rule to relate derivatives of U to derivatives of v and substitute these into the GS equation, then we find that the

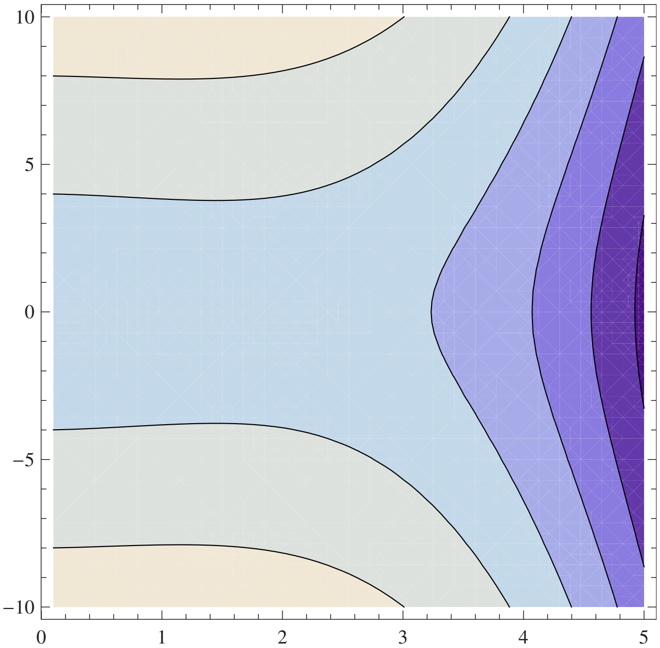

FIG. 5. (Color online) Flux surfaces for invariant solution corresponding to $U^{(3)}$.

variable r drops out of the equation and we are left with the following ODE:

$$y^2 v_{yy} + 2y v_y - \tfrac{3}{4} v = 0.$$

This has the solution

$$v(y) = c_1 \sqrt{y} + \frac{c_2}{y^{3/2}}.$$

We can now solve for U in terms of v from Eq. (8) and resubstitute $y=(r^2+z^2)/r$ to get the H-invariant solution to the GS equation corresponding to the subgroup H_1. This gives the following result:

$$U^{(1)}(r,z) = \frac{r}{\sqrt{r^2+z^2}}\left[c_1 + c_2 \frac{r}{r^2+z^2}\right] + \frac{Gr^2}{3} + \frac{Fr^4}{8} - \frac{Gr^2}{2}\log(r).$$

The invariant solutions corresponding to H_2 and H_3 may be computed similarly. The results are

$$U^{(2)}(r,z) = r^2\left[\frac{-Gz^2}{2r^2} + c_1 \frac{z}{2r}\sqrt{1+\frac{z^2}{r^2}} + c_1 \frac{1}{2}\sinh^{-1}(z/r) + c_2\right] - \frac{Fr^4}{8},$$

$$U^{(3)}(r,z) = c_1 + \frac{c_1 \frac{z}{r}}{\sqrt{1+\frac{z^2}{r^2}}} + \frac{Gr^2}{2}[1+\log(r)] - \frac{Fr^4}{8}.$$

The solutions $U^{(1)}$, $U^{(2)}$, and $U^{(3)}$ have each been verified to satisfy the GS equation. Level sets corresponding to these solutions are plotted in Figs. 3–5. Recall that in the case of a magnetized plasma described above, these surfaces represent magnetic surfaces.

V. SUMMARY

We have used Lie-symmetry methods[6] to uncover all the transformations of dependent and independent variables that leave the GS equation, with the conventional assumption that the pressure and current profiles are linear in the flux, invariant. Thus we have found the complete symmetry group for this basic equilibrium description. We then used the resulting symmetries in two ways. First, by applying the transformations to the known, exact solution due to Solev'ev,[3] we have generated families of new exact solutions, including some that are quite different in form and structure from the starting point. Second, we have used an invariant subgroup of the complete symmetry group to reduce the GS equation to an ODE. This equation has been found to have an analytic solution, allowing the generation of a family of entirely new GS equilibria. The outcome from both procedures is a major extension of the known analytic solutions to the basic equation of axisymmetric plasma equilibrium.

Although our presentation of the symmetry group is complete, we have not attempted to explore or categorize the enormous class of new equilibria that result, being content to display several examples. It seems likely, however, that equilibria generated using the symmetries found here could be useful in (1) developing analytic models for tokamak fluid stability and (2) testing numerical simulations of tokamak equilibrium.

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