Hamiltonian derivation of the Charney–Hasegawa–Mima equation

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The Charney–Hasegawa–Mima equation is an infinite-dimensional Hamiltonian system with dynamics generated by a noncanonical Poisson bracket. Here a first principle Hamiltonian derivation of this system, beginning with the ion fluid dynamics and its known Hamiltonian form, is given. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3194275]

I. INTRODUCTION

When dissipative terms are dropped, all of the important models of plasma physics are described by partial differential equations that possess Hamiltonian form in terms of noncanonical Poisson brackets. For example, this is the case for ideal magnetohydrodynamics,[1–3] the Vlasov–Maxwell equations,[4–6] and other systems (see Refs. 7–9 for review). Among these, there exist several reduced fluid models whose Hamiltonian structure has been derived *a posteriori*. These include the four-field model for tokamak dynamics of Hazeltine *et al.*,[10] models for collisionless magnetic reconnection derived and investigated by Schep *et al.*,[11] Kuvshinov *et al.*,[12] and Tassi *et al.*;[13] and the recent gyrofluid model of Waelbroeck *et al.*[14] The noncanonical Hamiltonian formulation has also been adopted to investigate the electron temperature gradient driven mode[15] and convective-cell formation in plasma fluid systems.[16] In addition to these fluid models, the Hamiltonian structure of kinetic and reduced kinetic equations has also been highlighted, for example, in guiding-center theory and gyrokinetics (see Refs. 17–21 for review).

This Hamiltonian form originates from the Hamiltonian and action principle forms of the basic electromagnetic interaction, i.e., the Hamiltonian form possessed by the equations that describe a system of charged particles coupled to Maxwell's equations (see, e.g., Ref. 9 for discussion). It is now well established that there exist numerous advantages of such a Hamiltonian formulation, among which are the identification of conserved quantities (that are important for the verification of numerical codes), the study of stability, the use of techniques for Hamiltonian systems like averaging and perturbation theory, etc. Here we perform a perturbative derivation within the noncanonical Hamiltonian context, which means the Poisson bracket as well as the Hamiltonian must be expanded.

In a nutshell, a Hamiltonian system is a system whose dynamics of any observable F (depending on a finite or infinite number of variables) can be written using a Hamiltonian (scalar) function H and a Poisson bracket $\{\cdot,\cdot\}$ as

$$\frac{\partial F}{\partial t} = \{F,H\},$$

where the Poisson bracket satisfies the following properties: bilinearity, antisymmetry, Leibniz rule, and Jacobi identity. Given a reduced model whose dynamics is given by a partial differential equation, it is in general difficult to guess whether or not the model is a Hamiltonian system, and if it is, finding the Hamiltonian and the Poisson bracket may be similarly difficult. There are basically two methods for finding Hamiltonian structure: the first method is to use physical intuition to obtain the Hamiltonian (energy) and to construct a general class of antisymmetric operators which, when acting on the gradient of the Hamiltonian, produces the equations of motion. Then, the Jacobi identity is used to select from the class the desired operator that is the essence of the noncanonical Poisson bracket. This method has been used to obtain a large number of basic and approximate Poisson brackets for fluid and plasma dynamics, examples being the reduced fluid models cited above. The second method begins from a known or postulated action principle, in the latter case obtained by using physical intuition to obtain the "energies" of the Lagrangian. Usually associated with the action principle is a canonical Hamiltonian description, which can be written by means of the chain rule in terms of physical variables of interest (e.g., Refs. 5 and 9) resulting in a noncanonical Poisson bracket.

If one begins from some Hamiltonian parent model, some basic starting point in the derivation, and introduces crude approximations suggested, e.g., by physical considerations of some experimental setup, then the Hamiltonian structure can be easily destroyed. The Hamiltonian form of the resulting system must therefore be verified, in particular, the Jacobi identity for the Poisson bracket. Given this verification, the reduced model is naturally equipped with a Hamiltonian structure since the Poisson bracket and the Hamiltonian function are provided by the derivation process (for an example of this derivation process, see Ref. 22).

In this paper we consider the derivation of the Charney–Hasegawa–Mima (CHM) equation,[23,24] which describes both the dynamics of Rossby waves of geophysical fluid dynamics (see, e.g., Ref. 25) and drift waves in inhomogeneous plasmas (see, e.g., Ref. 26). We focus on the derivation in the plasma physics context but our analysis can be easily adapted to the geophysical context. In particular, we show how the Hamiltonian structure is preserved in the derivation of the CHM equation starting from a fluid parent model. In the present approach the Hamiltonian structure is provided by the derivation process and the Jacobi identity need not be checked.

We consider a plasma under the influence of a constant and uniform magnetic field $\mathbf{B}=B\hat{\mathbf{z}}$. The relevant dynamics occurs in the (two-dimensional) transverse plane whose coordinates in a given basis are denoted by x and y. Under some assumptions, the CHM equation gives the following evolution of the electrostatic potential $\phi(x,y,t)$ generated by the plasma:

$$\frac{\partial}{\partial t}(\phi-\Delta\phi)=[\phi,\Delta\phi+\lambda], \tag{1}$$

where the bracket $[\cdot,\cdot]$ is given by

$$[f,g]=\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}=\hat{\mathbf{z}}\cdot\nabla f\times\nabla g,$$

and λ is any function of x and y (related to the equilibrium configuration). The infinite-dimensional phase space is composed of the variables $\phi(x,y)$ for any point (x,y) in the transverse plane. The space of observables, F, for this system is composed of functionals of ϕ. It has been shown in Ref. 27 that this equation possesses an infinite-dimensional Hamiltonian structure where the Hamiltonian is

$$H(\phi)=\frac{1}{2}\int d^2x(\phi^2+|\nabla\phi|^2),$$

and the noncanonical Poisson bracket is

$$\{F,G\}=-\int d^2x(\phi-\Delta\phi-\lambda)[(1-\Delta)^{-1}F_\phi,(1-\Delta)^{-1}G_\phi], \tag{2}$$

where F_ϕ denotes the functional derivative of the functional F with respect to the variable ϕ. This Hamiltonian structure was found *ad hoc* in Ref. 27 by an educated guess in analogy with the vorticity equation for two-dimensional incompressible flow (see, e.g., Ref. 7). This analogy is rather straightforward if we consider the dynamics for the field $q=\Delta\phi-\phi+\lambda$, which is given by the Hamiltonian

$$H=\frac{1}{2}\int d^2x(q-\lambda)(1-\Delta)^{-1}(q-\lambda),$$

and the Lie–Poisson bracket

$$\{F,G\}=\int d^2xq[F_q,G_q],$$

which is of the same form as that for the Vlasov–Poisson system[4] and a quite general class of systems.[28] In what follows, we start by considering a Hamiltonian formulation for the fluid equations for the ions (in Sec. II) and derive the above Hamiltonian and Poisson bracket from this formulation (in Sec. III).

II. ION FLUID EQUATIONS AS A HAMILTONIAN SYSTEM

We start the derivation of the CHM equation from two dynamical equations: one describing the transverse dynamics of the ion velocity field $\mathbf{v}(x,y,t)$ and the other describing the dynamics of the ion density field $n(x,y,t)$,

$$M(\dot{\mathbf{v}}+(\mathbf{v}\cdot\nabla)\mathbf{v})=-e\nabla\phi+e\mathbf{v}\times\mathbf{B}, \tag{3}$$

$$\dot{n}=-\nabla\cdot(n\mathbf{v}), \tag{4}$$

where the dot indicates the partial derivative with respect to time t. The electrostatic potential ϕ is obtained from the dynamics of the electrons: by neglecting their inertia, the electron density obeys the Boltzmann law,

$$n_e=n_0\exp(e\phi/T), \tag{5}$$

where T is the electron temperature and $n_0=n_0(x,y)$ is the electron density at equilibrium. From the quasineutrality condition, we obtain that $n=n_e$. The total energy of the ions, given by the sum of their kinetic energy plus the potential energy provided by the electric field, is a conserved quantity that is also a good candidate for the Hamiltonian of the system of Eqs. (3) and (4). This Hamiltonian is written as

$$H(n,\mathbf{v})=\int d^2x\left[n\frac{v^2}{2}+\frac{T}{M}n\left(\ln\left(\frac{n}{n_0}\right)-1\right)\right]. \tag{6}$$

The dynamics is determined by the Poisson bracket

$$\{F,G\}=-\int d^2x\left[F_{\mathbf{v}}\cdot\nabla G_n-G_{\mathbf{v}}\cdot\nabla F_n -\left(\frac{\nabla\times\mathbf{v}}{n}+\frac{\omega_c}{n}\hat{\mathbf{z}}\right)\cdot(F_{\mathbf{v}}\times G_{\mathbf{v}})\right], \tag{7}$$

where $\omega_c=eB/M$. The bracket of Eq. (7) is identical to a portion of that of Ref. 1 with the inclusion of an additional "vorticity" term, $\omega_c\hat{\mathbf{z}}/n$; consequently, it is known to satisfy the Jacobi identity. It is easy to verify that the ion momentum equation is obtained from the bracket of the velocity field with the Hamiltonian (6):

$$\dot{\mathbf{v}}\equiv\{\mathbf{v},H\}=-(\mathbf{v}\cdot\nabla)\mathbf{v}-\frac{T}{M}\nabla\ln\left(\frac{n}{n_0}\right)+\omega_c\mathbf{v}\times\hat{\mathbf{z}},$$

and, similarly, the ion continuity equation is given by

$$\dot{n}\equiv\{n,H\}=-\nabla\cdot(n\mathbf{v}).$$

III. CHARNEY–HASEGAWA–MIMA EQUATION

Without loss of generality we write the vector field $\mathbf{v}(x,y,t)$ in terms of two scalar fields ϕ and Υ as

$$\mathbf{v}=\hat{\mathbf{z}}\times\nabla\phi+\nabla\Upsilon, \tag{8}$$

where one function is related to $\nabla\cdot\mathbf{v}$ and the other to $\nabla\times\mathbf{v}$ by the relations: $\Delta\phi=\hat{\mathbf{z}}\cdot\nabla\times\mathbf{v}$ and $\Delta\Upsilon=\nabla\cdot\mathbf{v}$. In fact, we find it more convenient to consider a related change of variables $(n,\mathbf{v})\mapsto(\tilde{n},q,D)$ defined by

$$\tilde{n} = n,$$

$$q = \frac{\hat{\mathbf{z}} \cdot \nabla \times \mathbf{v} + \omega_c}{n},$$

$$D = \nabla \cdot \mathbf{v}.$$

The above equations are incomplete because they do not possess a unique inverse. However, a unique inverse is defined by the following:

$$n = \tilde{n},$$

$$\mathbf{v} = \hat{\mathbf{z}} \times \nabla \Delta^{-1}(q\tilde{n} - \omega_c) + \nabla \Delta^{-1} D,$$

where

$$\Delta^{-1} F = -\frac{1}{2\pi} \int d^2x' \ \ln \|\mathbf{x} - \mathbf{x}'\| F(\mathbf{x}').$$

In terms of the new variables $(\tilde{n}, q, D)$, Hamiltonian (6) becomes

$$H(\tilde{n},q,D) = \int d^2x \left[\tilde{n} \left(\frac{|\nabla \Delta^{-1}(q\tilde{n} - \omega_c)|^2}{2} + [\Delta^{-1}(q\tilde{n} - \omega_c), \Delta^{-1} D] + \frac{|\nabla \Delta^{-1} D|^2}{2} \right) + \frac{T}{M} \tilde{n} \left(\ln\left(\frac{\tilde{n}}{n_0}\right) - 1 \right) \right], \quad (9)$$

since $|\hat{\mathbf{z}} \times \nabla f|^2 = |\nabla f|^2$ for any function f of x and y and bracket (7) becomes

$$\{F,G\} = -\int d^2x \left(-\nabla F_D \cdot \nabla G_{\tilde{n}} + \nabla G_D \cdot \nabla F_{\tilde{n}} + \frac{G_q}{\tilde{n}} \nabla F_D \cdot \nabla q - \frac{F_q}{\tilde{n}} \nabla G_D \cdot \nabla q - q \left[\frac{F_q}{\tilde{n}}, \frac{G_q}{\tilde{n}} \right] - q[F_D, G_D] \right).$$

It should be noted that Casimir invariants of such a bracket, which are the functionals that Poisson commute with all the other functionals ($\{C,G\}=0$ for all functionals G) are given by

$$C = \int d^2x \tilde{n} \mathcal{F}(q),$$

where $\mathcal{F}$ is any function of q.

We first assume that the variables evolve slowly with time, which is equivalent to adding a factor of $1/\epsilon$ in front of the Hamiltonian,

$$H(\tilde{n},q,D) = \frac{1}{\epsilon} \int d^2x \left[\tilde{n} \left(\frac{|\nabla \Delta^{-1}(q\tilde{n} - \omega_c)|^2}{2} + [\Delta^{-1}(q\tilde{n} - \omega_c), \Delta^{-1} D] + \frac{|\nabla \Delta^{-1} D|^2}{2} \right) + \frac{T}{M} \tilde{n} \left(\ln\left(\frac{\tilde{n}}{n_0}\right) - 1 \right) \right],$$

then we introduce an ϵ-ordering for the dynamical variables. The hypothesis is that the system of interest is near an equilibrium state whose spatial variations are of the order of ϵ,

$$n(\mathbf{x},t) = n_0(\epsilon \mathbf{x}) + \epsilon n_1(\mathbf{x},t),$$

$$\mathbf{v}(\mathbf{x},t) = \epsilon \mathbf{v}_1(\mathbf{x},t),$$

which translates into an assumption on the new variables $(\tilde{n}, q, D)$ and, in particular, on the definition of new dynamical variables $(\tilde{n}_1, q_1, D_1)$,

$$\tilde{n} = \tilde{n}_0 + \epsilon \tilde{n}_1,$$

$$q = q_0 + \epsilon q_1,$$

$$D = \epsilon D_1,$$

where $q_0 = \omega_c / \tilde{n}_0$ and $\tilde{n}_0 = n_0(0,0)$ are constant (the spatial variations of n_0 are included in $\tilde{n}_1$). Notice that the potential energy can be rewritten as

$$\tilde{n} \left(\ln\left(\frac{\tilde{n}}{n_0}\right) - 1 \right) = \tilde{n} \left(\ln\left(\frac{\tilde{n}}{\tilde{n}_0}\right) - 1 \right) - \tilde{n} \ln \frac{n_0}{\tilde{n}_0},$$

with the following expansion:

$$\tilde{n} \left(\ln\left(\frac{\tilde{n}}{n_0}\right) - 1 \right) = -\tilde{n}_0 - \tilde{n}_0 \ln \frac{n_0}{\tilde{n}_0} + \epsilon^2 \frac{\tilde{n}_1^2}{2\tilde{n}_0} - \epsilon \tilde{n}_1 \ln \frac{n_0}{\tilde{n}_0} + O(\epsilon^3).$$

The term $-\epsilon \tilde{n}_1 \ln(n_0/\tilde{n}_0)$ is of the order of ϵ^2, due to the spatial variations of n_0, which can be seen by writing $n_0 = \tilde{n}_0 + \epsilon \delta n_0$,

$$-\epsilon \tilde{n}_1 \ln \frac{n_0}{\tilde{n}_0} = -\epsilon^2 \frac{\tilde{n}_1 \delta n_0}{\tilde{n}_0} + O(\epsilon^3).$$

Next, we expand the Hamiltonian and the Poisson bracket: the Hamiltonian is

$$H = \epsilon \int d^2x \tilde{n}_0 \left[\frac{|\nabla \Delta^{-1}(q_1 \tilde{n}_0 + q_0 \tilde{n}_1)|^2}{2} + \frac{|\nabla \Delta^{-1} D_1|^2}{2} + \frac{T}{2M} \frac{\tilde{n}_1^2 - 2\tilde{n}_1 \delta n_0}{\tilde{n}_0^2} \right] + O(\epsilon^2),$$

since $\int d^2x \tilde{n}_0 [\Delta^{-1}(q_1 \tilde{n}_0 + q_0 \tilde{n}_1), \Delta^{-1} D_1] = 0$ and the Poisson bracket is

$$\{F,G\} = \frac{1}{\epsilon^2} \int d^2x (\nabla F_{D_1} \cdot \nabla G_{\tilde{n}_1} - \nabla F_{\tilde{n}_1} \cdot \nabla G_{D_1}) - \frac{1}{\epsilon} \int d^2x \left(\frac{G_{q_1}}{\tilde{n}_0} \nabla F_{D_1} \cdot \nabla q_1 - \frac{F_{q_1}}{\tilde{n}_0} \nabla G_{D_1} \cdot \nabla q_1 - q_1 \left[\frac{F_{q_1}}{\tilde{n}_0}, \frac{G_{q_1}}{\tilde{n}_0} \right] - q_1 [F_{D_1}, G_{D_1}] \right) + O(\epsilon^0).$$

Thus the dynamics emerges to leading order at ϵ^{-1} (which gives the dynamics on a time-scale of order ϵ) and to next order at ϵ^0 (whose influence happens on a time-scale of order one).

First we study the dynamics given by the leading order. It should be noticed that q_1 is constant, since the leading order Poisson bracket does not contain any functional derivatives with respect to q_1, and that

$$\dot{\tilde{n}}_1 = \frac{1}{\epsilon^2}\Delta H_{D_1} + O(\epsilon)$$
$$= -\frac{\tilde{n}_0}{\epsilon}D_1 + [\tilde{n}_1, \Delta^{-1}(q_1\tilde{n}_0 + q_0\tilde{n}_1)] - \nabla \cdot (\tilde{n}_1 \nabla \Delta^{-1} D_1) + O(\epsilon),$$

$$\dot{D}_1 = -\frac{1}{\epsilon^2}\Delta H_{\tilde{n}_1} + \frac{1}{\epsilon}\nabla \cdot \left(\frac{H_{q_1}}{\tilde{n}_0}\nabla q_1\right) + \frac{1}{\epsilon}[H_{D_1}, q_1] + O(\epsilon).$$

If we impose the following constraints on the initial conditions:

$$\Delta H_{\tilde{n}_1} = 0,$$

$$\Delta H_{D_1} = 0,$$

then these constraints are preserved by the leading order flow. These constraints are equivalent to the following:

$$D_1 = 0,$$

$$\frac{T}{M}\frac{\Delta \tilde{n}_1}{\tilde{n}_0} - q_0\tilde{n}_0(q_0\tilde{n}_1 + q_1\tilde{n}_0) = 0.$$

Note that, from expanding $n(\mathbf{x},t) = n_0(\epsilon\mathbf{x}) + \epsilon n_1(\mathbf{x},t)$ about $\epsilon=0$, it follows that δn_0 is a linear function of $\mathbf{x}$ and, as a consequence, it does not appear in the equations for the constraints. A generalization to the case of non-harmonic δn_0 is, however, possible. Even if we neglect higher order terms (ϵ^2 in the Hamiltonian and ϵ^0 in the Poisson bracket), these constraints are not preserved by the Poisson bracket. Therefore, these quantities are approximately conserved on a time scale of order ϵ. Next, we approximate the dynamics on a time scale of order 1 by inserting the constraints on D_1 and n_1 into the second order Poisson bracket. By dropping all dependence on D_1 and n_1, the dynamics is thus equivalently given by the Hamiltonian

$$H_1 = \int d^2x \frac{\tilde{n}_0}{2}\left(-(q_0\tilde{n}_1 + q_1\tilde{n}_0)\Delta^{-1}(q_0\tilde{n}_1 + q_1\tilde{n}_0) + \frac{T}{M}\frac{\tilde{n}_1^2 - 2\tilde{n}_1\delta n_0}{\tilde{n}_0^2}\right),$$

where $\tilde{n}_1$ is a function of q_1 given by

$$\tilde{n}_1 = -\frac{\tilde{n}_0}{q_0}\left(1 - \frac{T}{M\omega_c^2}\Delta\right)^{-1} q_1,$$

and the Poisson bracket

$$\{F,G\}_1 = \int d^2x q_1\left[\frac{F_{q_1}}{\tilde{n}_0}, \frac{G_{q_1}}{\tilde{n}_0}\right], \tag{10}$$

which satisfies the properties of a Poisson bracket—in particular, the Jacobi identity. Using this condition on n_1, the Hamiltonian H_1 can be rewritten as

$$H_1 = \frac{T\tilde{n}_0}{2Mq_0^2}\int d^2x\left[q_1\left(1 - \frac{T}{M\omega_c^2}\Delta\right)^{-1} q_1 - 2\lambda\left(1 - \frac{T}{M\omega_c^2}\Delta\right)^{-1} q_1\right], \tag{11}$$

where λ contains the spatial variations of the equilibrium density as follows:

$$\lambda = -\frac{q_0}{n_0}\delta n_0(\mathbf{x}).$$

Using the symmetry of the operator $(1-(T/M\omega_c^2)\Delta)^{-1}$, Hamiltonian (11) can be rewritten as

$$H_1 = \frac{T\tilde{n}_0}{2Mq_0^2}\int d^2x(q_1 - \lambda)\left(1 - \frac{T}{M\omega_c^2}\Delta\right)^{-1}(q_1 - \lambda). \tag{12}$$

Up to some constants, the Poisson bracket (10) and the Hamiltonian (12) are indeed the same as those presented in Ref. 27. Thus we provided a derivation process that leads to dynamics, on time scales of order 1, which is still generated by a Hamiltonian and a Poisson bracket.

IV. CONCLUSION

An important issue in the derivation of reduced models for plasma physics is avoiding the introduction of fake dissipative terms, which may result from uncontrolled approximations and truncations in the derivation process. In particular, if the parent model has a Hamiltonian structure, we argue that the final reduced model should also have a Hamiltonian structure.

In this paper we examined, in this spirit, the case of the CHM equation. In particular we showed how the fundamental elements, i.e., the Hamiltonian functional and the Poisson bracket, of the Hamiltonian formulation of the CHM equation, emerge from the Hamiltonian structure of a parent model, which is the starting point of the derivation commonly adopted in the plasma physics literature. The appearance of the Hamiltonian and the bracket of the CHM equation in the derivation process was seen to be facilitated by adopting the new set of variables $(q, \tilde{n}, D)$. In terms of these variables, the part of the bracket of the parent model that becomes the CHM bracket can be easily identified. Indeed, what our paper shows is how the ordering adopted in the derivation is able to reduce the bracket of the parent model to the CHM bracket, without compromising the fundamental properties of a Poisson bracket, such as for instance the Jacobi identity. A further new element of our analysis is the way the plasma compressibility is treated. Without invoking the drift approximation and the polarization drift, the divergence-free condition on the plasma velocity appears as a solution for the variable D_1 on a time scale of the order of ϵ. Such a solution is used in order to approximate the dynamics of order 1, assuming for such dynamics that an equilibrium solution exists. A similar argument is used for the dynamics of $\tilde{n}_1$, which at the lowest order is constrained to be a function of q_1 or, more precisely, to be proportional to the plasma stream function.

We believe that the method adopted in this paper is a framework for deriving the Hamiltonian structure in other reduced models of plasma physics, and for deriving new models while avoiding the risk of introducing fake dissipative terms.

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