Journal of Physics: Conference Series **169** (2009) 012006 doi:10.1088/1742-6596/169/1/012006

Thoughts on brackets and dissipation: old and new

P J Morrison

Department of Physics and Institute for Fusion Studies, The University of Texas at Austin, Austin, TX 78712

E-mail: `morrison@physics.utexas.edu`

Abstract.

Bracket formulations of two kinds of dynamical systems, called incomplete and complete, are reviewed and developed, including double bracket and metriplectic dynamics. Dissipation based on the Cartan-Killing metric is introduced. Various examples of incomplete and complete dynamics are discussed, including dynamics associated with three-dimensional Lie algebras.

1. Introduction

The idea of extending the Hamiltonian or Lagrangian framework to include dissipation dates to Rayleigh, but active interest in this idea was renewed in 1981. At that time, Allan Kaufman had the idea to formulate dissipation in terms of brackets with some algebraic properties, an extension of the work on noncanonical Poisson brackets that I had begun a couple of years earlier with John Greene [1, 2]. Allan invited me to join him, and this resulted in a paper on quasilinear theory [3], which unfortunately is the only paper I have published with Allan. By 1984 we had gone our separate ways on this topic ([4, 5, 6, 7, 8]). Others became interested too (e.g. then [9] and afterwards [10, 11]) and, in particular, in two papers [12, 13] the double bracket for describing a particular kind of dissipation was introduced. In this work I will review some of the above, and describe and develop a few old ideas I had back in the mid 1980's, but never published.

The point of a bracket formulation is to imbue a dynamical system with certain structural properties that are generally of a geometrical flavor. For example, the ordinary Poisson bracket ensures the invariance of a closed, nondegenerate two-form, and the geometrical ramifications, such as the existence of the Poincare' invariants, that follow from this. Dynamical systems theory is a huge endeavor, encompassing mappings, a variety of ordinary and partial differential equations, and other more exotic equations. Here, encompassing statements applicable to finite and infinite-dimensional systems are sought, and thus the discussion need be of a general and formal nature. Two kinds of systems are described: incomplete and complete, a terminology similar to but different from the open, closed, and isolated systems of thermodynamics. Incomplete systems are ones that are not complete, but of special interest here are systems described by gradient dynamics of some type, while complete systems are described by metriplectic dynamics, a terminology introduced in [7]. Because incomplete and complete systems may have a Hamiltonian component, in the remainder of this introduction some features of Hamiltonian systems are described first, as needed for the discussions of incomplete and complete systems that follow.

Hamiltonian systems serve as the archetype of energy conserving nondissipative systems, and can be viewed as a dynamical extension of the first law of thermodynamics. Here the Hamiltonian will be a time-independent function, $H(z)$, that can be identified with energy. The Hamiltonian dynamics considered will be noncanonically Hamiltonian, which for finite-dimensional systems means it has the form

$$\dot{z}^i = [z^i, H] = J^{ij}\frac{\partial H}{\partial z^j}\,, \qquad i,j = 1,2,\ldots,m\,, \tag{1}$$

where repeated up and down indices are summed, $z = (z^1, z^2, \ldots, z^m)$ denotes phase space coordinates, and the Poisson bracket is defined by

$$[f,g] = \frac{\partial f}{\partial z^i} J^{ij}\frac{\partial g}{\partial z^j}\,, \qquad i,j = 1,2,\ldots m\,, \tag{2}$$

for some cosymplectic form, $J(z)$, is a Lie algebra realization over $\mathbb{R}$, i.e. it is (i) bilinear, (ii) antisymmetric, and (iii) satisfies the Jacobi identity. Much has been written about this kind of dynamics and the interested reader is referred to [14, 15, 16]. Canonical Hamiltonian systems are the special case where $J = J_c$ with

$$J_c = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}\,, \tag{3}$$

where $n = m/2$ denotes the number of degrees of freedom, 0_n denotes an $n\times n$ block of zeros, and I_n is the $n \times n$ identity.

Associated with noncanonical Hamiltonian systems are two kinds of invariants: those that commute with the particular H of interest, which will be denoted by $\mathcal{P} = \{P_1(z), P_2(z), \ldots\}$, where each P_i satisfies $[\mathcal{P}, H] = 0$, and a special class of invariants, distinct from $\mathcal{P}$, that are built into the phase space. These invariants, called Casimir invariants and denoted by $\mathcal{C} = \{C_1(z), C_2(z), \ldots\}$, satisfy $[\mathcal{C}, f] \equiv 0$ for any phase space function f. Thus they are invariants for any Hamiltonian. Casimir invariants require $\det J \equiv 0$, while $H \in \mathcal{P}$ is always the case.

For canonical Hamiltonian systems, a phase space point z_e is an equilibrium point iff it satisfies $\partial H/\partial z^i|_{z_e} = 0$ for all $i = 1,2,\ldots n$. For noncanonical Hamiltonian systems, the Hamiltonian is not unique because $F := H + \sum_i \lambda_i C_i$ produces the same equations of motion for any $\lambda_i \in \mathbb{R}$ when inserted in the Poisson bracket. However, different equilibria are obtained from $\partial F/\partial z^i|_{z_e} = 0$ for all $i = 1,2,\ldots n$, for different choices of the λ_i.

Equations that describe macroscopic media in terms of Eulerian variables are noncanonically Hamiltonian systems. This includes Euler's equations for fluid motion, magnetohydrodynamics, the Vlasov equation, etc. (see e.g. [15, 17] and many references therein).

The *Incomplete systems* of interest here dissipate energy and possibly other physical quantities. Energy, $E(z)$, leaves the system at a rate prescribed at each phase space point z. Thus, $\dot{E}(z) \leq 0$ for all z. Systems governed by gradient flows (see e.g. [18]), which for finite m-dimensional systems have the form

$$\dot{z} = -\nabla_z E\,, \tag{4}$$

where $\nabla_z = (\partial/\partial z^1, \partial/\partial z^2, \ldots, \partial/\partial z^m)$, are incomplete systems. Evidently,

$$\dot{E} = -|\nabla_z E|^2 \leq 0\,, \tag{5}$$

and so by Lyapunov's theorem, gradient flows have built-in asymptotic stability, i.e. a temporal relaxation to an equilibrium point z_e that satisfies $\partial E/\partial z^i|_{z_e} = 0$. This follows because E serves as a Lyapunov function.

Stokes flow, which describes motion in the low Reynolds number limit of the the Navier-Stokes equation, is a physical example of an infinite-dimensional gradient flow. Other examples of infinite-dimensional gradient flows are Cahn-Hilliard systems [19] and Ricci flows [20, 21], which are nonlinear diffusion-like equations. By Serrin's theorem [22], the asymptotic stability of Stokes flow also occurs for the full nonlinear Navier-Stokes equation, provided the Reynolds number is small enough, yet this system, because it has a Hamiltonian component, is not a gradient flow. This suggests that there may be some more general structure lurking. Indeed, this is the subject matter of Sec. 2, where generalizations of gradient flows are described.

As basic theories, incomplete systems are incomplete: they do not properly account for the lost energy, which would require coupling to the outside dynamics. The system's loss is the outside's gain, and the gain is not accounted for in a dynamical sense, but just assumed to vanish in a prescribed way. They are also incomplete because they do not describe the dynamics of both energy and entropy. One or the other does not enter into the formulation in a physically precise manner. Nevertheless, such systems that do not conserve energy or ignore entropy (or vice versa) are quite useful in engineering and practical applications, the Navier-Stokes equation being a case in point. Also, dissipative dynamics can be useful for proving theorems and constructing numerical algorithms.

Complete systems conserve energy and produce entropy. They can be viewed as dynamical extensions of thermodynamics that possess built-in structure emblematic of the first and second laws of thermodynamics and, as such, possess both Hamiltonian and dissipative components. The first law of energy conservation is embodied in the requirement that H be conserved under the full dynamics. It is also assumed that the set $\mathcal{P}$ is conserved. In addition, complete systems are assumed to possess an entropy function, $S(z)$, that satisfies $\dot{S}(z) \geq 0$ for all z, which is an embodiment of the second law. Thus the system may produce entropy, but it is possible there could be motions where entropy remains constant.

As defined, complete systems do not exchange entropy, energy, or other physical quantities with the outside. Indeed one could say there is no outside. For partial differential equations this would mean, depending on the topology of the domain, the existence of certain boundary conditions at infinity. However, to avoid being overly restrictive for infinite degree-of-freedom systems (e.g. partial differential equations) this is relaxed to allow for finite boundaries by only requiring local conservation of all physical quantities except those that are entropy-like. With this alteration of the definition, boundary fluxes of $\mathcal{P}$ may exist and entropy could also traverse the boundary. Archetype complete systems are the Boltzmann equation and the Vlasov equation with collisions [4, 5, 7], which conserve mass, momentum and energy, but produce entropy.

Complete systems with no boundary fluxes are kindred dynamical extensions of the isolated systems of thermodynamics, systems that do not exchange heat, work, or matter with an environment. The closed systems of thermodynamics exchange energy but not matter with an environment. Complete systems can accomplish this by suitable boundary conditions on $\mathcal{P}$ and the allowance of boundary entropy fluxes. Similarly, complete systems can embody the open systems of thermodynamics that allow, in addition to heat and work, matter fluxes. Incomplete systems do not naturally line up with ideas from thermodynamics because either entropy or energy dynamics is missing in the formulation.

In Sec. 3 metriplectic dynamics, the paradigmatic complete system, is discussed. As stated, metriplectic systems possess a dynamical first law of conservation of energy and a second law of entropy production. It will be seen that the Casimir invariants of the noncanonical Hamiltonian description are candidate entropy functions and the Hamiltonian plays the role of the thermodynamic internal energy. A shortcoming of metriplectic formulation is the apparent stitching together of the Hamiltonian and dissipative parts. An attempt to build a more unified structure is considered in Sec. 4, where a special kind of dissipation based on the Cartan-Killing metric of a Lie algebra is introduced. This metric can be utilized in the context of incomplete

Journal of Physics: Conference Series **169** (2009) 012006 doi:10.1088/1742-6596/169/1/012006

or complete systems. Some special cases corresponding to the three-dimensional Lie algebras are worked out. Finally, in Sec. 5, we conclude and summarize.

2. Incomplete Systems

To begin with, consider the general finite-dimensional incomplete system of the form

$$\dot{z}^i = [z^i, H] + \mathcal{D}^i(z)\,, \qquad i = 1, 2, \ldots, m\,, \tag{6}$$

where $\mathcal{D}$ represents dissipation of some kind. Analogous infinite-dimensional systems are written as

$$u_t = [u, H] + \mathcal{D}(u)\,, \tag{7}$$

where $u(x,t)$ depends on independent variables $x \in \Omega$ and $t \in \mathbb{R}$, and

$$[F, G] = \int_\Omega dx \frac{\delta F}{\delta u} \mathcal{J} \frac{\delta G}{\delta u}\,, \tag{8}$$

where $\mathcal{J}$ is a cosymplectic operator and $\delta F/\delta u$ denotes the function derivative of the arbitrary functional F with respect to the dependent variable u (see e.g. [14, 15]). It is of interest to construct 'collision operators' i.e. dissipative terms $\mathcal{D}$ that relax to steady states for future times, $t \geq 0$. In Sec. 2.1 one way to do this is shown [23], while in Sec. 2.2 the double bracket formulation of [12, 13] is described.

2.1. Constructing H-theorems: general projection

Consider a partial differential equation of the form (7). It is not difficult to construct dissipative terms $\mathcal{D}$ that formally relax to solutions associated with the Hamiltonian system without the dissipation. The Boltzmann H-theorem serves as a guide. Because the Hamiltonian part conserves $\mathcal{P} = \{P_1(z), P_2(z), \ldots\}$, the quantity

$$E := \sum_i \lambda_i P_i\,, \tag{9}$$

where the $\lambda_i \in \mathbb{R}$ are arbitrary, satisfies the following when $\mathcal{D} \neq 0$:

$$\frac{dE}{dt} = \int_\Omega dx\, \frac{\delta E}{\delta u} u_t = \int_\Omega dx\, \frac{\delta E}{\delta u} \mathcal{D}\,. \tag{10}$$

This suggests an obvious choice for $\mathcal{D}$, viz. $\mathcal{D}_1 := -\delta E/\delta u$, which means (7) has the form of a gradient flow and

$$\frac{dE}{dt} = -\int_\Omega dx \left(\frac{\delta E}{\delta u}\right)^2 \leq 0\,. \tag{11}$$

Whence one infers that the system could relax to states that satisfy $\delta E/\delta u = 0$.

As an example consider the KdV equation with dissipation,

$$u_t + u u_x + u_{xxx} = \mathcal{D}_1\,, \tag{12}$$

and recall Kruskal's soliton variational principle $\delta E/\delta u = 0$ with

$$E = \lambda_0 P_0 + \lambda_1 P_1 + \lambda_2 P_2 \tag{13}$$

where

$$P_0 = \int_{\mathbb{R}} dx\, u\,, \quad P_1 = \int_{\mathbb{R}} dx\, \frac{u^2}{2}, \quad P_2 = \int_{\mathbb{R}} dx \left(\frac{u^3}{6} - \frac{u_x^2}{2}\right)\,. \tag{14}$$

Journal of Physics: Conference Series **169** (2009) 012006 doi:10.1088/1742-6596/169/1/012006

Extrema of this version of his principle are the single soliton solutions. Assuming

$$\mathcal{D}_1 = -\eta \frac{\delta E}{\delta u}\,, \tag{15}$$

where η is a parameter that sets the time scale for the relaxation, gives the following equation of motion:

$$u_t + uu_x + u_{xxx} = -\eta\left(\lambda_0 + \lambda_1 u + \lambda_2 \frac{1}{2}u^2 + \lambda_2 u_{xx}\right)\,. \tag{16}$$

Here the first three terms on the right correspond to quadratically non-linear damping and the last is Burger's dissipation. When evaluated on the single soliton solution the right-hand-side vanishes, as of course does the left.

Generalizing a bit, suppose $\mathcal{D}_2 = -\mathcal{L}\delta E/\delta u$, where $\mathcal{L}$ is a non-negative operator, i.e. it satisfies $\langle \phi, \mathcal{L}\phi\rangle = ||T\phi||^2 \geq 0$ for all ϕ for some pairing or inner product. It is easy to construct such operators by simply writing $\mathcal{L} = \mathcal{A}^\dagger\mathcal{A}$ which could be a positive operator, i.e. satisfy $\langle \phi, \mathcal{L}\phi\rangle > 0$ for all $\phi \neq 0$, or it could have some desired degeneracies built-in that result in the conservation of some of the P_i's in the presence of this dissipation. With this choice

$$\frac{dE}{dt} = -\int_\Omega dx\, \left(\mathcal{A}\frac{\delta E}{\delta u}\right)^2 \leq 0\,. \tag{17}$$

Similarly, as a final possibility consider $\mathcal{D}_3 = -\mathcal{A}^\dagger\mathcal{K}\mathcal{A}\delta E/\delta u$, where $\mathcal{K}$ is some other known positive or non-negative operator.

The above constructions suggest a degenerate metric form for dissipation. For finite systems this would be

$$\dot{z}^i = [z^i, H] + (z^i, E)\,, \qquad i = 1, 2, \ldots, m\,, \tag{18}$$

with

$$(f, g) = -\frac{\partial f}{\partial z^i} g^{ij} \frac{\partial g}{\partial z^j}\,, \qquad i, j = 1, 2, \ldots m\,, \tag{19}$$

where the 'metric' $g_{ij}(z)$ is symmetric and non-negative, i.e. it has positive eigenvalues and is possibly degenerate ($\det g = 0$).

There is freedom in two respects with the above construction: one can choose E to obtain desired solutions to which the system relaxes, and one can design the degeneracies of g to preserve certain elements of $\mathcal{P}$. For example, g can be constructed using a projection operator so that for some $P \in \mathcal{P}$, $g^{ij}\partial P/\partial z^j = 0$ for all i. Then if $E = H$, the equation of motion follows from

$$\dot{z}^i = [z^i, E] + (z^i, E)\,, \qquad i = 1, 2, \ldots m\,, \tag{20}$$

and the system can relax to solutions of $\partial(H + P)/\partial z^i|_{z_e} = 0$. Evidently, the gradient flows of Sec. 1 correspond to the case where g is the Kronecker delta, i.e. $g^{ij} = \delta_{ij}$.

2.2. Double bracket dynamics

If the operator $\mathcal{A}$ of Sec. 2.1 above is chosen to be the cosymplectic form, $\mathcal{J}$, then a Casimir preserving relaxation to equilibria z_e that satisfy $\partial F/\partial z^i|_{z_e} = 0$ is obtained. This nice idea was introduced in [12, 13], and this special form of dissipative bracket is now referred to as the double bracket, which for finite-dimensional systems has the form

$$\{\{f, g\}\} = \sum_{k=1}^{m} J^{ik} J^{jk} \frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^j}\,. \tag{21}$$

The equations of motion follow from

$$\dot{z}^i = [z^i, F] + \{\{z^i, F\}\}\,, \qquad i = 1, 2, \ldots m\,, \tag{22}$$

where [,] is a noncanonical Poisson bracket with Casimir invariants $\mathcal{C}$ and $F = H + \mathcal{C}$ (a notation that means F is the sum of H plus a linear combination of elements of $\mathcal{C}$). Evidently,

$$\dot{\mathcal{C}} = \{\{\mathcal{C}, F\}\} = 0\,, \tag{23}$$

for any $\mathcal{C}$, and

$$\dot{F} = \{\{F, F\}\} = \{\{H, H\}\} \leq 0\,. \tag{24}$$

As an example, consider again the KdV equation which is noncanonically Hamiltonian with the Gardner bracket [24],

$$[F, G] = -\int_\Omega dx \frac{\delta F}{\delta u}\frac{\partial}{\partial x}\frac{\delta G}{\delta u}\,, \tag{25}$$

and the Hamiltonian P_2 of (14). The only Casimir of the Gardner bracket is P_0 of (14). The double bracket based on (25) is

$$\{\{F, G\}\} = \int_\Omega dx \frac{\delta F}{\delta u}\frac{\partial^2}{\partial x^2}\frac{\delta G}{\delta u} \tag{26}$$

and the resulting equation of motion is

$$u_t + uu_x + u_{xxx} = \left(\frac{\delta H}{\delta u}\right)_{xx} = (uu_x + u_{xxx})_{xx} \tag{27}$$

When $\Omega = \mathbb{R}$ and the boundary conditions $u = 0$ at $\pm$ infinity is assumed, this results in the trivial solution.

However, in the case of the ideal fluid, the situation is much richer and this double bracket construction has been used as a means of numerically obtaining equilibrium solutions that preserve the Casimir invariants for Euler's fluid equations [13] and for calculating the V-states of contour dynamics [17]. It is noted in passing that the double bracket formalism has been explored in a deeper and quite interesting sense in the context of the Toda lattice [25].

3. Complete Systems

Metriplectic dynamics [7], a systemization and formalization of the work of Allan Kaufman, the author, and others, describes complete systems. As stated in Sec. 1, it is a dynamic generalization of thermodynamics that embodies both the first law of conservation of energy and a second law of internal entropy production. Because of this, it could be argued that it is a paradigm for the most basic classical dynamics. The structure of metriplectic dynamics is similar to that above for incomplete systems, in that metriplectic dynamics possess both Hamiltonian and dissipative components, but the dynamics is generated by both a Hamiltonian and an entropy function and the metric is designed to have degeneracies of a particular type.

The metriplectic equation of motion has the form

$$\dot{z}^i = \{z^i, F\}_M = [z^i, F] + (z^i, F)_M \tag{28}$$

where [,] is a noncanonical Poisson bracket and $F = H + S$, where S is the entropy function. As stated in Sec. 1, Casimir invariants are candidate entropies, i.e. S is chosen from $\mathcal{C}$, and this choice determines the equilibrium state z_e. Given that Casimir invariants are related to relabeling symmetries [26, 27], and thus a counting of states, it is not too surprising that entropies should come from $\mathcal{C}$. As before, $[f, \mathcal{C}] = 0$ implies

$$[f, S] = 0 \qquad \text{for all f}\,, \tag{29}$$

and, as in (19), dissipation is described by the bracket

$$(f, g)_M = \frac{\partial f}{\partial z^i} g_M^{ij} \frac{\partial g}{\partial z^j}\,, \qquad i, j = 1, 2, \ldots m\,, \tag{30}$$

but with a sign change and the following additional requirement:

$$(f, \mathcal{P})_M = 0 \qquad \text{for all f}\,. \tag{31}$$

Thus upon identifying H with the energy and the entropy S with an element of $\mathcal{C}$, the dynamical embodiment of the first and second laws are as follows:

$$\dot{H} = \{H, F\}_M = [H, F] = 0 \qquad \text{and} \qquad \dot{S} = \{S, F\}_M = (S, S)_M \geq 0\,. \tag{32}$$

As noted above, the brackets for this have been worked out for many systems, including kinetic theories [4, 5, 7, 8], the compressible Navier-Stokes equation [6], and other systems [9, 10, 11]. Here the case of the free rigid body as given in [7] is briefly reviewed.

The free rigid body noncanonical Poisson bracket has Lie-Poisson form (e.g. [15, 16]), denoted $[f, g]_{LP}$, which means its cosymplectic form has the form

$$J^{ij} = c_k^{ij} z^k\,, \tag{33}$$

with c_k^{ij} being the structure constants of some Lie algebra. For classical theories this is generally a real Lie algebra. If $\{e^1, e^2, \ldots, e^m\}$ denotes an m-dimensional Lie algebra basis, then the structure constants satisfy

$$[e^i, e^j]_{Lie} = c_k^{ij} e^k\,. \tag{34}$$

where $[\,,\,]_{Lie}$ is the Lie algebra nonassociative product. (Note, for consistency our index placement is dual to what is typical.) It can be shown that the Jacobi identity for the Poisson bracket defined by (33) follows from that for $[\,,\,]_{Lie}$.

For the rigid body [7], the Lie algebra is $so(3)$ with $c_k^{ij} = \epsilon_{ijk}$, the Levi-Civita symbol, and the coordinates $z = (z^1, z^2, z^3)$ correspond to the three components of the body (principal axes) frame angular momenta. The Hamiltonian is the rotational kinetic energy

$$H = \frac{1}{2} A_{ij} z^i z^j \qquad i, j = 1, 2, 3\,, \tag{35}$$

where $A = \mathrm{dia}(1/I_1, 1/I_2, 1/I_3)$ is a diagonal matrix that depends on the principal moments of inertia, I_i. The Casimir invariant is proportional to the square of the angular momentum

$$C = \frac{1}{2} \delta_{ij} z^i z^j \qquad i, j = 1, 2, 3\,, \tag{36}$$

which serves as the entropy S, and the bracket $(f, g)_M$ is given by

$$(f, g)_M = -\sum_{i,j,k=1}^{3} \left[\frac{\partial H}{\partial z^i} \frac{\partial H}{\partial z^j} - \delta_{ij} \frac{\partial H}{\partial z^k} \frac{\partial H}{\partial z^k} \right] \frac{\partial f}{\partial z^i} \frac{\partial g}{\partial z^j}\,, \tag{37}$$

where we have scaled out η, the parameter that sets the time scale for relaxation. Observe (37) is a simple projection designed so that $\partial H/\partial z$ is a null eigenvector of g_M.

4. Cartan-Killing Dissipation

Metriplectic dynamics and its concomitant metriplectic manifold defined in [7] have an unnatural quality: despite the complementary alignment of the degeneracies of J and g, there is no deep intrinsic gemetrical connection between the Hamiltonian and dissipative components. One would hope for a structure with both components arising out of some common geometrical principle. As it stands, metriplectic dynamics has a clumsy and piecemeal quality with its brute force projection method, as evidenced by the lack of tensorial form of (37). For incomplete systems the double bracket formalism addresses this deficiency somewhat, in that the dissipation relies in an essential way on the cosymplectic form. However, (21) also lacks tensorial form and has an unnatural quality.

An origin of the problem is that symplectic manifolds have no natural metric, while for dissipation described by a gradient flow a metric is the essential ingredient. Obviously (31) would benefit from a natural metric. However, in addition, both the projection of (37) for complete systems and the double bracket expression of (21) for incomplete systems can be viewed as incorporating a Euclidean metric, and if a natural metric were available then they could be replaced by the following:

$$(f,g)_M = -\frac{\partial H}{\partial z^m}\frac{\partial H}{\partial z^n}\left[g^{mi}g^{nj} - g^{nm}g^{ij}\right]\frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^j}\,, \qquad (38)$$

$$\{\{f,g\}\} = J^{in}g_{mn}J^{jn}\frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^j}\,. \qquad (39)$$

Noncanonical Hamiltonian systems with Lie-Poisson brackets arise from a reduction based on symmetry (e.g. [15, 16]) and consequently are described by a particular Lie algebra. Thus, the idea comes to mind to incorporate the trace form used by Cartan to classify Lie algebras as a natural metric for dissipation. The trace form, sometimes called the Cartan-Killing metric, is defined in coordinates by

$$g^{ij}_{CK} = c^{in}_{m}c^{jm}_{n}\,, \qquad (40)$$

where the structure constants $c^{ij}_k \in \mathbb{R}$ are defined as in Sec. 3. So, a dissipation naturally mated to a Lie-Poisson bracket appears to be

$$(f,g)_{CK} = \frac{\partial f}{\partial z^i}g^{ij}_{CK}\frac{\partial g}{\partial z^j}\,, \qquad i,j = 1,2,\ldots m\,. \qquad (41)$$

This choice comes, so to speak, 'as is'. Because it is essentially a kinematic construction, there is no a priori guarantee that $(f,H)_{CK} = 0$ for all f. Thus a bracket of the form $\{f,g\} = [f,g]_{LP} + (f,g)_{CK}$ might not work, and for a metriplectic formulation one would need to use a bracket of the form of (38). Similarly, in general $(f,H)_{CK} \neq 0$ for all f, and so for an incomplete description of dissipation one might need to use (39). In any event, both these constructions seem to be superior.

Another problem with (41), a potential major problem, is that the eigenvalues of g_{CK} can have either sign. Because g_{CK} may also have zero eigenvalues, there could be expanding, contracting, and null directions, depending on the choices of H and S. It is possible that this seeming disadvantage could be turned into an advantage in selecting out preferential systems that possess the desired degeneracies, in which case, $\{f,g\} = [f,g]_{LP} + (f,g)_{CK}$ would work.

For semisimple Lie algebras, $\det g \neq 0$ and an inverse, g^{CK}_{jk}, can be defined, i.e.

$$g^{ik}_{CK}g^{CK}_{kj} = \delta^i_j\,. \qquad (42)$$

The standard quadratic Casimir invariant is given in terms of this inverse as follows:

$$C = \frac{1}{2}g^{CK}_{ij}z^i z^j\,. \qquad (43)$$

Journal of Physics: Conference Series **169** (2009) 012006 doi:10.1088/1742-6596/169/1/012006

If the Lie algebra is compact as well as semisimple then g_{CK} generates a gradient flow. In this case there are no degeneracies in g_{CK} and consequently the system relaxes without preserving an H nor an S.

Darboux's theorem of geometric mechanics reveals that all symplectic manifolds are equivalent in the sense that they all can be described locally in terms of canonical coordinate systems. In this vein, there is a similar structure theorem for metriplectic manifolds and double bracket dissipation with a Lie-Poisson bracket and Cartan-Killing dissipation. In the classification of Lie algebras, it is known that a basis exists such that the CK metric can be written in standard form where $g_{CK} = dia(\epsilon_1, \epsilon_2, \ldots,)$ with $\epsilon_i \in \{-1, 0, 1\}$. A linear change of phase space coordinates, $\bar{z}^i = A^i_j z^j$ induces a basis change for the Lie algebra, which amounts to a usual tensorial change of coordinates for the structure constants and the Cartan-Killing metric. Thus, the work of Cartan and others can be carried over directly to give a structure theorem.

To illustrate some of the ideas above, consider the Lie algebras of dimension three [28]. The designation used here will be the same as that used for the homogeneous Bianchi cosmologies [29]. There are nine real three-dimensional Lie algebras, but only Types III, VIII, and IX will be considered for the purpose of demonstration. The structure constants, for these three cases are as follows:

(i) Type IX $c^{ij}_k = \epsilon_{ijk}$,
(ii) Type VIII $c^{23}_1 = -c^{32}_1 = -1\,, c^{31}_2 = -c^{13}_2 = 1\,, c^{12}_3 = -c^{21}_3 = 1$, otherwise 0,
(iii) Type III $c^{13}_1 = -c^{31}_1 = +1$ otherwise 0.

Type IX is $so(3)$ and Type VIII is $sl(2,1)$, which arises in the context of a vortex reduction of the Hamiltonian description of fluid mechanics, e.g. the Kida vortex [30, 31]. The cosymplectic forms for the Lie-Poisson brackets for these algebras are

$$J_{IX} = \begin{pmatrix} 0 & z^3 & -z^2 \\ -z^3 & 0 & z^1 \\ z^2 & -z^1 & 0 \end{pmatrix}, \tag{44}$$

$$J_{VIII} = \begin{pmatrix} 0 & -z^3 & -z^2 \\ z^3 & 0 & z^1 \\ z^2 & -z^1 & 0 \end{pmatrix}, \tag{45}$$

$$J_{III} = \begin{pmatrix} 0 & 0 & z^1 \\ 0 & 0 & 0 \\ -z^1 & 0 & 0 \end{pmatrix}, \tag{46}$$

which have the Casimir invariants

$$C_{IX} = (z^1)^2 + (z^2)^2 + (z^3)^2\,, \tag{47}$$

$$C_{VII} = (z^1)^2 + (z^2)^2 - (z^3)^2\,, \tag{48}$$

$$C_{III} = f(z^1, z^2)\,, \tag{49}$$

respectively. The Cartan-Killing metrics, g_{CK}, for these algebras are

$$g_{IX} = -2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{50}$$

$$g_{VIII} = -2\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{51}$$

$$g_{III} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{52}$$

Note, Type IX is compact and semisimple, Type VIII is semisimple and noncompact, while Type III is not semisimple. To within a scale factor these are already in the standard form described above.

Because g_{IX} is the Euclidean metric, Type IX metriplectic dynamics and double bracket dynamics are exactly as before. It is, however, somewhat interesting to see what kind of dissipation is generated by the standard Casimir invariant of (43) when it is inserted into $(f,g)_{CK} = (f,g)_{IX}$, the dissipative bracket for this case. This gives

$$(z^i, C_{IX})_{IX} = g_{IX}^{ij} g_{jk}^{IX} z^k = z^i \,, \tag{53}$$

which corresponds to isotropic linear damping or growth. Note, this holds true for any compact semisimple Lie Group of any dimension.

The lack of definiteness for Type IIV is problematic if one seeks relaxation. However, if one chooses an S independent of z^1, then partial relaxation to the $z^2 - z^3$ plane is possible. Or, conversely, one could choose S independent of z^2 and z^3 and consider a relaxation to the z^1 axis. If such an S does not appear to have a mathematical or physical origin, then this is not too appealing.

Because of the degeneracy, Type III is more interesting. From (52) and (49), it is seen that $\partial C_{III}/\partial z$ is a null eigenvector of g_{III}. Consequently, the dynamics

$$\dot{z}^i = [z^i, F]_{III} + (z^i, F)_{III} \,, \tag{54}$$

with $F = H + C_{III}$, satisfies $\dot{C} = 0$, $\dot{F} = \dot{H}$ and

$$\dot{H} = (H,H)_{III} \leq 0 \,. \tag{55}$$

Thus, this is an alternative to double bracket dynamics for incomplete systems.

One could continue and construct metriplectic dynamics for complete systems by projection etc., but this will not be pursued further now.

5. Summary and Conclusions

In this work, a variety of bracket formulations of dynamical systems have been described, with a division into the two categories of incomplete and complete systems. Incomplete systems do not describe in entirety the dynamics of both entropy and energy, but are quite useful for practical situations. Complete systems, as described by metriplectic dynamics, are a dynamical extension of thermodynamics that describes energy conservation and entropy production dynamics. An new kind of dissipation based on the Cartan-Killing metric was introduced and described for examples associated with Lie algebras of dimension three. The results about Cartan-Killing dissipation are preliminary: only the surface has been scratched and there are many directions in which one could proceed. This might be the subject of a future publication.

Journal of Physics: Conference Series **169** (2009) 012006 doi:10.1088/1742-6596/169/1/012006

Appendix A. Reminiscence

For many decades Allan Kaufman has been a most important presence in plasma physics. He has a widespread reputation for crisp, clear, and beautiful physics at the forefront of plasma science, with a theoretical physics flavor that belies his training at the University of Chicago under Gregor Wentzel. Allan has also had a profound influence by advising generations of students and by encouraging other young physicists. I personally owe him a debt of gratitude and I am glad to have this opportunity to express it and to reminisce a bit.

In the Fall of 1979 I moved to Princeton to accept a post doctoral position and while there I was most fortunate to eventually work with John Greene, another giant of our field. But, this did not happen right away. After a month or two at Princeton, I summoned the courage to poke my head into John's office and utter something that had been on my mind for a while, namely, that I thought MHD was a Hamiltonian theory. I am not exactly sure of the origin of this idea, but I had studied Hamiltonian dynamics as a graduate student and knew that this Berkeley plasma physics professor by the name of Kaufman had an interest in it and its applications to plasma physics. I remember in 1973 I wanted to know more about and possibly even be part of Allan's Berkeley school. In any case, my thought about MHD was met by John's 'go-away' stare – and so I left. Nevertheless, now and then I began toiling away on my own on this idea, totally ignorant of any literature on the subject. A while later, I returned to John and repeated that I thought MHD was a Hamiltonian theory – but added, here is why. John was more receptive this time and let me explain that based on a scaling argument the velocity field could not be a canonical field and that one needed to add a gradient and that this could be seen in Fourier space etc. John's response was, "Oh That." It turned out that I was well on my way to rediscovering the Clebsch representation of fluid mechanics, a potential representation of the velocity field dating from the mid-nineteenth century (see e.g. [22]). This was a big disappointment to me, but as things usually go, not a total loss. John suggested some literature, one thing led to another, and eventually this led to our fairly influential paper [1] on noncanonical brackets for MHD that was submitted to PRL in April of 1980. My Clebsch material was used as one of two means for proving the Jacobi identity by exploiting the connection between the canonical Clebsch variables and the usual but noncanonical Eulerian variables, which John insisted we do prior to publication. I wanted to give some backstory about how this came to pass, since many people have asked me about it and because Allan played a role in this story.

In January of 1980 there was a buzz around the theory wing of the Princeton Plasma Physics Laboratory that one of Allan's students was coming to give a seminar. Allan's students were (and are) viewed as gems: smart, highly polished, articulate, and universally well-educated. Robert Littlejohn came to Princeton and didn't disappoint when he spoke about his dissertation work on guiding center perturbation theory using noncanonical variables [32] that he did under Allan's supervision. Robert gave a sparkling and highly motivating talk. His talk and one of the papers that John suggested to me by Clifford Gardner on the Hamilton description of the KdV equation [24] were the inspiration for our work. Both concern Hamiltonian structure in terms of Poisson brackets that obey the Lie algebraic properties, but the latter was for an infinite-dimensional system and possesses a degeneracy and, hence, what we now call a Casimir invariant. This served as the prototype of what we called the noncanonical Hamiltonian formulation in our paper.

Allan immediately took an active interest in this work and gave me more encouragement than anyone. Many of us early in our careers have doubts and insecurities, and having an esteemed scientist like Allan take an interest was flattering, motivating, and also somewhat overwhelming. Allan's efforts were most important to me, because my style of mathematical physics was uncommon in the plasma community and my paper was met with criticism. Without his help, I may not have continued. Allan also played an active role in the dissemination of this work. Through him, Jerry Marsden and Alan Weinstein became aware of it, and this started

Journal of Physics: Conference Series **169** (2009) 012006 doi:10.1088/1742-6596/169/1/012006

their interest in noncanonical brackets for fluids and plasmas, followed by many others. Without question, because of Allan, my work rapidly cut a much wider swath.

Finally, I would like to mention another debt I owe Allan. Before I left Princeton he nominated me for a fellowship; although I didn't receive it, his action had consequences. First, he sent me a copy of the nomination letter, which was quite flattering. I remember carrying it around, rereading it, unbelieving that he had written such words. Second, he sent it to several senior scientists, and it was evident to me that their attitudes toward me changed because of this. I believe this action of Allan's jump-started my career. Last, at some point this letter got sent to my mother-in-law, an outspoken Cal graduate, and I do believe her attitude toward me also improved after that. So, Allan, thanks again for all you've done, for being a generous and kind mentor, and I wish you the happiest of Happy Birthdays.

Acknowledgment
This work was supported by U.S. Dept. of Energy Contract No. DE-FG05-80ET-53088.

References

[1] Morrison P J and Greene J M 1980 *Phys. Rev. Lett.* **45** 790–793
[2] Morrison P J 1980 *Phys. Lett. A* **80** 383–386
[3] Kaufman A N and Morrison P J 1982 *Phys. Lett. A* **88** 405–406
[4] Kaufman A N 1984 *Phys. Lett. A* **100** 419–422
[5] Morrison P J 1984 *Phys. Lett. A* **100** 423–427
[6] Morrison P J 1984 *Center for Pure and Applied Mathematics Report* **PAM-228** University of California at Berkeley
[7] Morrison P J 1986 *Physica D* **18** 410–419
[8] Turski L A and Kaufman A N 1987 *Phys. Lett. A* **120** 331–333
[9] Grmela M 1986 *Physica D* **21** 179–212
[10] Beris A N and Edwards B J 1990 *J. Rheol.* **34** 55–78
[11] Holm D, Putkaradze V and Tronci C 2008 *J. Phys. A* **41** 344010
[12] Brockett R W 1988 *Proc. IEEE* **27** 799
[13] Vallis G K, Carnevale G and Young W R 1989 *J. Fluid Mech.* **207** 133–152
[14] Morrison P J 1982 *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems* vol 88 ed Tabor M and Treve Y (New York: Am. Inst. Phys.) pp 13–46
[15] Morrison P J 1998 *Rev. Mod. Phys.* **70** 467–521
[16] Marsden J E and Ratiu T S 1999 *Introduction to Mechanics and Symmetry* (New York: Springer)
[17] Morrison P J 2005 *Phys. Plasmas* **12** 058102–1–13
[18] Bloch A M 2003 *Nonholonomic Mechanics and Control* (New York: Springer)
[19] Otto F 2001 *Com. Partial Diff. Eqs.* **26** 101–174
[20] Hamilton R 1982 *J. Diff. Geom.* **17** 255–306
[21] Chow B and Knopf D 2004 *The Ricci Flow: An Introduction* (Providence: American Mathematical Society.)
[22] Serrin J 1959 *Handbuch der Physik* vol VIII, pt. 1 ed Flügge S (Berlin: Springer) pp 25–263
[23] Morrison P J 1985 *Talk presented at Kruskal festshrift, unpublished* (Institute for Theoretical Physics, Santa Barbara, Ca)
[24] Gardner C S 1971 *J. Math. Phys.* **12** 1548–1551
[25] Bloch A M, Flaschka H and Ratiu T 1990 *Duke Math.* **61** 41–65
[26] Padhye N and Morrison P J 1996 *Plasma Phys. Reports* **22** 869–877
[27] Morrison P J 1987 *Zeitschrift f. Naturforschung* **42a** 115–123
[28] Jacobson N 1962 *Lie Algebras* (New York: Interscience)
[29] Ryan M P and Shepley L C 1975 *Homogeneous Relativistic Cosmologies* (Princeton: Princeton University Press)
[30] Ngan K, Meacham S and Morrison P J 1995 *Phys. Fluids* **4** 896
[31] Meacham S P, Morrison P J and Flierl G R 1997 *Phys. Fluids* **9** 2310–2328
[32] Littlejohn R G 1979 *J. Math. Phys.* **20** 2445–2458