J. Plasma Physics (2009), *vol.* 75, *part* 2, *pp.* 145–152.
doi:10.1017/S0022377808007770 Printed in the United Kingdom

Letter to the Editor

Linear superposition of nonlinear waves

SWADESH MAHAJAN[1] and HIDEAKI MIURA[2]

[1] Institute for Fusion Studies, The University of Texas at Austin, Austin, TX 78712, USA
(mahajan@mail.utexas.edu)

[2] Division of Simulation Science, National Institute for Fusion Science, 322-6 Oroshi-cho, Toki, Gifu 509-5292, Japan

(Received 8 October 2008, first published online 11 December 2008)

Abstract. Exact nonlinear (arbitrary amplitude) wave-like solutions of an incompressible, magnetized, non-dissipative two-fluid system are found. It is shown that, in 1-D propagation, these fully nonlinear solutions display a rare property; they can be linearly superposed.

The Alfvénic systems—like magnetohydrodynamics (MHD) and Hall MHD—have the remarkable property that plane-wave-like solutions for fluctuations remain valid for arbitrary levels of perturbations provided that there exist definite relations between the perturbed velocity ($\mathbf{v}$) and magnetic field ($\mathbf{b}$) [1, 2]; these relationships render the nonlinear terms strictly zero so that a linear solution like a plane wave becomes an exact nonlinear solution. It is obvious but nonetheless interesting that the 'dispersion relation' for these arbitrary amplitude waves (to be called linear–nonlinear (LNL) waves) is exactly the same as the linear dispersion relation.

What is even more remarkable is that, under suitable conditions, the LNL waves obey a linear superposition principle, i.e. the sum of two or more fully nonlinear solutions is also a solution. To the best of our knowledge, this property is rarely found in non-trivial physical systems [3].

These rare cases, however, are extremely important and interesting; the most spectacular being the class of fully integrable nonlinear evolution equations which yield 'solitons'—localized, non-dispersive, propagating lumps of energy that not only retain their shape and velocity individually, but also asymptotically revert to the original shape after collisions with other solitons. Solitons, thus, constitute a class of nonlinear solutions that display asymptotic superposition [4].

What happens when a system is not 'completely integrable' or it has not been possible to explicitly demonstrate its integrability (in spite of the existence of very elegant tests for complete integrability)? In a set of extremely interesting recent papers, Babin and Figotin [5] have examined a wide class of systems including nonlinear Schrödinger and Maxwell equations; their main conclusion, stated in their own words, is 'we have discovered that the *superposition principle holds with a high accuracy* for general dispersive nonlinear wave systems provided that the initial data are a sum of generic wave packets'.

In this background it is easy to define the scope of this work. We have analyzed a rather complicated real physical system of two interacting vector fields, and found an exact solution by a non-trivial choice that makes the nonlinearities disappear. We have further shown that this exact solution, under appropriate conditions (1-D propagation), obeys the superposition principle exactly.

We will demonstrate explicitly the linear superposition of LNL waves for a two-fluid plasma—a system more general than MHD or Hall MHD. We begin with deriving LNL waves in a two-fluid plasma of equal but oppositely charged particles. Assuming constant densities, the equations of motion may be cast in the vortex form

$$\frac{\partial \mathbf{\Omega}_\pm}{\partial t} = \nabla \times [(\mathbf{V}_\pm \times \mathbf{\Omega}_\pm)], \tag{1}$$

where

$$\mathbf{\Omega}_\pm = \mathbf{B} \pm \frac{cm_\pm}{q_\pm} \nabla \times \mathbf{V}_\pm, \tag{2}$$

and quasineutrality relates $n_+ q_+ = -n_- q_- \equiv nq$.

The following normalizations lead to a dimensionless system: $\mathbf{B}$ to $\mathbf{B}_0$, $\mathbf{V}$ to $V_\mathrm{A} = B_0/\sqrt{4\pi(n_+ m_+ + n_- m_-)}$, lengths to $\lambda = c/\overline{\omega}_\mathrm{p} = c/\sqrt{4\pi n^2 q^2/(n_+ m_+ + n_- m_-)}$, and t to λ/V_A, an effective inverse cyclotron time. In this notation, $\mathbf{J}$ is normalized to $nq\,V_\mathrm{A}$. The dimensionless vorticities become

$$\mathbf{\Omega}_\pm = \mathbf{B} \pm \mu_\pm \nabla \times \mathbf{V}_\pm, \tag{3}$$

with $\mu_\pm = m_\pm n_\pm/(m_+ n_+ + m_- n_-)$, $\mu_+ + \mu_- = 1$. Now we wish to go over to the one-fluid variables, the mean velocity $\mathbf{V} = (n_+ m_+ \mathbf{V}_+ + n_- m_- \mathbf{V}_-)/(n_+ m_+ + n_- m_-)$, and the current $\mathbf{J} = \mathbf{V}_+ - \mathbf{V}_-$. We find that $\mathbf{V}_\pm = \mathbf{V} \pm \mu_\mp \mathbf{J}$, which, when used in (3), yields for the vorticities

$$\mathbf{\Omega}_\pm = \mathbf{B} + \mu_+ \mu_- \nabla \times \mathbf{J} \pm \mu_\pm \nabla \times \mathbf{V}. \tag{4}$$

The equation of motion may now be rewritten as

$$\frac{\partial \mathbf{\Omega}_\pm}{\partial t} = \nabla \times [(\mathbf{V} \pm \mu_\mp \mathbf{J}) \times \mathbf{\Omega}_\pm]. \tag{5}$$

Splitting the system into the ambient and fluctuating fields: $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \mathbf{V} = \mathbf{v}$, and $\mathbf{J} = \mathbf{J}$, (5) becomes

$$\frac{\partial}{\partial t} \widehat{\mathbf{\Omega}}_\pm = \nabla \times [(\mathbf{v} \pm \mu_\mp \mathbf{J}) \times (\mathbf{B}_0 + \widehat{\mathbf{\Omega}}_\pm)], \tag{6}$$

with $\widehat{\mathbf{\Omega}}_\pm = \mathbf{b} + \mu_+ \mu_- \nabla \times \mathbf{J} \pm \mu_\pm \nabla \times \mathbf{v}$ representing the perturbed vorticity. The condition for the disappearance of the nonlinear terms (second on the right-hand side of (6)) is

$$\mathbf{v} + \mu_- \mathbf{J} = \alpha(\mathbf{b} + \mu_+ \mu_- \nabla \times \mathbf{J} + \mu_+ \nabla \times \mathbf{v}), \tag{7}$$

$$\mathbf{v} - \mu_+ \mathbf{J} = \alpha(\mathbf{b} + \mu_+ \mu_- \nabla \times \mathbf{J} - \mu_- \nabla \times \mathbf{v}), \tag{8}$$

where we have taken the separation constants to be equal anticipating the eventual requirements of consistency [2]. Conditions (7) and (8) are manipulated to give

$$\mathbf{J} = \alpha \nabla \times \mathbf{v}, \tag{9a}$$

$$[\mathbf{v} - \alpha^2 \mu_+ \mu_- \nabla \times (\nabla \times \mathbf{v})] = \alpha \mathbf{b} + \alpha(\mu_+ - \mu_-)(\nabla \times \mathbf{v}). \tag{9b}$$

At this stage it is worthwhile to distinguish between the standard linear waves and the LNL waves under investigation. For both these classes, the nonlinear terms in (6) are effectively zero. The linear waves, then, are fully determined by what is left of (6). The LNL waves, in contradistinction, are the solutions of what is left of (6) subject to the conditions (7) and (8); the latter constraint, for example, imposes definite polarization requirements on the allowed fluctuating fields.

Manipulating (6), (7), and (8) gives

$$\frac{\partial \mathbf{J}}{\partial t} = \alpha \nabla \times [\mathbf{J} \times \mathbf{B}_0], \tag{10}$$

$$\frac{\partial \mathbf{v}}{\partial t} = \alpha \nabla \times (\mathbf{v} \times \mathbf{B}_0), \tag{11}$$

to be solved simultaneously with (9a), (9b), and the Maxwell equation (in fact its curl) expressed in the dimensionless form

$$\nabla \times \nabla \times \mathbf{b} = \nabla \times \mathbf{J} - \frac{V_{\mathrm{A}}^2}{c^2} \frac{\partial^2 \mathbf{b}}{\partial t^2}. \tag{12}$$

Since the simplified (exact) system is linear, we try the plane-wave form

$$\mathbf{X} = \widehat{\mathbf{X}} e^{i\mathbf{k}\cdot\mathbf{x} + i\alpha k_{\mathrm{s}} t}, \tag{13}$$

where $\mathbf{X}$ is $\mathbf{J}$, $\mathbf{b}$, or $\mathbf{v}$, and $k_{\mathrm{s}} = \mathbf{k} \cdot \mathbf{B}_0$. For (7) to qualify as a solution, (9a), (9b), and (12) will have to be compatible.

Equation (7) converts the Maxwell equation (12) to

$$\left(k^2 - \alpha^2 k_{\mathrm{s}}^2\, v_{\mathrm{A}}^2/c^2\right)\widehat{\mathbf{b}} = i(\mathbf{k} \times \widehat{\mathbf{J}}), \tag{14}$$

that may be inverted to obtain ($\mathbf{k} \cdot \widehat{\mathbf{J}} = 0$)

$$\widehat{\mathbf{J}} = i\left[1 - \alpha^2 \frac{V_{\mathrm{A}}^2}{c^2} \frac{k_{\mathrm{s}}^2}{k^2}\right](\mathbf{k} \times \widehat{\mathbf{b}}). \tag{15}$$

Similar operations on (9a) and (9b) reduce the system to two basic equations: the first relating the magnetic and the velocity perturbations

$$\widehat{\mathbf{b}} = \frac{\alpha}{1 - \alpha^2\left(V_{\mathrm{A}}^2/c^2\right)\left(k_{\mathrm{s}}^2/k^2\right)}\widehat{\mathbf{v}}, \tag{16}$$

and the second for $\widehat{\mathbf{v}}$ alone,

$$\widehat{\mathbf{v}}\left[1 - \alpha^2 \mu_- \mu_+ k^2 - \frac{\alpha^2}{1 - \alpha^2\left(V_{\mathrm{A}}^2/c^2\right)\left(k_{\mathrm{s}}^2/k^2\right)}\right] = i\alpha(\mu_+ - \mu_-)\mathbf{k} \times \widehat{\mathbf{v}}. \tag{17}$$

Solving (17) to obtain the dispersion relation $\alpha = \alpha(k)$ (note from (13) that the wave frequency $\omega = -\alpha k_{\mathrm{s}}$) completes the description of the LNL supported by a two-fluid plasma. Two distinct cases arise:

(1) The fluid particles have the same mass ($\mu_+ = \mu_- = 0.5$); the right-hand side of (17) vanishes and the dispersion relation is obtained by simply equating the coefficient of $\widehat{\mathbf{v}}$ to zero:

$$\left(1 - \frac{\alpha^2 k^2}{4}\right)\left(1 - \alpha^2 \frac{V_{\mathrm{A}}^2}{c^2} \frac{k_{\mathrm{s}}^2}{k^2}\right) = \alpha^2. \tag{18}$$

(2) The fluid particles have different masses. In this case, (17) is the Fourier transform of the Beltrami equation, and its solution is known to be a circularly

polarized wave with α and k related by (see [2] for details of the solution of the generic equation (17))

$$\left(1-\alpha^2\frac{V_A^2}{c^2}\frac{k_s^2}{k^2}\right)(1-\alpha\mu_+k)(1+\alpha\mu_-k)=\alpha^2. \tag{19}$$

It is not our intention, here, to analyze the dispersion relation (19). It was derived simply to establish the existence of LNL waves for the two-fluid plasma system (more general than MHD or Hall MHD). The important point is that the separation constant α is fully determined in terms of k, the modulus of the wavenumber $\mathbf{k}$; the latter labels a given mode. Since α is a complicated function of k, these waves are highly dispersive. From (9a), (16), and (17), LNL waves obey: for a given $\mathbf{k}$, $\mathbf{b_k}=g_k\mathbf{v_k}$ and $\mathbf{J_k}=f_k\mathbf{v_k}$, where g_k and f_k are known functions of $\mathbf{k}$ and the plasma parameters.

We are now ready to pursue the main goal of this paper—the establishment of a linear superposition principle. Let

$$\mathbf{X_k}e^{i\cdot\mathbf{k}\cdot\mathbf{x}+ik(\widehat{e}_k\cdot\widehat{e}_s)t}\equiv\mathbf{X_k}e^{Q_k(\mathbf{x},t)} \tag{20}$$

represent the LNL wave solution of the nonlinear time-dependent equations (6). We now wish to examine if and when a linearly superposed solution

$$\sum_{\mathbf{k}}\lambda_k\ \mathbf{X_k}e^{Q_k(\mathbf{x},t)}, \tag{21}$$

where $\lambda_\mathbf{k}$ are arbitrary functions, could exactly solve (6). To establish the superposition, all we need to find are the conditions under which the cross terms (coming from $\mathbf{k}\neq\mathbf{k}'$) in the nonlinear term vanish. For (21), the nonlinear terms in (6) read

$$\begin{aligned}\mathrm{NL}&\equiv\nabla\times[(\mathbf{v}\pm\mu_\mp\mathbf{J})\times\widehat{\mathbf{\Omega}}_\pm]\\&=\nabla\times\sum_{\mathbf{kk}'}(\lambda_{1k}\mathbf{v_k}\pm\mu_\mp\lambda_{2k}\mathbf{J_k})\widehat{\mathbf{\Omega}}_{\pm\mathbf{k}}\lambda_{3k'}Q_kQ_{k'}.\end{aligned} \tag{22}$$

To simplify (22), we remember that for the modes under consideration (a restatement of (7) and (8))

$$\widehat{\Omega}_{\pm\mathbf{k}'}=\alpha^{-1}(\mathbf{k}')[\mathbf{v_{k'}}\pm\mu_\mp\mathbf{J_{k'}}] \tag{23}$$

and $\mathbf{J_{k'}}=f_{k'}\mathbf{v_{k'}}$. Equation (22) then becomes

$$\mathrm{NL}=\sum_{\mathbf{kk}'}Q_\mathbf{k}(\lambda_{1\mathbf{k}}\pm\mu_\mp\lambda_{2\mathbf{k}}f_k)\alpha^{-1}(\mathrm{k}')\lambda_{3\mathbf{k}'}(1\pm\mu_\pm f_{k'})Q_{\mathbf{k}'}\nabla\times(\mathbf{v_k}\times\mathbf{v_{k'}}). \tag{24}$$

The cross nonlinear terms $\nabla\times(\mathbf{v_k}\times\mathbf{v_{k'}})$ will trivially go to zero if $\mathbf{v_k}$ is parallel to $\mathbf{v_{k'}}$. For the pair plasma, this can always be arranged because $\mathbf{v_k}$ can have arbitrary polarization; the arbitrary amplitude modes with parallel perturbations are, in this case, always superposable.

The situation is somewhat more complex and interesting for plasmas whose constituents have unequal masses. We will show that, even for this case, the waves propagating in one spatial direction can be added like linear waves. The simplest arbitrary amplitude nonlinear circularly polarized waves propagating in the z direction are [2]

$$\mathbf{X}_k=A_k(\widehat{e}_x+i\widehat{e}_y)e^{ikz+i\alpha(\widehat{e}_s\cdot\widehat{e}_z)kt}+c.c. \tag{25}$$

For this class of waves, we cannot rely on $\mathbf{v_k}$ being parallel to $\mathbf{v_{k'}}$ because the polarization is a function of time, and if the waves started parallel they will not remain so with the passage of time. However, since all the $\mathbf{v_k}$s are in the x–y plane and $\nabla = \widehat{e}_z \partial/\partial z$, $\nabla\times(\mathbf{v_k}\times\mathbf{v_{k'}}) = 0$. The nonlinearity, therefore, identically vanishes. Thus, we have established the principle of linear superposition of nonlinear waves propagating in one spatial direction.

Let us work out a simple consequence of the linear superposability of LNL waves—the time evolution, for instance, of an initial Gaussian (in space) pulse. Let us concentrate on the $\mathbf{b}$ and $\mathbf{v}$ fields only ($\mathbf{J}$ can be derived from them). For simplicity and for a comparison with a Hall MHD code, let us take the $\mu_- = 0, \mu_+ = 1$, and $V_{\mathrm{A}}^2/c^2 \ll 1$ (neglecting electron inertia and the displacement current) limit of the system we derived. The LNL solutions reduce to (from now on it will be understood that the physical wave is the real part of the $\mathbf{b}$s and $\mathbf{v}$s)

$$\mathbf{b}_k = \alpha(k)\mathbf{v}_k,$$

$$\mathbf{v}_k = g_k(\widehat{e}_x + i\widehat{e}_y)e^{ik(z+\alpha a t)}, \tag{26}$$

where $a = \widehat{e}_s\cdot\widehat{e}_z$ is the angle between the magnetic field and the direction of propagation. The linear superposition principle, just established, allows us to superpose (26) to construct the general solution:

$$\mathbf{b} = (\widehat{e}_x + i\widehat{e}_y)\int dk\alpha(k)g_k e^{ik(z+\alpha a t)} \equiv (\widehat{e}_x + i\widehat{e}_y)b,$$

$$\mathbf{v} = (\widehat{e}_x + i\widehat{e}_y)\int dk g_k e^{-k(z+\alpha a t)} \equiv (\widehat{e}_x + i\widehat{e}_y)v. \tag{27}$$

Let us now calculate the explicit space–time behavior of (27) for an initial pulse

$$v(z,0) = \sqrt{\frac{w}{2\pi}}e^{-wz^2/2}. \tag{28}$$

The coefficient g_k, being the Fourier transform of (28), is $g_k = \exp[-k^2/2w]$. Substituting g_k in (27), we find that

$$v(z,t) = \int_{-\infty}^{+\infty} dk\ e^{-k^2/2w+ik(z+a\,\alpha(k)t)}. \tag{29}$$

This integral can be readily calculated numerically, but an analytic approximation reveals essential and fundamental features. Note that the MHD limit is $\alpha = -1 (k \ll 0)$. In that case the integral is trivial, and is a Gaussian packet moving with a speed a (speeds are normalized to the Alfvén speed). Introducing only the first-order dispersion, i.e. taking $\alpha \simeq k/2 - 1$, can be justified if w is sufficiently small, i.e. the initial width of the spatial Gaussian is large. In this case, the integral again allows an analytic evaluation, and we find that

$$v = \sqrt{\frac{2\pi}{(1/w^2) + a^2t^2}}\left(\frac{1}{w} + i\alpha t\right)^{1/2} e^{-(z-ta)^2/2((1/w^2)+a^2t^2)((1/w)+i\alpha t)}. \tag{30}$$

The time behavior of the initial pulse has become much richer; the main peak does still propagate at speed a, but the amplitude begins to oscillate as well as decrease in magnitude as time advances. Equation (30), of course, holds for arbitrary

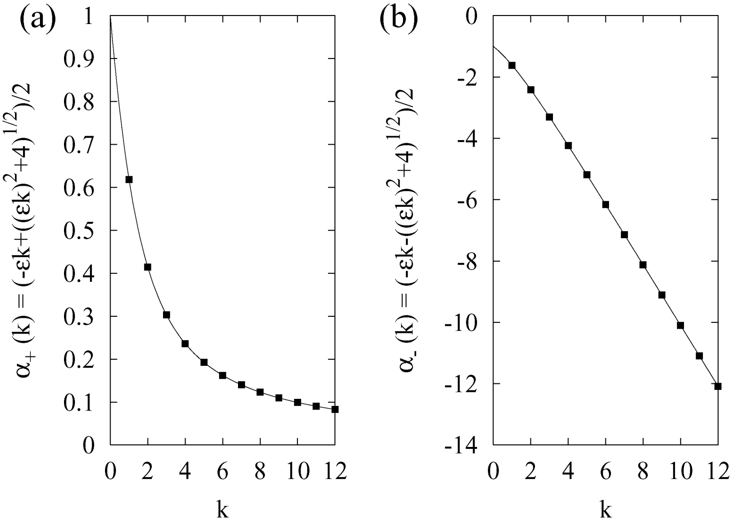

Figure 1. Comparisons of wave-propagation speeds between the analytical expression and numerical observations.

amplitudes. Thus, the LNL, like its linear dispersive counterpart, does also suffer from damping caused by phase mixing.

Although most results of the paper are rigorously derived from the model and do not really need any numerical support, we have still checked the basic features of the results by resorting to an initial value Hall MHD code (in which a very small dissipation is kept to remove numerical noise).

The Hall MHD equations are solved by means of the pseudo-spectral and Runge–Kutta–Gill methods for a $(2\pi)^3$ triply-periodic boundary condition. The number of grid points is $N^3 = 128^3$. The aliasing error is removed by the 2/3-rule truncation so that the maximum wavenumber in each of the three directions is $N/3$. We restrict ourselves to $a = 1$, that is the wave-propagation vector is parallel to the mean magnetic field. The initial velocity,

$$\mathbf{v} = \sum_k g_k(\widehat{e}_x \sin(kz) + \widehat{e}_y \cos(kz)), \tag{31}$$

is the imaginary part of the complex velocity given in (26). The periodic boundary conditions allow the replacement of the integral in (29) by a summation over k. Note that $\alpha(k)$ has two branches, $\alpha_\pm = -\frac{1}{2}k \pm \sqrt{((k^2/4)+1)}$. The solid lines in Fig. 1(a) and (b) show the α_+ and α_- branches, respectively. The black boxes represent the numerically calculated dispersion (the αs). We will come back to this figure later.

In Fig. 2, the time evolution of the velocity profile as a function of (z, t) is plotted for the α_+ branch. Initially, the velocity is chosen to have a Gaussian profile, with $w = 8$ (28). The waves are modulated in time with local amplification or attenuation. The numerical analysis below reveals that the modulations are caused by the superposition of waves each of which propagates according to its own nonlinear dispersion relation.

In Fig. 3, we display the amplitudes of the Fourier coefficients $\widetilde{v}_k(t)$. In all the figures, the curve (a) ((b)) represents the numerical results for the branch α_+ (α_-). To avoid clutter, we show only a few Fourier components ($k \leqslant 10$) in all the figures.

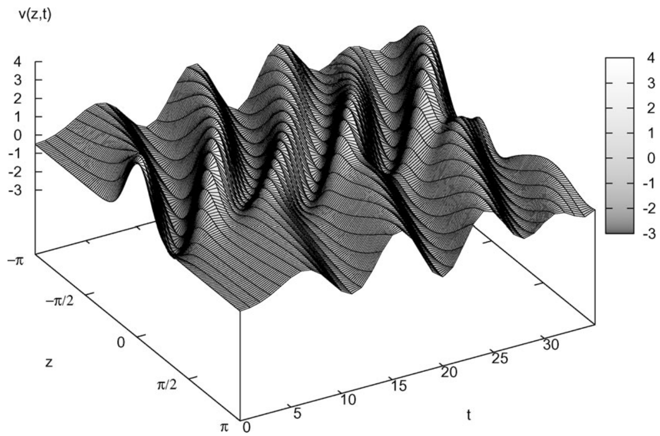

Figure 2. Wave propagation in the z–t plane. Waves which consist initially of the Gaussian profile propagate with their own propagation speed, which is in turn described by the analytical dispersion relation given by Mahajan and Krishnan [2].

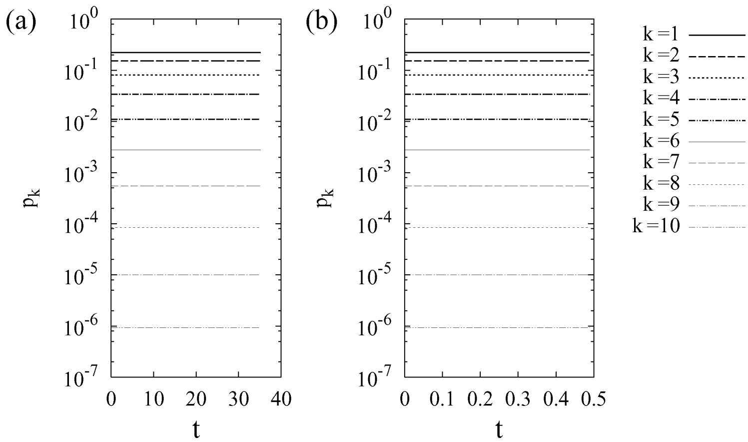

Figure 3. Temporal evolution of the amplitudes of each Fourier mode for (a) $\alpha_+(k)$ and (b) $\alpha_-(k)$ branches.

For both branches, displayed respectively in Fig. 3(a) and (b), the amplitudes $P_k = |\widetilde{v}_k(t)|^2$ remain almost constant in time. Since the time scale associated with the α_- branch for $k \gg 1$ is very small, we observe a fast decay in Fig. 3b.

Figure 4 represents the phase of the Fourier coefficients, $\phi_k = \tan^{-1}[\Re(\widetilde{v}_k(t))/\Im(\widetilde{v}_k(t))]$, defined to vary from $-\pi/2$ to $\pi/2$. In Fig. 4(a) and (b), ϕ_k is obviously a linear function of the wavenumber k. While ϕ_k of the slow waves changes gradually, ϕ_k of the fast waves ($k \simeq 1$ in (a) and $k \gg 1$ in (b)) traverses the $-\pi/2 \leqslant \phi_k \leqslant \pi/2$ region repeatedly, showing that their propagation speeds are totally independent of each other.

Since ϕ_k is a linear function of time for each k, it is easy to compute the propagation speed $d\phi/dt$ from the data. In Fig. 1, the phase speed given by the analytic dispersion relation $\omega_\pm = -\alpha_\pm k$ (see (16)) is compared with $d\phi/dt$ obtained numerically (actually $-\alpha_\pm$ (solid line) is compared to $(d\phi/dt)/k$ (black boxes)). It is clearly

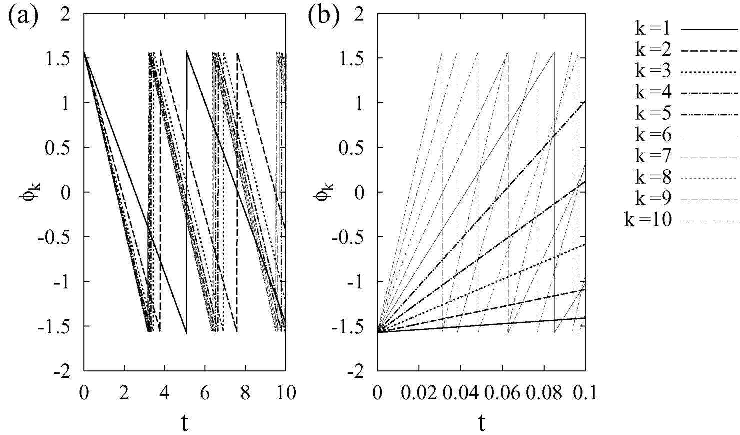

Figure 4. Temporal evolution of the phases of each Fourier mode for (a) α_+ and (b) α_- branches.

observed that the propagation speeds of the nonlinear waves well coincide with the analytical expression, showing that the analytical solution (30) is well realized in the numerical simulations.

It is shown that the general two-fluid system provides an invaluable and fascinating laboratory for investigating the characteristics of arbitrary amplitude fluctuations – the proof that nonlinear waves, under some conditions, can be linearly superposed just gives us a glimpse of what may be waiting to be explored.

Acknowledgements

SM is grateful to Prof. Z. Yoshida for helpful discussions. The research of SM was supported by the USDOE contract DE-FG03-96ER-54366. The research of HM was partially supported by 'Application of Fusion Theory to Space Plasmas' under the project 'Formation of International Network for Scientific Collaborations' at the National Institute for Fusion Science (NIFS)/National Institute for Natural Sciences (NINS), Japan. The numerical work was performed on the NEC SX-7 'Plasma Simulator' of NIFS.

References

[1] Walén, C. 1994 *Ark. Mat. Astron. Fys.* **30A**(15) and **31B**(3).
Alfvén, H. and Falthämmer, C. 1963 *Cosmical Electrodynamics*. Oxford: Clarendon Press.
Parker, E. N. 1979 *Cosmic Magnetic Fields*. Oxford: Clarendon Press.

[2] Mahajan, S. M. and Krishnan, V. 2005 *Mon. Not. R. Astron. Soc.* **359**(1), L27–L29.

[3] Whitham, G. B. 1974 *Linear and Nonlinear Waves*. New York: John Wiley.

[4] Newell, A. C. 1985 *Solitons in Mathematics and Physics*. Philadelphia: Society for Industrial and Applied Mathematics.
Rajaraman, R. 1982 *Solitons and Instantons*. Amsterdam: North-Holland.

[5] Babin, A. and Figotin, A. 2006 *Rev. Math. Phys.* **18**(9), 971–1053.