Effects of Orbit Squeezing on Ion Transport Processes Close to Magnetic Axis

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Abstract

It is shown that ion thermal conductivity close to the magnetic axis in tokamaks is reduced by a factor of $|S|^{5/3}$ if $(M_i/M_e)^{2/3}(T_e/T_i)^{4/3}/|S|^{5/3} \gg 1$. Here, S is the orbit squeezing factor, $M_i(M_e)$ is the ion (electron) mass, and $T_i(T_e)$ is the ion (electron) temperature. The reduction reflects both the increase of the fraction of trapped particles by a factor of $|S|^{1/3}$, and the decrease of the orbit size in units of the poloidal flux ψ by a factor of $|S|^{2/3}$.

I. Introduction

It is known that tokamak particle orbits that are close to the magnetic axis scale differently with respect to energy and magnetic field strength than those far away from the axis.[1,2] The consequence of these orbits on neutral beam heating, α-particle heating and transport, and magnetohydrodynamic instabilities has been studied.[3−4] The importance of these particles to plasma transport is emphasized by Politzer and Lin *et al.*[5,6] The transport processes for these particles have been calculated from the solution of the drift kinetic equation it is found that they can be understood in terms of a random walk in the poloidal flux function ψ.[7] Here, we extend the transport calculations to include the effects of orbit squeezing associated with $d^2\Phi/d\psi^2$ where Φ is the equilibrium electrostatic potential.[8] It is shown that ion thermal conductivity is reduced by a factor of $|S|^{5/3}$ reflecting an increase of the fraction of trapped particles by a factor of $|S|^{1/3}$, and the reduction of the orbit size in ψ by a factor of $|S|^{2/3}$. Thus

$$\chi_\psi = 0.8\nu_i \left(\frac{Iv_t}{\Omega_0}\right)^{7/3} \left(\frac{q}{IR}\right)^{1/3} \frac{1}{|S|^{5/3}}, \tag{1}$$

where the orbit squeezing factor S is defined as $S = [1 + I^2 e\Phi''/(\Omega_0^2 M)]$, $I = R^2\boldsymbol{\nabla}\zeta \cdot \mathbf{B}, \mathbf{B}$ is the magnetic field, R is the major radius, ζ is the toroidal angle, ν_i is the ion collision freuqency, e is the ion charge, M is the ion mass, Ω_0 is the ion gyrofrequency on axis, v_t is the ion thermal speed, q is the safety factor, and $\Phi'' = d^2\Phi/d\psi^2$. This result is obtained under the assumption that $(M/M_e)^{2/3}(T_e/T_i)^{4/3}/|S|^{5/3} \gg 1$, so that the ion viscous coefficient is larger than the electron viscous coefficient. Here, M_e is the electron mass, T_e is the electron temperature, and T_i is the ion temperature.

The remainder of the paper is organized as follows. In Sec. II, we solve the squeezed particle orbits close to the magnetic axis using three constants of motion: P_ζ, the toroidal canonical momentum; $E = v^2/2 + e\Phi/M$, the total energy; and μ the magnetic moment. Here, v is the particle speed. The poloidal angular speed ω is calculated in Sec. III. The linearized drift kinetic equation is solved in Sec. IV. The ion poloidal or parallel flow and ion heat flux are calculated in Sec. V. The concluding remarks are given in Sec. VI.

II. Particle Orbits Close to Magnetic Axis

Particle orbits in an axisymmetric tokamak can be determined from three constants of motion: total particle energy $E = v^2/2 + e\Phi/M$, magnetic moment $\mu = v_\perp^2/2B$, and toroidal canonical momentum $P_\zeta = \psi - Iv_\parallel/\Omega$. Here, $B = |\mathbf{B}|$, $v_\perp$ is the perpendicular (to $\mathbf{B}$) particle speed, $v_\parallel$ is the parallel particle speed, Ω is the ion gyrofrequency.

Assuming $B = B_0/(1 + \epsilon\cos\theta)$ where $\epsilon = r/R_0$ is the inverse aspect ratio and B_0 is B on the axis, we find from P_ζ conservation that

$$\psi - \psi_0 = \frac{Iv_\parallel}{\Omega_0}(1 + \epsilon\cos\theta) - \frac{Iv_{\parallel 0}}{\Omega_0}(1 + \epsilon_0\cos\theta_0), \tag{2}$$

where (ψ_0, θ_0) is a convenient reference point, and $v_{\parallel 0} = v_\parallel(\psi_0, \theta_0)$. Solving Eq. (2) for $v_\parallel$ and substituting into energy and magnetic moment conservation laws, we obtain

$$S(\psi-\psi_0)^2 + \left[2\frac{Iv_{\parallel 0}}{\Omega_0}(1 + \epsilon_0\cos\theta_0) + 2\,\frac{I^2}{\Omega_0^2}\,\frac{e\Phi_0'}{M}\,(1 + \epsilon\cos\theta)^2\right](\psi-\psi_0) + \frac{2I^2v_{\parallel 0}^2}{\Omega_0^2}(\epsilon_0\cos\theta_0 - \epsilon\cos\theta)$$
$$+ \frac{2I^2}{\Omega_0^2}\mu B_0(\epsilon_0\cos\theta_0 - \epsilon\cos\theta) = 0 + \mathcal{O}(\epsilon). \tag{3}$$

Here, we have approximated Φ by $\Phi = \Phi_0 + \Phi_0'(\psi-\psi_0) + (1/2)\Phi_0''(\psi-\psi_0)^2 + \ldots$. Equation (3) evidently uses a large aspect ratio approximation, $\epsilon \ll 1$. Note that crucial ψ-dependence enters through

$$\epsilon = C_1\sqrt{\psi}$$

where $C_1 = \sqrt{2q/\delta IR}$, and δ is the elongation parameter. Note that in distinction to the off-axis, conventional orbit description, the relation is *not* quadratic in ψ. The point is that the ψ dependence of ϵ must be treated more carefully near the axis. We simplify by choosing $\psi_0 = \cos\theta_0 = 0$ and neglecting $dq/d\psi$. The result is

$$Sx^3 + 2\left(\frac{Iv_{\|0}}{\Omega_0} + \frac{I^2 e\Phi_0'}{M\Omega_0^2}\right)x - \frac{2I^2C_1}{\Omega_0^2}(v_{\|0}^2 + \mu B_0)\cos\theta = 0, \tag{4}$$

where $x = \sqrt{\psi}$. Equation (4) describes the squeezed trajectory (ψ, θ) for particles close to the magnetic axis.

Because $\psi > 0$ for real orbits, only solutions that keep $x > 0$ are acceptable. Also, all the imaginary solutions must be discarded. Thus we seek real positive solutions of Eq. (4).

A cubic equation is characterized by the sign of $(\mathbf{r}^2 + \mathbf{q}^3)$, where, for Eq. (4)

$$\mathbf{r} = \frac{I^2C_1}{S\Omega_0^2}(v_{\|0}^2 + \mu B_0)\cos\theta, \tag{5}$$

$$\mathbf{q} = \frac{2}{3}\frac{I\left(v_{\|0} + \frac{Ic\Phi_0'}{B_0}\right)}{\Omega_0 S}, \tag{6}$$

and $(\mathbf{r}^2 + \mathbf{q}^3)$ is

$$\mathbf{r}^2 + \mathbf{q}^3 = \left(\frac{I^2C_1}{S\Omega_0^2}\right)^2 (v_{\|0}^2 + \mu B_0)^2(\sigma\kappa + \cos^2\theta) \tag{7}$$

where $\sigma = \mathrm{sgn}[(v_{\|0} + Ic\Phi_0'/B)/(S\Omega_0)], \mathrm{sgn}(x) = x/|x|$, and

$$\kappa = \frac{8}{27}\left(\frac{I}{|S|\Omega_0}\right)^3 \frac{|v_{\|0} + Ic\Phi_0'/B_0|^3}{\left(\frac{I^2C_1}{S\Omega_0^2}\right)^2 (v_{\|0}^2 + \mu B_0)^2}. \tag{8}$$

The solution of Eq. (4) can be written in terms of S_1 and S_2 where

$$S_1 = \left[\mathbf{r} + (\mathbf{q}^3 + \mathbf{r}^2)^{1/2}\right]^{1/3}, \tag{9}$$

$$S_2 = \left[\mathbf{r} - (\mathbf{q}^3 + \mathbf{r}^2)^{1/2}\right]^{1/3}. \tag{10}$$

The general solutions for Eq. (4) are

$$Z_1 = (S_1 + S_2), \tag{11}$$

$$Z_2 = -\frac{1}{2}(S_1 + S_2) + \frac{i\sqrt{3}}{2}(S_1 - S_2), \tag{12}$$

$$Z_3 = -\frac{1}{2}(S_1 + S_2) - \frac{i\sqrt{3}}{2}(S_1 - S_2). \tag{13}$$

Note that to obtain Eqs. (5)–(13) we have used the fact that the coefficient of x^2 term vanishes in Eq. (4). This simplifies the general results in Ref. 9.

It is obvious that the sign of $(\mathbf{r}^2 + \mathbf{q}^3)$ is determined by the magnitude and the sign of $\sigma\kappa$. We shall find that for circulating particles $-\infty < \sigma\kappa < -1$ and $0 < \sigma\kappa < \infty$, and for trapped particles $-1 \leq \sigma\kappa \leq 0$. We will assume $S > 0$. For $S < 0$, particles are trapped on the strong magnetic field side of the torus.

A. Circulating particles $-\infty < \sigma\kappa < -1$

If $(\mathbf{r}^2 + \mathbf{q}^3)$ is negative, all solutions of Eq. (4) are real. But we only choose the positive solutions. If $\cos\theta > 0$, there is only one positive solution

$$x = 2\widehat{x}(-\sigma\kappa)^{1/6} \cos\left(\frac{\beta}{3}\right), \tag{14}$$

where $\cos\beta = \cos\theta/\sqrt{-\sigma\kappa}$, and $\widehat{x} = \left[\left(I^2 C_1/S\Omega_0^2\right)(v_{\|0}^2 + \mu B_0)\right]^{1/3}$. If $\cos\theta < 0$, there are two positive solutions

$$x_\pm = 2\widehat{x}(-\sigma\kappa)^{1/6} \sin\left(\frac{\pi}{6} \pm \frac{\beta}{3}\right), \tag{15}$$

where $\cos\beta = |\cos\theta|/\sqrt{-\sigma K}$.

From Eqs. (14) and (15), we can construct two classes of circulating particles. The ones that encircle the magnetic axis are described by

$$x = 2\widehat{x}(-\sigma\kappa)^{1/6} \begin{cases} \cos\left(\dfrac{\beta}{3}\right) \quad , & \cos\theta > 0 \\[2ex] \sin\left(\dfrac{\pi}{6} + \dfrac{\beta}{3}\right) \quad , & \cos\theta < 0. \end{cases} \tag{16}$$

The ones that intersect but do not encircle the magnetic axis are described by

$$x = 2\widehat{x}(-\sigma\kappa)^{1/6} \sin\left(\frac{\pi}{6} - \frac{\beta}{3}\right), \quad \cos\theta < 0. \tag{17}$$

Note that this class of circulating particles only exist on the second and the third quadrant where $\cos\theta < 0$.

B. Circulating particles $0 < \sigma\kappa < \infty$

When $(\mathbf{r}^2 + \mathbf{q}^3)$ is positive, and there is only one real solution. If $\cos\theta > 0$, there is a real positive solution

$$x = 2\widehat{x}(\sigma\kappa)^{1/6} \sinh\left(\frac{\beta}{3}\right), \tag{18}$$

where $\sinh\beta = \cos\theta/\sqrt{\sigma\kappa}$. If $\cos\theta < 0$, there is no real positive solution. Therefore, there is only one class of circulating particles described by Eq. (18). These circulating particles intersect (but do not encircle) the magnetic axis and exist only in the first and the fourth quadrant.

C. Trapped particles $-1 \leq \sigma\kappa \leq 0$

In this parameter regime, there is a critical value of the poloidal angle θ_c such that $\sigma\kappa + \cos^2\theta_c = 0$.

i) $-\theta_c < \theta < \theta_c$, and $\cos\theta > 0$

$(\mathbf{r}^2 + \mathbf{q}^3)$ is positive in this case and there is only one real solution. It is a positive solution

$$x = 2\widehat{x}(-\sigma\kappa)^{1/6}\cosh(\beta/3) \tag{19}$$

where $\cosh\beta = \cos\theta/\sqrt{-\sigma\kappa}$.

ii) $\pi/2 > \theta > \theta_c$, or $-\theta_c > \theta > -\pi/2$, and $\cos\theta > 0$:

Because $(\mathbf{r}^2 + \mathbf{q}^3)$ is negative, there are three real solutions. But there is only one positive real solution

$$x = 2\widehat{x}(-\sigma\kappa)^{1/6} \cos\left(\frac{\beta}{3}\right), \tag{20}$$

where $\cos\beta = \cos\theta/\sqrt{-\sigma\kappa}$.

iii) $\pi - \theta_c > \theta > \pi/2$, or $-\pi/2 > \theta > -(\pi - \theta_c)$, and $\cos\theta < 0$:

There are three real solutions because $(\mathbf{r}^2 + \mathbf{q}^3) < 0$, and two of them are positive

$$x_\pm = 2\widehat{x}(-\sigma\kappa)^{1/6} \sin\left(\frac{\pi}{6} \pm \frac{\beta}{3}\right) \tag{21}$$

where $\cos\beta = |\cos\theta|/\sqrt{-\sigma\kappa}$.

iv) $\pi > \theta > \pi - \theta_c$ or $-(\pi - \theta_c) > \theta > -\pi$ and $\cos\theta < 0$:

In this case $(\mathbf{r}^2 + \mathbf{q}^3) > 0$, there is only one real solution. However, this real solution is negative and is not acceptable.

From solutions in Eqs. (19)–(21), we can construct a class of trapped particles characterized by

$$x = 2\hat{x}(-\sigma\kappa)^{1/6} \begin{cases} \cosh\left(\dfrac{\beta}{3}\right), & (i) \\ \cos\left(\dfrac{\beta}{3}\right), & (ii) \\ \sin\left(\dfrac{\pi}{6} + \dfrac{\beta}{3}\right), & (iii) \end{cases} \tag{22}$$

for the outer-half of the trapped particle trajectory and

$$x = 2\hat{x}(-\sigma\kappa)^{1/6}\sin(\pi/6 - \beta/3), \qquad (iii) \tag{23}$$

for the inner-half of the trapped particle trajectory. These two halves are separated by the turning points $\pm\theta_t = \pm(\pi - \theta_c)$. As we shall see in Sec. III, the poloidal angular speed ω vanishes at turning points. Note, however, the inner-half of the trapped particle trajectory intersect (but does not encircle) the magnetic axis.

III. Poloidal Angular Speed ω

The poloidal angular speed ω that appears in the drift kinetic equation is

$$\omega = v_\parallel - v_\parallel \frac{\partial}{\partial\psi}\left(\frac{I v_\parallel}{\Omega}\right), \tag{24}$$

where $\mathbf{E} \times \mathbf{B}$ drift and curvature and grad B drifts are included. Here, $\mathbf{E}$ is the electric field. In this section we express ω in terms of x and β. Assuming $\epsilon \ll 1$ and employing $\epsilon = C_1\sqrt{\psi}$, we can write ω as

$$\omega = v_\parallel - \frac{v_\parallel^2 + \mu B}{\sqrt{\psi}\Omega B}\frac{I}{2}C_1 B_0\cos\theta + \frac{Ic}{B_0}(\Phi_0' + \Phi_0''\psi). \tag{25}$$

From Eq. (2), we find, by assuming $\psi_0 = 0$ and $\epsilon \ll 1$,

$$v_\parallel \approx \frac{\Omega_0}{I}\left(\psi + \frac{I v_{\parallel 0}}{\Omega_0}\right). \tag{26}$$

Substituting Eq. (26) into Eq. (25), we have

$$\omega \approx \frac{\Omega_0}{I\sqrt{\psi}}\left[S\psi^{3/2} + \frac{I(v_{\parallel 0} + Ic\Phi_0'/B_0)}{\Omega_0}\psi^{1/2} - \frac{I^2 C_1}{2\Omega_0^2}(v_{\parallel 0}^2 + \mu B_0)\cos\theta\right]. \tag{27}$$

Note that the last term in the square brackets of Eq. (27) can be simplified with Eq. (4) and ω is reduced to

$$\omega = \frac{3}{4}\frac{\Omega_0}{I}\left[S\psi + \frac{2}{3}\frac{I}{\Omega_0}\left(v_{\parallel 0} + Ic\Phi_0'/B_0\right)\right]. \tag{28}$$

Now $\psi = x^2$ has the general form

$$\psi = 4\widehat{x}^2\kappa^{1/3}T^2, \tag{29}$$

where T is one of the following functions: $\cos(\beta/3)$, $\sin(\pi/6 \pm \beta/3)$, $\sinh(\beta/3)$, and $\cosh(\beta/3)$. Substituting Eq. (29) into Eq. (28), we can express ω as

$$\omega = \widehat{\omega}(4T^2 + \sigma), \tag{30}$$

where

$$\widehat{\omega} = \frac{3}{4}\frac{\Omega_0}{I}S\widehat{x}^2\kappa^{1/3}. \tag{31}$$

The expression of ω given in Eq. (30) plays a role in our theory similar to that of $v_\parallel$ in the conventional theory. To see this, let us consider ω of the trapped particles in the region where $\theta \approx \pm\theta_t$. In this case, ω is

$$\omega = \widehat{\omega}\left[4\sin^2\left(\frac{\pi}{6} \pm \frac{\beta}{3}\right) + \sigma\right]. \tag{32}$$

At $\theta = \pm\theta_t$, $\beta = 0$, and $\omega = 0$. Thus, indeed at $\theta = \pm\theta_t$, the poloidal angular speed ω vanishes as expected. It is straightforward to show that ω does not vanish anywhere on the circulating particle's trajectory.

IV. Solution of the Drift Kinetic Equation

The drift kinetic equation for ions in tokamaks is

$$v_\parallel \widehat{n}\cdot\boldsymbol{\nabla} f + \mathbf{v}_d\cdot\boldsymbol{\nabla}\theta\,\frac{\partial f}{\partial\theta} + \mathbf{v}_d\cdot\boldsymbol{\nabla}\psi\,\frac{\partial f}{\partial\psi} = C(f), \tag{33}$$

where $\widehat{n} = \mathbf{B}/B$, $C(f)$ is the Coulomb collision operator, and $\mathbf{v}_d$ is the drift velocity which for low-β (β is the ratio of the plasma pressure to the magnetic field pressure) plasma is

$$\mathbf{v}_d = -v_\parallel \widehat{n}\times\boldsymbol{\nabla}\left(\frac{v_\parallel}{\Omega}\right). \tag{34}$$

The independent variables in Eqs. (33) and (34) are (ψ, θ, E, μ). It is useful to note that

$$\mathbf{v}_d\cdot\boldsymbol{\nabla}\theta = -v_\parallel \widehat{n}\cdot\boldsymbol{\nabla}\theta\,\frac{\partial}{\partial\psi}\left(\frac{Iv_\parallel}{\Omega}\right) \tag{35}$$

and

$$\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi = v_\parallel \widehat{n}\cdot\boldsymbol{\nabla}\theta\,\frac{\partial}{\partial\theta}\left(\frac{Iv_\parallel}{\Omega}\right). \tag{36}$$

Let ψ_e be the equilibrium scale length for the equilibrium distribution function f_0. Assuming $\nu_i \sim (v_\parallel \widehat{n}\cdot\boldsymbol{\nabla}\theta + \mathbf{v}_d\cdot\boldsymbol{\nabla}\theta) \gg v_d\cdot\boldsymbol{\nabla}\psi/\psi_e$, we obtain from Eq. (33) to the lowest order in $\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi/(\psi_e\nu_i)$,

$$v_\parallel \widehat{n}\cdot\boldsymbol{\nabla}\theta\,\frac{\partial f_0}{\partial\theta} + \mathbf{v}_d\cdot\boldsymbol{\nabla}\theta\,\frac{\partial f_0}{\partial\theta} = C(f_0). \tag{37}$$

A solution to Eq. (37) is a Maxwellian distribution,

$$f_0 = f_M(\psi). \tag{38}$$

The next order linearized equation is then

$$v_\parallel \widehat{n}\cdot\boldsymbol{\nabla}\theta\,\frac{\partial f_1}{\partial\theta} + \mathbf{v}_d\cdot\boldsymbol{\nabla}\theta\,\frac{\partial f_1}{\partial\theta} + \mathbf{v}_d\cdot\boldsymbol{\nabla}\psi\,\frac{\partial f_1}{\partial\psi} + \mathbf{v}_d\cdot\boldsymbol{\nabla}\psi\,\frac{\partial f_0}{\partial\psi} = C(f_1), \tag{39}$$

where f_1 is the perturbed distribution function. The reason that $\partial f_1/\partial\psi$ terms must be kept is that, although $f_1 \sim \Delta\psi\,\partial f_0/\partial\psi \ll f_0$, we also have $\partial f_1/\partial\psi \sim \partial f_0/\partial\psi$, as pointed out in Refs. 10 and 11.

The solution to Eq. (39) can be expressed as

$$f_1 = -\frac{Iv_\parallel}{\Omega} f_M \left(\frac{p'}{p} + \frac{e\Phi'}{T} + y\,\frac{T'}{T} \right) + g, \tag{40}$$

where the prime denotes $d/d\psi$, p is the ion pressure, and y is a parameter to be determined. The function g is localized in pitch angle: the pitch angle derivative of g becomes small when away from the trapping boundary. Note that for Eq. (40) to be valid ion viscous coefficient must be larger than electron viscous coefficient. This requirement is satisfied if $(M/M_e)^{2/3}(T_e/T_i)^{4/3}/|S|^{5/3} \gg 1$. The function g satisfies

$$v_\parallel \hat{n}\cdot\boldsymbol{\nabla}\theta\frac{\partial g}{\partial\theta} + \mathbf{v}_d\cdot\boldsymbol{\nabla}\theta\frac{\partial g}{\partial\theta} + \mathbf{v}_d\cdot\boldsymbol{\nabla}\psi\frac{\partial g}{\partial\psi} + (\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi) f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T} = C(g), \tag{41}$$

where $x = v^2/v_t^2$, and $v_t = \sqrt{2T/M}$. The first term on the right-hand side of Eq. (40) gives rise to a parallel flow

$$V_\parallel = -\frac{IcT}{eB}\left(\frac{p'}{p} + \frac{c\Phi'}{T} + y\frac{T'}{T} \right). \tag{42}$$

where c is the speed of light. Note that in obtaining Eq. (42), the radial gradient scale length of the parallel flow is assumed to be smaller than the size of the orbits. The parallel flow and poloidal flow are also assumed to be subsonic.

Equation (41) can be solved by a method developed in Ref. 10. Transforming the independent variables from (ψ, θ, E, μ) to $(E, \mu, P_\zeta, \theta)$, Eq. (41) can be simplified to

$$\omega\hat{n}\cdot\boldsymbol{\nabla}\theta\frac{\partial g}{\partial\theta} + (\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi) f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T} = C(g). \tag{43}$$

Note that the $\partial/\partial\theta$ in Eq. (43) is evaluated at fixed (E, μ, P_ζ). Both ω and $(\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi)$ can be expressed in terms of the gradients of P_ζ:

$$\omega = -\frac{I}{\Omega}\frac{\partial P_\zeta/\partial\psi}{\partial P_\zeta/\partial E}, \tag{44}$$

and

$$\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi = \frac{I}{\Omega}\frac{\partial P_\zeta/\partial\theta}{\partial P_\zeta/\partial E}\hat{n}\cdot\boldsymbol{\nabla}\theta. \tag{45}$$

We are interested in the banana regime, where the effective collision frequency $\nu_{\text{eff}} = \nu_i/f_t^2$ is less than the bounce frequency ω_b of the trapped particles. The fraction of the trapped particles can be estimated from the definition of κ in Eq. (8) and is found to be

$$f_t \sim \left(\frac{I v_t}{\Omega_0}|S|C_1^2\right)^{1/3}. \tag{46}$$

Thus, the fraction of the trapped particles is increased by a factor of $|S|^{1/3}$. The bounce frequency ω_b is basically $\widehat{\omega}/Rq$ with $\kappa \sim 1$

$$\omega_b \sim \frac{v_t}{Rq} f_t. \tag{47}$$

Thus $\nu_{\text{eff}} < \omega_b$ implies $\nu_i < f_t^3 v_t/Rq$.

In the banana regime, g can be expanded as $g = g_0 + g_1 + \cdots$, where

$$\omega\widehat{n}\cdot\boldsymbol{\nabla}\theta\frac{\partial g_0}{\partial\theta} + (\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi)f_M\left(x - \frac{5}{2} - y\right)\frac{T'}{T} = 0, \tag{48}$$

and

$$\omega\widehat{n}\cdot\boldsymbol{\nabla}\theta\frac{\partial g_1}{\partial\theta} = C(g_0). \tag{49}$$

The solution to Eq. (48) is, by utilizing Eqs. (28), (44), and (45),

$$g_0 = -\frac{4}{3}\frac{I\omega}{\Omega_0 S}f_M\left(x - \frac{5}{2} - y\right)\frac{T'}{T} + h(E,\mu,P_\zeta), \tag{50}$$

where h is an integration constant to be determined from Eq. (49).

Because g is localized, the collision operator is dominated by the pitch angle scattering. Thus

$$C(g) = \nu_D\frac{B_0 v_\parallel}{BE}\frac{\partial}{\partial\lambda}\left(\lambda v_\parallel\frac{\partial g}{\partial\lambda}\right), \tag{51}$$

where ν_D is the deflection frequency, and $\lambda = \mu B_0/(v^2/2)$. The dominant scattering process occurs across the $\omega \approx 0$ boundary in the velocity space. Hence Eq. (51) can be reduced to

$$C(f) \approx \nu_D\frac{v^2}{2}\frac{\partial^2 f}{\partial\omega^2}. \tag{52}$$

With Eq. (52), we can write Eq. (49) as

$$\omega \widehat{n} \cdot \boldsymbol{\nabla}\theta \frac{\partial g_1}{\partial \theta} = \nu_D \, \frac{v^2}{2} \, \frac{\partial^2 g_0}{\partial \omega^2}. \tag{53}$$

Substituting Eq. (50) into Eq. (53) and annihilating the left-hand side of Eq. (53) by averaging over the particle trajectory, we find

$$\oint \frac{d\theta}{\omega \widehat{n} \cdot \boldsymbol{\nabla}\theta} \frac{\partial^2 h}{\partial \omega^2} = 0, \tag{54}$$

where for circulating particles $\oint d\theta/\omega$ is defined as integration along the complete poloidal trajectory, and for trapped particles $\oint d\theta/\omega$ is defined as $\int_{-\theta_t}^{\theta_t} d\theta/\omega_+ + \int_{\theta_t}^{-\theta_t} d\theta/\omega_-$ with ω_+ indicating $\omega > 0$ and ω_- indicating $\omega < 0$.

It is convenient to solve Eq. (54) by changing variables from ω to κ, with the relation

$$d\omega \cong \frac{4}{3} \frac{\widehat{\omega}^2}{2\kappa\omega} d\kappa, \tag{55}$$

which is valid in the region where $\omega \approx 0$. Substituting Eq. (55) into Eq. (54), and solving $\partial h/\partial \kappa$, we obtain

$$\frac{\partial h}{\partial \kappa} = \frac{C}{\frac{\kappa}{\widehat{\omega}^2} \oint \frac{d\theta \omega}{\widehat{n} \cdot \boldsymbol{\nabla}\theta}}, \tag{56}$$

and

$$\frac{\partial g_0}{\partial \omega} = -\frac{4}{3} \frac{I}{\Omega_0} f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T} + \frac{4}{3} \frac{2|\omega| C}{\oint \frac{d\theta}{\widehat{n} \cdot \boldsymbol{\nabla}\theta} |\omega|}. \tag{57}$$

For circulating particles, we impose the boundary condition that $\partial g_0/\partial \omega$ vanishes when $\kappa \to \infty$. For trapped particles, h must be even in ω because of the reflection boundary condition at the turning points. Thus $\partial h/\partial \kappa$ must be even in ω for trapped particles. This condition can be satisfied if $C = 0$ for trapped particles. We conclude that $\partial g/\partial \omega$ is

$$\frac{\partial g_0}{\partial \omega} = -\frac{4}{3} \frac{I}{\Omega_0 S} f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T} \left(1 - H \frac{|\omega|}{\langle |\omega| \rangle_\theta} \right), \tag{58}$$

where $H = 1$ for circulating particles and $H = 0$ for trapped particles, and $\langle |\omega| \rangle_\theta = \oint d\theta |\omega| / \widehat{n} \cdot \boldsymbol{\nabla}\theta \Big/ \oint d\theta / \widehat{n} \cdot \boldsymbol{\nabla}\theta$. To calculate the transport flux, we only need to know $\partial g_0/\partial \omega$.

V. Heat Flux and Parallel Flow

The ion particle and heat fluxes are

$$\Gamma_j = \frac{1}{\Delta\psi} \int d\psi \oint \frac{d\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} \int d^3v \, \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi f_1 \Sigma_j \Big/ \oint \frac{d\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta}, \tag{59}$$

where $j = 1$, or 2. Here $\boldsymbol{\Gamma}_{\mathbf{1}} = \langle \boldsymbol{\Gamma} \cdot \boldsymbol{\nabla}\psi \rangle$, where $\boldsymbol{\Gamma}$ is the particle flux, and $\Gamma_2 = \langle \boldsymbol{q} \cdot \boldsymbol{\nabla}\psi \rangle / T$, $\boldsymbol{q}$ is the heat flux. Also, $\Sigma_1 = 1$, $\Sigma_2 = x - 5/2$ and $\Delta\psi$ is the typical orbit width. It is obvious that the first term on the right-hand side of Eq. (40) does not contribute to Γ_j because of the θ average; only g contributes to Γ_j. The $\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi$ in Eq. (59) can be expressed in terms of g from Eq. (41) to obtain

$$\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi = \left[C(g) - \left(v_\parallel \widehat{n} \cdot \boldsymbol{\nabla}\theta + \mathbf{v}_d \cdot \boldsymbol{\nabla}\theta \right) \frac{\partial g}{\partial \theta} - \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi \frac{\partial g}{\partial \psi} \right] \\ \times \left[f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T} \right]^{-1}. \tag{60}$$

Substituting Eq. (60) into Eq. (59), we can express Γ_j in terms of the Coulomb collision operator

$$\Gamma_j = \frac{1}{\Delta\psi} \int d\psi \oint \frac{d\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} \int d^3v \Sigma_j g C(g) \left[f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T} \right]^{-1} \Big/ \left(\oint \frac{d\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} \right)^{-1}. \tag{61}$$

Employing Eq. (51) into Eq. (61), integrating by parts in λ, and changing the variable from λ to ω, we can express Γ_j in terms of $\partial g / \partial \omega$

$$\Gamma_j = -\frac{1}{\Delta\psi} \int d\psi \oint \frac{d\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} \int d^3v \Sigma_j \left[f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T} \right]^{-1} \nu_D \, E \left(\frac{\partial g}{\partial \omega} \right)^2 \left(\oint \frac{d\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} \right)^{-1}. \tag{62}$$

Equation (62) can be evaluated to obtain

$$\Gamma_j = -\frac{8}{2^{2/3} 9 \sqrt{\pi}} N \nu_i \left(\frac{I v_t}{\Omega_D} \right)^{7/3} \frac{C_1^{2/3}}{|S|^{5/3}} \, I_p \left[\int_0^\infty dx \, x^{5/3} \, \frac{\nu_D}{\nu_i} \Sigma_j e^{-x} \left(x - \frac{5}{2} - y \right) \right] \frac{T'}{T}, \tag{63}$$

where N is the plasma density, and the pitch angle integral I_p is

$$I_p = \sum_\alpha \int_0^\infty \frac{d\kappa}{\kappa^{2/3}} \left(\left\langle \frac{\widehat{\omega}}{|\omega|} \right\rangle_\theta - H \frac{\widehat{\omega}}{\langle |\omega| \rangle_\theta} \right). \tag{64}$$

Note that $I_p = 2.77$.[7] The parameter y is determined by the approximate ambipolarity constraint that $\Gamma_1 \cong 0$, which is valid if $(M/M_e)^{2/3}(T_e/T_i)^{4/3}/|S|^{5/3} \gg 1$. Thus

$$y = \frac{\mu_2}{\mu_1} = -1.021, \tag{65}$$

where $\mu_1 = \int_0^\infty dx\, x^{5/3}(\nu_D/\nu_i)e^{-x} = 0.531$ and $\mu_2 = \int_0^\infty dx\, x^{5/3}(x-5/2)(\nu_D/\nu_i)e^{-x} = -0.542$. The ion heat flux is then

$$\Gamma_2 = -\frac{8}{2^{2/3}9\sqrt{\pi}} N\nu_i \left(\frac{Iv_t}{\Omega_0}\right)^{7/3} \frac{C_1^{2/3}}{|S|^{5/3}}\, I_p \cdot \frac{T'}{T}\left(\mu_3 - \frac{\mu_2^2}{\mu_1}\right), \tag{66}$$

where $\mu_3 = \int_0^\infty dx\, x^{5/3}(x-5/2)^2(\nu_D/\nu_i)e^{-x} = 1.282$. In terms of ion thermal conductivity χ_ψ,

$$\langle \boldsymbol{q}\cdot\boldsymbol{\nabla}\psi\rangle = -N\chi_\psi T_i', \tag{67}$$

where

$$\chi_\psi = 0.8\nu_i\left(\frac{Iv_t}{\Omega_0}\right)^{7/3}\left(\frac{q}{IR}\right)^{1/3}\frac{1}{|S|^{5/3}}. \tag{68}$$

When $|S| \to 1$, Eq. (68) reproduces the expression of the ion heat conductivity in Ref. 7.

Note that ion parallel flow in Eq. (42) is completely determined once y is known. Because ion parallel flow is modified, the electron bootstrap current is also modified accordingly.

VI. Concluding Remarks

The scaling of χ_ψ in Eq. (68) can be understood from a random walk process in ψ,

$$\chi_\psi \sim f_t \cdot \frac{\nu_i}{f_t^2}\cdot(\Delta\psi)^2. \tag{69}$$

Since $\psi = x^2$ we have

$$\Delta\psi \sim \left(\frac{I^2 C_1}{\Omega_0^2}\,\frac{v_t^2}{|S|}\right)^{2/3} \tag{70}$$

which indicates that the size of the orbits in ψ is reduced by a factor of $|S|^{2/3}$. Combining Eqs. (46) and (70), we find

$$\chi_\psi \sim \nu_i\left(\frac{Iv_t}{\Omega_0}\right)^{7/3}\frac{C_1^{2/3}}{|S|^{5/3}}, \tag{71}$$

which is the same as Eq. (68) except for the numerical coefficient and the plasma density.

The reduction in χ_ψ due to the effects of orbit squeezing may be responsible for the better than conventional neoclassical ion confinement observed in Enhanced-Reverse-Shear (ERS) mode.[12]

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