ENERGETIC PARTICLE STABILIZATION OF BALLOONING MODES IN TOKAMAKS

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Abstract:

Introduction of an anisotropic, highly energetic trapped particle species into a tokamak may allow direct stable access to the high-beta regime of second stability. Under certain conditions, the mode at marginal stability acquires a real frequency close to the precessional drift frequency of the energetic particles, perhaps correlating with recent "fishbone" observations on PDX.
Plasma stabilization by an energetic particle component has been proposed and analyzed in the Astron$^1$ and ion ring devices$^{2,3}$ and in the ELMO bumpy torus.$^4$ In the latter, annuli of hot electrons provide stability for the toroidal core plasma. Because these hot electrons precess so rapidly, they tend to be rigid with respect to usual $E \times B$ fluid displacements and hence create a stabilizing diamagnetic well. In the letter we suggest that energetic particles could have similar value if introduced into a tokamak. Whereas continuous introduction of hot particles is essential for stability in the bumpy torus, in a tokamak they may only be required until the plasma reaches the second stability region$^{5,6}$ where stability may improve with increasing beta, as has been shown at least with respect to ballooning and internal kink modes.

The stabilizing effects of fast ions have been pointed out recently by Connor et al.,$^7$ who perturbatively analyzed isotropic circulating particles in the zero-bounce-frequency limit. The ballooning stability of anisotropic tokamaks has also been examined, without kinetic effects$^8$ but with finite gyroradii.$^9$

Here, we analyze ideal magnetohydrodynamic (MHD) ballooning stability when a fairly anisotropic population of energetic particles is mirror trapped on the outer, unfavorable curvature side of a tokamak. These particles are assumed to drift across field lines rapidly: $\bar{w}_{dh} \gg |\omega|$ where $\bar{w}_{dh}$ is their bounce-averaged magnetic drift frequency and $\omega$ is the frequency (or growth rate) for the perturbation of interest. Also, since they are trapped on the outside, we assume that $\omega_{th}/\bar{w}_{dh} > 0$, with $\omega_{th}$ their diamagnetic frequency.
Under these assumptions, we can investigate linear stability by means of the low-frequency kinetic energy principle

\[ \delta W = \delta W_f + \delta W_k, \]

where the fluid term is

\[
\begin{align*}
\delta W_f &= \frac{1}{2} \int \frac{ds}{B} \left[ \sigma |\nabla S|^2 (\delta \cdot \nabla \phi)^2 + \tau (Q_\parallel - \sigma \frac{\tau}{t} \delta \cdot \kappa \phi)^2 \right. \\
&\left. - (\delta \cdot \kappa) (\delta \cdot \nabla P_{\parallel} + \sigma \frac{\tau}{t} \delta \cdot \nabla P_L) \phi^2 \right]
\end{align*}
\]

and the kinetic term is

\[
\delta W_k = \frac{1}{2} \int dE d\mu \left[ \frac{d(\sigma Q_\parallel + \nu_0^2 \delta \cdot \kappa \phi)}{\nu_0^2 (\delta \cdot \nabla B + \nu_0^2 \delta \cdot \kappa)} \right]^2 \\

\int \frac{d(\sigma Q_\parallel + \nu_0^2 \delta \cdot \kappa \phi)}{\nu_0^2 (\delta \cdot \nabla B + \nu_0^2 \delta \cdot \kappa)}
\]

Here, \( Q_\parallel \) is the (Lagrangian) magnetic field perturbation parallel to the equilibrium field \( \mathbf{b} = \delta \mathbf{B} \) and \( \phi \) is the perturbed electrostatic potential; \( P_{\perp,\parallel} \) are the total pressure components; \( s \) is the arc length along a field and \( \nabla = \nabla - (\nabla B) \partial / \partial B \), \( \sigma = 1 + (P_{\perp} - P_{\parallel})/B^2 \), \( \tau = 1 + (\nabla P_{\perp}/B \partial B) \), \( \kappa = (\delta \cdot \nabla)\delta \), \( \mu = \nu_0^2 / 2B \), and \( E = \nu_0^2 / 2 + \mu B \). We have restricted attention to high-mode-number interchange-ballooning modes, whose transverse variation is in the eikonal \( S \), where \( \mathbf{b} \cdot \nabla S = 0 \) and \( \delta = \delta \mathbf{B} \times \nabla S / B^2 \). Eq. (2) pertains to the high bounce frequency limit, appropriate for trapped fast particles, in which their distribution function \( F_h \) is constant on a field line. Hot particles trapped on the outside of a tokamak stabilize through \( \delta W_k \), but are destabilizing in \( \delta W_f \).
To simplify the analysis of $\delta W_{k^*}$, we invoke the Schwartz inequality to obtain a lower bound: $\delta W_k \geq \delta W_1$, with

$$\delta W_1 = \frac{1}{2} \frac{\left[ \frac{1}{B} (\vec{\epsilon} \cdot \vec{\nabla} P_{\perp h} + (\vec{\epsilon} \cdot \vec{k})(\vec{\epsilon} \cdot \vec{\nabla} P_{\parallel h}) \phi \right]^2}{\int \frac{ds}{B} \left[ \frac{1}{B} (\vec{\epsilon} \cdot \vec{\nabla} B)(\vec{\epsilon} \cdot \vec{\nabla} P_{\perp h}) + (\vec{\epsilon} \cdot \vec{k})(\vec{\epsilon} \cdot \vec{\nabla} P_{\parallel h}) \right]} \cdot \tag{3}$$

A pessimistic estimate of stability can then be obtained by first minimizing $\delta W_k + \delta W_1$ with respect to $Q_{\parallel}$ to obtain

$$Q_{\parallel} = \frac{\sigma}{\tau} B \phi (\vec{\epsilon} \cdot \vec{k}) - \frac{1}{\tau B} (\vec{\epsilon} \cdot \vec{\nabla} P_{\perp h}) \Lambda, \text{ with}$$

$$\Lambda(j) = \frac{\int \frac{ds}{B} (\vec{\epsilon} \cdot \vec{k})(\vec{\epsilon} \cdot \vec{\nabla} P_{\perp h} + \frac{\sigma}{\tau} \vec{\epsilon} \cdot \vec{\nabla} P_{\parallel h}) \phi}{\int \frac{ds}{B} (\vec{\epsilon} \cdot \vec{k})(\vec{\epsilon} \cdot \vec{\nabla} P_{\perp h} + \frac{\sigma}{\tau} \vec{\epsilon} \cdot \vec{\nabla} P_{\parallel h}) - \frac{1}{\tau B^2} (\vec{\epsilon} \cdot \vec{\nabla} P_{\perp h})(\vec{\epsilon} \cdot \vec{\nabla} P_{\parallel h})]} \tag{4}$$

where $P_{\perp h}$ and $P_{\parallel}$ are the hot and (isotropic) core plasma pressures. The line integrals in Eqs. (3) and (4) are to be performed over the $j^{th}$ trapped particle region. Next we vary with respect to $\phi$ to obtain the inhomogeneous ballooning equation

$$\vec{B} \cdot \nabla \left( \frac{\sigma |V_S|^2}{B^2} \vec{B} \cdot \nabla \phi \right) + (\vec{\epsilon} \cdot \vec{k})(\vec{\epsilon} \cdot \vec{\nabla} P_{\parallel} + \frac{\sigma}{\tau} \vec{\epsilon} \cdot \vec{\nabla} P_{\perp}) \phi = (\vec{\epsilon} \cdot \vec{k})(\vec{\epsilon} \cdot \vec{\nabla} P_{\parallel h} + \frac{\sigma}{\tau} \vec{\epsilon} \cdot \vec{\nabla} P_{\perp h}) \Lambda. \tag{5}$$

The general solution of Eq. (5) is $\phi = \phi_0 + c \phi_1$, where $\phi_0$ is the homogeneous solution and $\phi_1$ is the particular solution for $\Lambda=1$, with $c$ then determined by Eq. (6). The solution is successively generated by solving for $\phi_0$ and $\phi_1$ in each trapped/untrapped region. If $\phi$ is well
behaved at infinity and does not change sign for \(|s| < \infty\), the 
equilibrium is stable.

Interestingly, if the core pressure vanishes, \(\psi = \text{const.}\) is an 
equal solution of Eq. (7); i.e., the hot particles precess too rapidly 
to be affected by curvature. For small core plasma beta, the 
right-hand side of Eq. (7) can be expanded to show that ballooning 
instability is then driven only by the core pressure gradient, while 
the hot particles contribute a stabilizing diamagnetic well. For large 
\(\beta_c\), the integrand in the denominator of \(A\) can vanish; this is related 
to the core beta limit predicted for bumpy tori\(^{10}\) but for our problem 
occurs after drift reversal.

In order to apply Eq. (5) to our stability problem, we must first 
obtain an appropriate equilibrium. With \(\mathbf{B} = \nabla\psi \times \nabla\mathbf{B}\) in Clebsch form, 
the poloidal flux \(\psi\) satisfies the anisotropic Grad-Shafranov 
equation\(^{12}\)

\[
\left( \frac{3^2}{3R^2} - \frac{1}{R} \frac{3}{3R} + \frac{3^2}{3Z^2} \right) \psi + \nabla\psi \cdot \nabla \ln \sigma = - \frac{1}{\sigma} \frac{3G}{3\psi} - \frac{R^2}{3^2} \frac{3P}{3\psi},
\]  

with \(G(\psi) = \frac{1}{2} (\sigma R B_T)^2\), \(B_T = R \mathbf{B} \cdot \nabla\phi\) the toroidal field with \(\phi\) the 
toroidal angle of symmetry, and \(R\) and \(Z\) the major radius and symmetry 
axis coordinates. Parallel pressure balance requires 
\(\hat{\mathbf{B}} \cdot \nabla (P_B/B^2) = (P_B/B^2) \hat{\mathbf{B}} \cdot \nabla \mathbf{B}\). For large-toroidal-mode-number ballooning 
 modes, we ignore radial variation and replace \(\nabla S\) by the gradient of the 
winding function, \(\nabla S = \hat{\mathbf{B}} \times \nabla \psi / |\nabla \psi|^2 + \lambda \nabla \psi\). Its covariant component 
normal to a flux surface, \(\lambda\), is the local shear\(^{6}\) and satisfies 
\(\hat{\mathbf{B}}_P \cdot \nabla \lambda = V \cdot (\nabla \psi B_T/R |\nabla \psi|^2)\), with \(\hat{\mathbf{B}}_P = \nabla \phi \times \nabla \psi\) the poloidal field.
To proceed further, we adopt a model equilibrium\textsuperscript{5,6,13} in which the aspect ratio is large ($r/R \ll 1$), the flux surfaces are shifted circles, and the plasma beta is small but has a finite gradient localized radially in a thin layer. Also, we take the convenient poloidal distribution $P_{ih} = \text{const.}$ for $\left|\hat{\theta}\right| < \theta_0$ and zero elsewhere. In this limit the equilibrium equations can be manipulated to give

$$\nabla \psi = \left(\frac{q}{r}\right)\left[\hat{\psi} + \hat{h}(\theta)\right], \quad \text{with} \quad h(\theta) = S(\theta-\theta_k) - \alpha_c(\sin \theta - \sin \theta_k) - \frac{1}{2} \alpha_h[\sin \theta - g(\theta) - g(\theta_k)],$$

$$\sin \theta - \frac{\hat{\theta}}{\pi}[\sin \theta_0 + (\pi-\theta_0)\cos \theta_0], \quad 0 < \hat{\theta} < \theta_0$$

$$g(\theta) = (1 - \frac{\theta}{\pi})(\sin \theta_0 - \theta_0\cos \theta_0), \quad \theta_0 < \hat{\theta} < 2\pi-\theta_0 \quad (7)$$

$$\sin \theta - (\frac{\theta}{\pi} - 2)[\sin \theta_0 + (\pi-\theta_0)\cos \theta_0], \quad 2\pi-\theta_0 < \hat{\theta} < 2\pi$$

Here, $\hat{\theta}$ is now the extended poloidal coordinate of the ballooning representation, with $\hat{\theta}$ its value modulo $2\pi$, and $\theta_k$ is the zero of the local shear. Also, we defined $S = rq'/q$, $q = RB_p/BB_p$, and $\alpha = -2Rq^2p'/B_T^2$, and primes mean $\partial/\partial r$. The ballooning equation (5) becomes

$$\frac{d}{d\theta} [1+h^2(\theta)] \frac{d\phi}{d\theta} + (\alpha_c + \frac{1}{2} \alpha_h)D(\theta)\phi = \frac{1}{2} \alpha_h D \int_{\Psi}^{\Psi_0} \frac{d\psi \Psi D}{D - \alpha_c/2q^2} \quad (8)$$

with $D(\theta) = \cos \theta + h(\theta)\sin \theta$.

Figure 1 shows various stability boundaries in shear $S$ and core beta $\alpha_c$, with the beta of the hot particles $\alpha_h$ and their degree of localization $\theta_0$ as parameters, for $q=2$ and $\theta_k = 0$. The two dashed
curves show the well-known boundaries for first and second ballooning stability (without hot particles). The dotted lines indicate where drift reversal occurs at zero \( \alpha_h \) according to the condition \( \tilde{\omega}_{d,h}(\theta_0) = 0 \) which is easily expressed in terms of elliptic integrals. Thus, use of the Schwartz inequality limits the validity of our stability analysis to the left of the dotted line for a given \( \theta_0 \). The solid curves in Fig. 1 are the stability boundaries in the presence of hot particles; at every point on these curves, \( \alpha_h \) is chosen to have its maximum value allowed by the condition \( \tilde{\omega}_{d,h}\omega_{*h} > 0 \).

Although this procedure underestimates stability, the results in Fig. 1 nevertheless indicate that energetic particles trapped, for example, between \( \theta = \pm\pi/4 \) are able to stabilize ballooning for shear values up to \( S = 0.9 \) and for core beta values beyond the second stability threshold. For \( q = 4 \) and \( \theta_0 = \pi/4 \), stabilization extends up to \( S = 1.9 \), appropriate near the plasma edge. Also, the value of \( \theta_k \) was varied, to consider modes peaked off the midplane. With \( q = 2 \) and \( \theta_0 = \pi/4 \), the stability boundary for \( \theta_k = 3\pi/8 \) (approximately the most unfavorable \( \theta_k \) value for large \( \alpha_c \)) has virtually the same minimum in shear at \( S = 0.9 \) as the curve for \( \theta_k = 0 \), but dips abruptly to \( S = 0.6 \) at the intersection of the second stability and drift reversal boundaries.

We conclude that it is possible to bridge the ballooning gap between first and second stability by means of energetic particles, which are no longer needed after second stability is attained. Presumably the same scheme could be used in other devices, as has been suggested for the heliac by Furth and Boozer. The technological requirements for injection or heating of the hot particles, as well as
their power balance, deserve further study. Rather high energies are required, as will be seen below. Microinstabilities, such as whistlers or modes near the ion cyclotron frequency, may be possible. On the other hand, finite gyroradius and banana width of the hot particles could improve stability, whereas their slowed-down component could be drift resonantly destabilizing. Moreover, the same theory as described in this Letter can be applied to "sloshing ions", i.e. with $\bar{\omega}_{dh}\omega_{ph} < 0$, which are found to provide an alternative means for stabilizing a tokamak; although more difficult to produce than hot particles, they do not lead to residual resonant destabilization.

Finally, we note that when the precessional drift of the energetic particles is not large enough to decouple them, marginal stability occurs with a real frequency close to $\bar{\omega}_{dh}$. A simple discussion of finite frequencies can be based on decomposing the energy $\delta W = -\omega^2 \delta W_1 + \delta W_{fc} + \delta W_{ph} + \delta W_{kh}$ into fluid and kinetic energies for the core and hot species, with $\delta W_1$ for ion inertia. Let $\omega^2$ determine the low frequency ($\omega/\bar{\omega}_{dh} \rightarrow 0$) stability as discussed previously, and let $\omega_{mhd}^2 = \omega^2 - \gamma_h^2$ determine the fluid stability, where $\gamma_h$ is the growth rate for an unstable flute driven by the hot pressure gradient. The kinetic energy can be approximated for monoenergetic hot particles as $\delta W_{kh} \propto \gamma_h^2 (\bar{\omega}_{dh}/\omega_{ph})(\omega-\omega_{ph})/(\omega-\bar{\omega}_{dh})$. For $\omega,\bar{\omega}_{dh} < \omega_{ph}$, we thus obtain a cubic dispersion relation: $\omega^3 - \omega^2 \bar{\omega}_{dh} - \omega \omega_{mhd}^2 + \omega^2 \bar{\omega}_{dh} = 0$. In the usual MHD limit of small $\bar{\omega}_{dh}$, the expected roots are $\omega^2 = \omega_{mhd}^2$ and a small real root at $\omega = \bar{\omega}_{dh}(\omega/\omega_{mhd})^2$. In the decoupled hot plasma limit of large $\bar{\omega}_{dh}$ analyzed earlier in this Letter, we find $\omega^2 = \omega^2$ and a real root at $\omega = \bar{\omega}_{dh}$. For finite $\bar{\omega}_{dh}$, stability requires $[1+3(\omega_{mhd}/\bar{\omega}_{dh})^2]^3 > [1+9(\omega_{mhd}^2 - 3\omega^2)/2\bar{\omega}_{dh}^2]^2$. Thus the condition
\( \omega_h^2 > 0 \) on which our earlier analysis was based is in fact a valid sufficient condition for stability only if \( \gamma_h/\bar{\omega}_{dh} < 0.5 \), which requires sufficiently hot particles to attain stabilization. For instance, if we estimate \( \gamma_h \approx 0.25(N_h T_h/N_i T_i R)^{1/2} \) applying for \( m=2 \) and \( \beta_{lh} = \beta_i \) with \( T_h \) and \( N_h \) the hot particle temperature and density, this condition becomes \( T_h/T_i > (r R)^{1/2}/2m \rho_i \), where \( T_i \) and \( \rho_i \) are the plasma ion temperature and gyroradius and \( m \) the poloidal mode number. For D-T reactor-like parameters (\( r=1.5 \text{ m}, \ R=5 \text{ m}, \ B=5 \text{ T}, \text{ and } T_i = 10 \text{ keV} \)), \( T_h \gtrsim 2.1 \text{ MeV} \) for \( m=2 \) is required. We also find that instability sets in at a finite frequency of order \( \bar{\omega}_{dh} \). For example, if \( \gamma_h^2 \) is very small, a resonant mode occurs when \( \omega_{mhd}^2 \approx \bar{\omega}_{dh}^2 \), with onset at \( \omega = \bar{\omega}_{dh} \).

A detailed evaluation for the case of internal kinks also predicts a similar onset.\(^{15}\) This result may correlate with recent observations\(^{16}\) on the PDX tokamak of beam-driven MHD oscillations referred to as fishbones that rotate toroidally at a rate approximately equal to the precession rate of the injected beam particles and are near marginal stability for internal kinking.

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References


Figure Caption

Marginal stability boundaries in shear $S$ and normalized core beta, $\alpha_c$, for maximal hot beta and various degrees of localization $\theta_0$. 