Equilibrium of a rotating mirror plasma

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The theory of axisymmetric equilibrium in a rotating mirror device is studied and compared with experimental data. In contrast to earlier studies that assumed constant along the magnetic field, the present formalism determines the two-dimensional structure of the electrostatic potential, which is nonlinearly coupled to the rotation. It is shown that parallel variation of the potential plays an important role in the resulting confinement. The limits of validity and possible applications of these results to centrifugally confined fusion plasmas and ion mass filters are considered.

I. INTRODUCTION

Axisymmetric mirror equilibria that are rotated by externally applied radial electric fields are of interest in applications to centrifugally confined fusion machines [1] and efficient ion mass filters [2]. The key issue in these devices is the two-dimensional structure of the electrostatic potential, which is nonlinearly coupled to the plasma rotation. However, in earlier treatments [1], this potential has been assumed to be constant along the field lines. In this paper, we make some reasonable simplifying assumptions to derive the electric potential and the nature and quality of the resulting confinement in the axial and radial directions. We find that the plasma is confined into a toroidal ring shape by centrifugal and electrostatic forces, and has a "spatial loss cone" along the axis. After discussing the boundary conditions that are required to produce these equilibria, and delineating the different rotation regimes where our calculations are valid, we show that our results qualitatively agree with initial data obtained on a new rotating mirror experiment. Extension of this work to the truly 3-dimensional equilibria and metastable states [3] that may arise as the rotation drive is increased will be discussed in a later paper.

II. BASIC VARIABLES

We consider a mirror system with azimuthal symmetry,

$$\frac{\partial}{\partial\theta} = 0. \tag{1}$$

Here θ is the azimuthal angle of cylindrical coordinates (r, θ, z); the corresponding unit vectors are

$$\hat{\boldsymbol{r}} \equiv \nabla r, \ \ \hat{\boldsymbol{\theta}} = r\nabla\theta, \ \ \hat{\boldsymbol{z}} = \nabla z.$$

The magnetic field $\boldsymbol{B} = B(r, z)\boldsymbol{b}$ is conveniently expressed in terms of its flux function, $\chi(r, z)$:

$$\boldsymbol{B} = \nabla\chi \times \nabla\theta. \tag{2}$$

Note that χ/r is the vector potential for $\boldsymbol{B}$.

We neglect parallel flow, but allow rapid perpendicular motion, given by the $E \times B$–drift:

$$\boldsymbol{V} = \boldsymbol{b} \times \nabla\Phi/B.$$

Because $\boldsymbol{b}\cdot\nabla\theta = 0$, an azimuthally symmetric Φ can give only an azimuthal drift,

$$\boldsymbol{V} = \hat{\boldsymbol{\theta}}V = \hat{\boldsymbol{\theta}}\left(\frac{b^z}{B}\frac{\partial\Phi}{\partial r} - \frac{b^r}{B}\frac{\partial\Phi}{\partial z}\right). \tag{3}$$

Thus the equilibrium is described by three scalar functions: the pressure $p(r, z)$, which determines the density n; the electrostatic potential $\Phi(r, z)$, which determines the velocity; and the flux function $\chi(r, z)$, which determines the magnetic field.

III. MAGNETIC FIELD GEOMETRY

A. Field–line average

We place the two ends of the mirror at $z = 0$ and $z = L$. The mirror ratio is conveniently measured by

$$R = B(r = 0, z = 0)/B(r = 0, z = L/2).$$

We will sometimes use the coordinate s measuring distance along the field line:

$$\nabla_{\parallel} = \hat{\boldsymbol{b}}\cdot\nabla = \frac{\partial}{\partial s}.$$

The average of some function $f(r, z)$ along a field line is denoted by

$$\bar{f} \equiv \langle f\rangle \equiv S^{-1}\int_0^L \frac{dz}{b^z} f(r_*(z), z) \tag{4}$$

where S is the length of the chosen field line and $r_*(z)$ is the cylindrical radius evaluated on the field–line trajectory. For any function g that is symmetric about the mirror mid–plane,

$$\left\langle \frac{\partial g}{\partial s}\right\rangle = 0.$$

An alternative version of the field–line average uses the coordinate transformation $(r, z) \rightarrow (\chi, s)$. Then we have

$$\bar{f}(\chi) \equiv \langle f\rangle \equiv S^{-1}\int_0^S ds f(\chi, s). \tag{5}$$

We will also use the notation

$$\tilde{f} \equiv f - \bar{f}.$$

B. Plasma current

The plasma current is computed from (2):

$$\nabla \times \boldsymbol{B} = -\nabla\theta(\nabla \cdot \nabla\chi) - (\nabla\chi \cdot \nabla)\nabla\theta + (\nabla\theta \cdot \nabla)\nabla\chi.$$

But

$$(\nabla\theta \cdot \nabla)\nabla\chi = -(\nabla\chi \cdot \nabla)\nabla\theta = \frac{\hat{\boldsymbol{\theta}}}{r^2}\frac{\partial\chi}{\partial r}$$

whence

$$\nabla \times \boldsymbol{B} = -\frac{\hat{\boldsymbol{\theta}}}{r}\left(\nabla \cdot \nabla\chi - \frac{2}{r}\frac{\partial\chi}{\partial r}\right).$$

We introduce the operator

$$\Delta^* \equiv \nabla \cdot \nabla - \frac{2}{r}\frac{\partial}{\partial r} = r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{6}$$

in order to write

$$\nabla \times \boldsymbol{B} = -\frac{\hat{\boldsymbol{\theta}}}{r}\Delta^*\chi. \tag{7}$$

C. Vacuum field

The magnetic flux corresponding to a vacuum field, $\nabla \times \boldsymbol{B}_v = 0$, is denoted by χ_v; it satisfies

$$\Delta^*\chi_v = 0.$$

Conventional separation of variables yields a solution

$$\chi_v = \frac{B_0}{2\pi}\left[\pi r^2 + qLrJ_1(kr)\cosh k(z - L/2)\right]$$

where $k = 2\pi/L$, the mirror ratio is measured by

$$R = \frac{1 + q\cosh\pi}{1 + q},$$

and J_1 is the Bessel function. We note that

$$J_1(kr) = \frac{kr}{2}\left[1 + \mathcal{O}(k^2r^2)\right]. \tag{8}$$

Hence for small kr (the so–called paraxial approximation),

$$\chi = B_0\frac{r^2}{2}[h(z) + \mathcal{O}(k^2r^2)]$$

where

$$h(z) \equiv 1 + q\cosh k(z - L/2) \tag{9}$$

is a dimensionless measure of the axial magnetic field near the cylindrical axis.

IV. BASIC EQUATIONS

A. Electron equilibration

The low collisionality of the experiment suggests that temperature gradients along the field are small; for simplicity we assume that gradients across the field are also negligible:

$$\nabla T_i = 0 = \nabla T_e.$$

Then the electron density n_e is related to the total pressure $p \equiv p_i + p_e$ by

$$n_e = p/(T_i + T_e). \tag{10}$$

Here the temperature is measured in energy units. We note that the isothermal assumption could be replaced by any equation of state, $p = p(n)$, without materially changing the analysis.

The same long–mean–free–path considerations lead to Maxwell–Boltzmann electron equilibration along $\boldsymbol{B}$:

$$\nabla_\parallel \log p = \nabla_\parallel \frac{e\Phi}{T_e} \tag{11}$$

where Φ is the electrostatic potential.

B. Force balance

Allowing for large plasma flow, $mV^2/T \sim 1$, we express equilibrium force balance as

$$m_i n \boldsymbol{V} \cdot \nabla \boldsymbol{V} + \nabla p = \mu_0^{-1}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}. \tag{12}$$

It is easily seen that for azimuthal, symmetric flow

$$\boldsymbol{V} \cdot \nabla \boldsymbol{V} = -V^2 \frac{\hat{\boldsymbol{r}}}{r}.$$

Hence, in view of (10), the convective inertial term has the form

$$m_i n \boldsymbol{V} \cdot \nabla \boldsymbol{V} = -\frac{m_i}{T_i + T_e} p V^2 \frac{\hat{\boldsymbol{r}}}{r}. \tag{13}$$

Regarding the magnetic force on the right–hand side of (12), we note that $\nabla\theta \cdot \nabla\chi = 0$ and quickly find

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = -\frac{\Delta^* \chi}{r^2} \nabla \chi. \tag{14}$$

Finally we substitute (13) and (14) into (12) in order to write the equation of motion as

$$\nabla p - \frac{m_i}{T_i + T_e}\frac{pV^2}{r}\hat{\boldsymbol{r}} = -\frac{\Delta^*\chi}{\mu_0 r^2}\nabla\chi. \tag{15}$$

Notice that

1. p cannot be a function of χ alone, because $\nabla\chi$ is not colinear with $\hat{\boldsymbol{r}}$; this fact distinguishes the mirror equilibrium from the toroidal case.

2. Force balance provides only two independent equations, since the θ–component of (15) is empty.

C. Normalized variables

At this point it is helpful to introduce normalized variables. We define the normalized potential ϕ by

$$\phi \equiv e\Phi/T_e$$

and the normalized flow velocity u by

$$u^2 \equiv \frac{m_i V^2}{T_i + T_e}.$$

Next we write the magnetic field as

$$\boldsymbol{B}(r, z) \equiv B_0\boldsymbol{H}(r, z)$$

where the constant B_0 is a representative field magnitude and $\boldsymbol{H}$ is dimensionless. Similarly we let

$$\alpha = \chi/B_0.$$

Then

$$\boldsymbol{H} = \nabla\alpha \times \nabla\theta.$$

The ion gyroradius is characterized by the constant

$$\rho^2 \equiv \frac{m_i T_e^2}{(T_i + T_e)e^2 B_0^2}.$$

and the pressure is conveniently measured by

$$\beta = p/B_0^2.$$

Finally we introduce the abbreviation

$$j \equiv \mu_0^{-1}\nabla \times \boldsymbol{H} \cdot \nabla\theta = -\frac{\Delta^*\alpha}{\mu_0 r^2}.$$

D. Normalized equations

Now parallel electron equilibration (11) is expressed by

$$\nabla_{\|}\phi = \nabla_{\|} \log \beta \tag{16}$$

and the force balance equation (15) takes the form

$$\nabla\beta = \beta u^2 \frac{\nabla r}{r} + j\nabla\alpha \tag{17}$$

where

$$u = \rho \left(\frac{\nabla\alpha \cdot \nabla\phi}{rH^2} \right) \tag{18}$$

is the normalized V_θ, and we have used the identity

$$\hat{\boldsymbol{b}} \times \nabla\phi \cdot r\nabla\theta = (rH)^{-1}\nabla\phi \cdot \nabla\alpha.$$

Equations (16)–(18) summarize our description of the rotating mirror equilibrium. We observe that

(a) (16) implies

$$\beta(\alpha, s) = \beta(\alpha, 0)e^{\Delta\phi(\alpha,s)} \tag{19}$$

where $\Delta\phi \equiv \phi(\alpha, s) - \phi(\alpha, 0)$.

(b) (17) has the solubility condition

$$\nabla \times \left(\beta u^2 \frac{\nabla r}{r} + j\nabla\alpha \right) = 0$$

or, equivalently,

$$r^2(\boldsymbol{H} \cdot \nabla)j = -\frac{\partial(\beta u^2)}{\partial z}. \tag{20}$$

V. ELECTROSTATIC POTENTIAL

A. Simplified equation

By combining (16), the parallel component of (17), and (18) we find

$$\nabla_{\|}\phi = \rho^2 \frac{b^r}{r} \left(\frac{\nabla\alpha \cdot \nabla\phi}{rH^2} \right)^2 \tag{21}$$

a nonlinear partial differential equation for the potential. To simplify the equation, we first assume $\nabla\alpha\cdot\nabla s\approx 0$, as pertains in the paraxial approximation. Then

$$\nabla\alpha\cdot\nabla\phi = |\nabla\alpha|^2\frac{\partial\phi}{\partial\alpha} = r^2H^2\frac{\partial\phi}{\partial\alpha}$$

and (21) becomes

$$\frac{\partial\phi}{\partial s} = \rho^2 r b^r\left(\frac{\partial\phi}{\partial\alpha}\right)^2.$$

However

$$rb^r = \hat{\boldsymbol{b}}\cdot\nabla\left(\frac{r^2}{2}\right) = \frac{\partial}{\partial s}(r^2/2), \tag{22}$$

so (21) can also be expressed as

$$\frac{\partial\phi}{\partial s} = \rho^2\frac{\partial(r^2/2)}{\partial s}\left(\frac{\partial\phi}{\partial\alpha}\right)^2. \tag{23}$$

The flow velocity is now given by

$$u = \rho r\frac{\partial\phi}{\partial\alpha}. \tag{24}$$

Notice that if the radial electric field vanishes at any point s_0 along a field line,

$$\frac{\partial\phi(\alpha,s_0)}{\partial\alpha} = 0,$$

then it will also vanish at all points interior to s_0. This fact prevents the radial electric field from changing direction.

We next display two distinct solutions to (23).

B. Fast rotation

When $\rho r(\partial\phi/\partial\alpha)\sim 1$, the rotation speed is fully comparable to the thermal speed. In that case we restrict attention to low–beta geometry and solve (23) using the vacuum magnetic field considered in subsection III C. From the paraxial expression,

$$\alpha\approx\frac{r^2}{2}h(z)$$

of subsection III C we notice that (23) can be expressed as

$$\frac{\partial\phi}{\partial s} = \rho^2\alpha\frac{\partial h^{-1}}{\partial s}\left(\frac{\partial\phi}{\partial\alpha}\right)^2.$$

This nonlinear equation has the exact solution

$$\phi = \phi_0 - \frac{\alpha h}{\rho^2} = \phi_0 - \frac{r^2}{2\rho^2}[1 + q \cosh k(z - L/2)]^2 \tag{25}$$

where ϕ_0 is an arbitrary constant.

As an exact solution to a nonlinear problem, (25) resembles a soliton and may approximate the observed field profile in some cases. However, its general physical significance is not clear; for example, it is not a realistic solution in the limit of vanishing gyroradius.

C. Slow rotation

The case of slow rotation appears more realistic; it corresponds to small gyroradius in the sense that

$$(\rho \nabla \phi)^2 \ll 1. \tag{26}$$

Then (21) implies

$$\phi = \bar{\phi}(\alpha) + \tilde{\phi}(\alpha, s)$$

where the second term is relatively small, and where s is the field–line coordinate defined in subsection III A. Now (21) has become a relation between $\tilde{\phi}$ and $\bar{\phi}$:

$$\frac{\partial \tilde{\phi}}{\partial s} = \rho^2 r b^r \left(\frac{\partial \bar{\phi}}{\partial \alpha}\right)^2. \tag{27}$$

But (22) allows us to express the right-hand side of (27) as an s-derivative, and thus to conclude that

$$\tilde{\phi} = \frac{\rho^2}{2}\left(\frac{\partial \bar{\phi}}{\partial \alpha}\right)^2 (r^2 - \langle r^2 \rangle). \tag{28}$$

Notice that this solution does not require small beta. However the function $\bar{\phi}(\alpha)$ must in this case be determined by boundary data.

VI. LOW BETA EQUILIBRIUM

A. Grad–Shafranov equation

Returning to the force balance relation (17), we express it in the form

$$\frac{\partial \beta}{\partial \alpha}\nabla \alpha + \frac{\partial \beta}{\partial s}\nabla s = \beta \frac{u^2}{r}\left(\frac{\partial r}{\partial \alpha}\nabla \alpha + \frac{\partial r}{\partial s}\nabla s\right) + j \nabla \alpha.$$

Here the terms proportional to ∇s are precisely those considered in the previous section. We now study the terms proportional to $\nabla\alpha$:

$$j = \frac{\partial\beta}{\partial\alpha} - \beta\frac{u^2}{r}\frac{\partial r}{\partial\alpha}. \tag{29}$$

We see that $j \propto \beta$. When β is small, first–order accuracy allows us to replace α by the known function α_v on the right–hand side. Thus, recalling (7), we need to solve

$$\Delta^*\alpha = -r^2\frac{\partial\beta}{\partial\alpha_v} + \beta u^2 r\frac{\partial r}{\partial\alpha_v}. \tag{30}$$

Equation (30) is the rotating mirror version of the Grad–Shafranov equation, used to study tokamak equilibrium. It has the same role as the conventional Grad–Shafranov equation: to determine the corrections to the magnetic field resulting from plasma current. The distinctive features of (30) are its lack of an azimuthal magnetic field (the toroidal field in a tokamak) and its sensitivity to rotation. Note that the latter enters explicitly and also implicity, through (19). We next consider the implicit effect in more detail.

B. Summary and discussion

We have considered an idealization of the rotating mirror experiment, omitting temperature variation, viscosity, impurities and such neutral gas effects as charge exchange. For this simplest case, we have determined the two-dimensional structure of the electrostatic potential $\Phi(r, z)$ in an azimuthally symmetric, rotating magnetic mirror. We have found two types of solution for the potential, given by (25) and (28). We have also outlined a formalism for studying mirror magnetic field at finite plasma beta; its essential feature is a rotating-mirror version of the Grad–Shafranov equation, (30).

The effects of rotation on confinement are expressed by (19),

$$\beta(\alpha, s) = \beta_0(\alpha)e^{\Delta\phi(\alpha,s)}$$

where

$$\beta_0(\alpha) \equiv \beta(\alpha, 0).$$

Notice that

1. The surface $s = 0$ is also that on which $z = 0$; we assume that the pressure on this surface—that is, the function $\beta_0(\alpha)$—is known.

2. The function $\phi(\alpha, s = 0)$ is not by itself significant, since it can be absorbed into $\beta_0(\alpha)$.

Confinement corresponds to a β that decreases as s approaches 0 or L. Since the only s–dependence occurs through ϕ, and since ϕ is driven exclusively by rotation, we describe this effect as "rotational confinement."

It is easily seen from (25) and (28) that rotational confinement does in fact pertain, in both cases of fast and slow rotation. In the fast rotation case, the decrease of β near the ends is clear from the minus sign in (25). In the slow rotation case, it follows from the fact that r^2 is decreasing and smaller than its average as the ends are approached. In fact the degree of confinement is not dramatically different in the two cases.

A special case of rotational confinement is Bernoulli confinement, based on the fact that flow kinetic energy $mn(V^2/2)$ adds to the pressure. But Bernoulli confinement does not pertain in the present configuration, where the $\nabla(V^2/2)$ contribution to $\boldsymbol{V} \cdot \nabla \boldsymbol{V}$ is precisely cancelled by part of $\boldsymbol{V} \times \nabla \times \boldsymbol{V}$. The confining mechanism relevant in our case (azimuthal flow and azimuthal symmetry) is simply centrifugal force.

Of course the centrifugal force from azimuthal rotation is purely radial. It has the desired parallel component only in the region between the mirror throats and the central plane, where the magnetic field has a substantial radial component. Figure 1 shows the three relevant forces. It is clear that, because of their negative charge, electrons will see a balance between (the parallel components of) the electric field and the pressure gradient; they are electrostatically confined. For the ions, however, both of these forces point out of the confinement region; ions are confined only by the parallel component of the centrifugal force (F_{cf} in the figure).

The confinement is far from perfect: there is substantial pressure at the mirror ends for either fast or slow rotation. The most glaring defect of rotational confinement in mirror geometry is intrinsic: its failure on the cylindrical axis, where both V and b^r vanish. (The situation is a coordinate–space analogy to the mirror loss cone in velocity space.) However, one can imagine a ring–shaped (toroidal!) plasma, without pressure on–axis, and with rapid rotation throughout its volume. This case seems consistent with the present analysis—it corresponds to a monotonically increasing function $\beta_0(\alpha)$ that vanishes on axis—and will be considered in subsequent work.

Our results point to an explanation of how ion mass separation in a plasma mass filter [2] may benefit from a mirror geometry: the heavier ions will be better confined by the centrifugal force near the center of the machine.

Thus centrifugal confinement can lead to mass-dependent spatial loss rate profiles that will enhance differential extraction of heavier mass ions near the machine center. This is a variation of the scheme proposed in Ref. [2]. We have obtained preliminary data on a new rotating mirror experiment at The University of Texas. In this experiment, a radial electric field is applied to a mirror plasma by using combination of 6 concentric end electrode rings and a center rod biased to different voltages (Figure 2). Radially movable Langmuir and Mach probes [4] in the central plane of the machine measure the floating potential as well as the azimuthal rotation speed of the plasma. The representative magnetic field B_0 is about 875 Gauss, corresponding to an electron cyclotron frequency of 2.45 GHz, the same as the one used by the microwave launched through one end of the machine to create a target plasma. The measured electron temperature is about 8 eV at the mid-plane, so that the ion Larmor radius is about 3 mm. The size of the plasma between mirror throats (L) is 1 m, and the perpendicular dimension where the representative B_0 is measured is .3 m, about 100 Larmor radii. The mirror ratio is 7.4, which corresponds to a value $q \cong 1$ in the vacuum field used by our theory. Figure 3 shows that our theoretical results qualitatively agree with the measured floating potential (dotted line with circles), which is $3.34 \cdot T_e$ lower than the plasma potential, while for zero biasing the floating potential (dotted line with squares) reflects no change as expected, since there is no plasma rotation, and in turn no centrifugal force involved.

These potential profiles and the calculated $\mathbf{E} \times \mathbf{B}$ drift are consistent with the direct measurements of the azimuthal rotation shown in Figure 4, which indicate rotation speeds of the order of the sound speed ($C_s = 3 \times 10^4\,\mathrm{m/s}$) when the plasma is biased. The net radial voltage drop near the center of the machine is smaller (about 100 V) than the externally applied voltage of 150 V, as we expect. However we cannot make a quantitative comparison at this time because we do not have floating potential measurements at different axial (z) locations.

Thus, while the potential on the electrodes is measured, its value in the plasma just beyond the sheath is not yet known. To resolve this issue, and to measure centrifugal confinement, we are adding an array of Langmuir probes near the mirror end (near $z = 0$).

The new probe array may also allow us to solve an interesting puzzle, not explained by the theory presented here: the potential drop depends strongly on the sign of the externally applied radial voltage. This asymmetry may be attributable to sheath structure, in particular shear differences on different parts of the ring electrodes. It may also be related to radial currents, whose magnetic forces are balanced by charge exchange. Since the sheath, as well as charge-exchange effects have not been included in the present idealized treatment, the asymmetry remains to be understood.

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FIGURE CAPTIONS

FIG. 1. The fluid forces and a field line, near the left mirror throat.

FIG. 2. Diagram of the experimental set up, which is symmetric about the z axis. The size of the plasma is given by the limiter, and represented here by the corresponding magnetic lines.

FIG. 3. Plot of the floating potential measured by the Langmuir probe at the mid-plane at different radial positions. In order to compare, the potential applied to the end rings is extrapolated following the corresponding magnetic line to the mid plane.

FIG. 4. Plot of the plasma drift velocity measured by the Mach probe at the mid plane. The Mach number is the ratio between the plasma velocity and the sound speed.