Forced magnetic reconnection in the inviscid Taylor problem

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The equations of incompressible inviscid 2-D MHD (magnetohydrodynamics) are numerically evolved in order to study a well-known model of forced magnetic reconnection. This problem, known as the Taylor problem, considers the response of a tearing-stable slab plasma equilibrium to a sudden, small amplitude boundary perturbation. The applied perturbation is such as to force magnetic reconnection and subsequent magnetic island formation within the plasma. The early dynamical phases of the reconnection process are investigated and found to be in good agreement with the analytic predictions of Hahm & Kulsrud [Phys. Fluids, **28**,2412 (1985)]. Recent criticisms of this analysis by Ishizawa & Tokuda [Phys. Plasmas, **8**, 376, (2001)] are shown to be unwarranted.

I. INTRODUCTION

This paper investigates a well-known model resistive MHD (magnetohydrodynamical) problem involving forced magnetic reconnection in 2-D slab geometry. This problem, generally known as the "Taylor Problem" (since it was first proposed by J.B. Taylor), deals with a tearing-stable slab plasma with an equilibrium magnetic field of the form

$$\mathbf{B}^{(0)} = \left[0, B_y^{(0)}(x), 0\right], \tag{1}$$

where $B_y^{(0)}(-x) = -B_y^{(0)}(x)$. The plasma is bounded by perfectly conducting walls located at $x = \pm a$, and is periodic in the y-direction with wavelength L. At $t = 0$, the walls are subject to a sudden deformation (in the x-direction) such that

$$x_{wall} \to \pm\left[a + \Xi_0 \cos(ky)\right], \tag{2}$$

where $k = 2\pi/L$, and $\Xi_0 \ll a$. The wall perturbation pushes oppositely directed magnetic field-lines together at the field null ($x = 0$), forcing them to reconnect and eventually form magnetic islands of wavelength L.

The inviscid Taylor problem was first investigated analytically by Hahm & Kulsrud,[1] who found five distinct dynamical phases in the reconnection process. The first four phases (labeled A, B, C, and D,) are governed by linear layer physics. The last (unlabeled) phase is described by the well-known nonlinear island dynamics of Rutherford.[2]

Recently, Ishizawa & Tokuda[3,4] have disputed the results of Hahm & Kulsrud, arguing that the initial phases A and B do not exist, and should be replaced by a quite different and controversial[5,6] reconnection phase. The essence of Ishizawa & Tokuda's criticism lies in the claim that Hahm & Kulsrud improperly used the well-known constant-ψ ordering[7] during their derivation of phases A and B.

The aim of this paper is to resolve the above mentioned controversy by showing that Hahm & Kulsrud's analysis does indeed correctly describe the initial phases of forced magnetic reconnection in the inviscid Taylor problem. Since the many steps employed in the standard analytic treatment of this problem are difficult for the non-expert to follow, we have chosen to employ a more transparent and direct route to calculating the magnetic reconnection rate: *i.e.*, numerical simulation. Now, Ishizawa & Tokuda's analysis is only concerned with forced reconnection in the *linear* regime, so in the following we shall numerically evolve the *linearized* equations of 2-D incompressible inviscid MHD, calculating the reconnection rate as a function of time, and making careful comparisons with Hahm & Kulsrud's predictions.

II. BASIC EQUATIONS AND GEOMETRY

Let us adopt standard Cartesian coordinates $(x\,,\,y\,,\,z)$. It is assumed that $\partial/\partial z \equiv 0$. The equilibrium magnetic field is specified in Eq. (1). For clarity, dimensionless variables are adopted in order to remove any unimportant plasma characteristics such as overall size, absolute magnetic field strength, *etc.* After this re-scaling, the perfectly conducting

walls are located at $x = \pm 1$, and the magnetic field is normalized such that $B_y^{(0)}(1) = 1$. Time is normalized to the characteristic Alfvén time evaluated at the walls.

Defining a magnetic flux function, $\boldsymbol{B} = \nabla\psi \wedge \hat{z}$, and a stream function, $\boldsymbol{V} = \nabla\phi \wedge \hat{z}$, the equations of incompressible inviscid 2-D MHD reduce to the following complete set:

$$\frac{\partial\psi}{\partial t} = [\phi, \psi] - \eta j + \eta, \tag{3}$$

$$\frac{\partial\omega}{\partial t} = [\phi, \omega] + [j, \psi], \tag{4}$$

$$j = -\nabla^2\psi, \tag{5}$$

$$\omega = -\nabla^2\phi, \tag{6}$$

where the standard Poisson bracket is given by $[A, B] = \partial_x A\, \partial_y B - \partial_y A\, \partial_x B$. The (constant) resistivity η is assumed to be much less than unity (*i.e.*, the Lundquist number $S = 1/\eta$ is assumed to be much greater than unity). The z-directed current density, j, is driven by a constant electric field $E^0 = \eta$ (pursuant to the boundary condition on the magnetic field). Finally, ω is the z-directed vorticity.

In dimensionless coordinates, the plasma lies in the rectangular region $-1 \leq x \leq +1$ and $-L/2 \leq y \leq +L/2$, where L is the normalized periodicity length. The tearing-stable plasma equilibrium is specified by $\psi^{(0)}(x) = -x^2/2$, $\omega^{(0)} = 0$, $j^{(0)}(x) = 1$, and $\phi^{(0)} = 0$.

Consider the wall displacement

$$x_{wall}(t) = \pm\left[1 + \Xi(t)\ \cos(ky)\right], \tag{7}$$

where $k = 2\pi/L$ is the normalized wave-number. Here,

$$\Xi(t) = \Xi_0\left[1 - e^{-t/\tau} - (t/\tau)\ e^{-t/\tau}\right], \tag{8}$$

for $t \geq 0$, and $\Xi(t) = 0$ for $t < 0$. This choice ensures $\Xi(t)$ and $d\Xi(t)/dt$ are both continuous at $t = 0$. The parameter τ represents the time-scale on which the wall displacement is switched on. Assuming that $\Xi_0 \ll 1$, appropriate boundary conditions at the walls are:

$$\psi(\pm 1, y, t) = -1/2 + \Xi(t)\ \cos(ky), \tag{9}$$

$$\omega(\pm 1, y, t) = 0, \tag{10}$$

$$j(\pm 1, y, t) = 1, \tag{11}$$

$$\phi(\pm 1, y, t) = \pm\frac{1}{k}\frac{d\Xi(t)}{dt}\ \sin(ky). \tag{12}$$

III. ANALYTIC THEORY

A. Introduction

The standard analytical treatment of the Taylor problem exploits the fact that in the high Lundquist number limit marginally-stable ideal-MHD only breaks down in a very narrow region centered on the field null (at $x = 0$). Thus, the plasma can be separated into two regions. In the "outer region," which comprises most of the plasma, both plasma resistivity and inertia can be neglected. However, resistivity and inertia play important roles in the "inner region," which is centered on the null surface, and is extremely narrow in the x-direction. Of course, it is necessary to asymptotically match analytic solutions obtained in both regions in order to obtain a complete solution.

B. Outer Region

Linearizing the MHD equations, we can write the perturbed flux- and stream-functions in the form $\delta\psi(x, y, t) = \psi(x, t)\ \cos(ky)$ and $\delta\phi(x, y, t) = \phi(x, t)\ \sin(ky)$. Neglecting resistivity, the linearized Ohm's law (3) yields

$$\phi(x, t) = \frac{1}{k\, x}\frac{\partial\psi(x, t)}{\partial t}. \tag{13}$$

Neglecting inertia, the vorticity equation (4) reduces to

$$\frac{\partial^2\psi(x,t)}{\partial x^2} - k^2\psi(x,t) = 0. \tag{14}$$

The solution of Eq. (14) which satisfies the boundary condition (9) and can be matched to the inner solution near the null surface is written

$$\psi^{out}(x,t) = \Psi(t)\left[\cosh kx - \frac{\sinh k\,|x|}{\tanh k}\right] + \Xi(t)\frac{\sinh k\,|x|}{\sinh k}. \tag{15}$$

At the boundary of the inner region the above expression reduces to

$$\psi^{out}(x\to 0,t) \;\longrightarrow\; \Psi(t) + \frac{1}{2}\Delta\Psi\,|x| + O(x^2), \tag{16}$$

where

$$\Delta\Psi = -\frac{2\;k\;\Psi(t)}{\tanh k} + \frac{2\;k\;\Xi(t)}{\sinh k}. \tag{17}$$

Naturally, the gradient discontinuity in $\psi^{out}(x,t)$ at $x=0$ is resolved in the inner region.

C. Inner Region

Let us linearize equations (3)-(6), and then exploit the narrowness of inner region by expanding all equilibirium quantities about $x=0$. All perturbed quantities are then Laplace transformed,

$$\bar{\psi}(x,g) = \int_0^\infty \psi(x,t)e^{-gt}dt, \tag{18}$$

and then Fourier transformed in the x-direction,

$$\bar{\psi}(x,g) = \int_{-\infty}^\infty \tilde{\psi}(\theta,g)e^{ikx\theta}d\theta. \tag{19}$$

After some algebra, we obtain the following *layer equation*

$$\frac{\partial}{\partial\theta}\left[\frac{\theta^2}{g+\eta\;k^2\;\theta^2}\frac{\partial\tilde{\phi}}{\partial\theta}\right] - g\;\theta^2\;\tilde{\phi} = 0. \tag{20}$$

A physically acceptable solution to the above equation must be well-behaved as $\theta\to\infty$. Fourier-Laplace transforming the outer region matching condition [Eqs. (13) and (16)] reveals the small-θ boundary condition satisfied by $\tilde{\phi}$:

$$\tilde{\phi}(\theta,g) \longrightarrow -\frac{g\bar{\Psi}(g)}{2}\left[\frac{\Delta(g)}{k\pi}\frac{1}{\theta} + 1 + O(\theta)\right] \tag{21}$$

as $\theta\to 0$. Here, $\Delta(g) = \Delta\bar{\Psi}(g)/\bar{\Psi}(g)$. (Note: in general, only the stream function is sufficiently well-behaved to be Fourier transformable, making it the preferred choice for asymptotic matching.) Solving the layer equation subject to the boundary condition at infinity, and then matching to the above expression as $\theta\to 0$, allows the layer quantity $\Delta(g)$ to be evaluated. The Laplace transformed perturbed magnetic flux at the *edge* of the inner region is then given by

$$\bar{\Psi}(g) = \frac{E_{sw}\bar{\Xi}(g)}{\Delta(g) - E_{ss}}, \tag{22}$$

where, for the sake of brevity, we have defined:

$$E_{ss} = -\frac{2k}{\tanh k}, \quad E_{sw} = \frac{2k}{\sinh k}. \tag{23}$$

Note that $E_{ss} < 0$ is the conventional tearing stability index.

Now, the true *reconnected magnetic flux*, $\psi_0(t)$, corresponds to the value of the perturbed flux-function at the *center*, rather than the *edge*, of the inner region. In fact, it can be demonstrated that

$$\bar{\psi}_0(g) = \frac{2\eta k^2}{g}\int_0^\infty \frac{\theta^2}{g+\eta\;k^2\;\theta^2}\;\frac{\partial\tilde{\phi}}{\partial\theta}\;d\theta. \tag{24}$$

D. Inertial Regime

When $t \ll \tau_1$, where

$$\tau_1 = \frac{1}{\eta^{1/3} k^{2/3}}, \tag{25}$$

resistivity may be neglected in the layer equation, which can be solved subject to the boundary conditions to give

$$\tilde{\phi}(\theta, g) = \frac{\bar{\Psi}(g)}{2} \frac{e^{-g\theta}}{\theta}, \tag{26}$$

with $\Delta(g) = -\pi k/g$. It follows from Eqs. (22) and (24) that

$$\bar{\psi}_0(g) = \frac{2\ \eta\ k\ E_{sw}\ \Xi_0}{\pi\,(1+g\,\tau)^2\ g^3}, \tag{27}$$

where we have neglected E_{ss} with respect to $\Delta(g)$ in accordance with the constraint that the width of the inner region remain small compared to unity in order for the asymptotic matching approach to be valid. Inverting (27) yields the reconnected magnetic flux as a function of time,

$$\psi_0(t) = \frac{2\eta k E_{sw} \Xi_0}{\pi} \left[\frac{t^2}{2} + 3\tau^2 - 2\tau t - \left(\tau t + 3\tau^2\right) e^{-t/\tau}\right]. \tag{28}$$

The associated reconnection rate is

$$J_I(t) \equiv \eta^{-1} \frac{d\psi_0}{dt} = \frac{2kE_{sw}\Xi_0}{\pi} \left[t + (t + 2\tau)\, e^{-t/\tau} - 2\tau\right]. \tag{29}$$

Note that $J_I(t)$ also measures the perturbed current density at the magnetic X-point. If the wall perturbation is switched on fairly suddenly (*i.e.*, $\tau \ll 1$), which is consistent with the assumptions of Hahm & Kulsrud, our results for the "inertial regime" correspond exactly to regimes A and B of Hahm & Kulsrud, although our analysis is somewhat different. Our analytic results do not agree with those of Ishizawa & Tokuda. Finally, our analysis reveals that the inertial regime is non-constant-ψ regime: *i.e.* $|\psi_0| \ll |\Psi|$. At no stage have we employed the constant-ψ ordering during the derivation of the above results.

E. Resistive Inertial Regime

In the limit $t \gg \tau_1$, the layer equation (20) can be solved in a two-step process,[8] yielding the constant-ψ result

$$\bar{\Psi}(g) = \bar{\psi}_0(g) = \frac{E_{sw}\Xi_0}{g\,(1+g\tau)^2 \left[\tau_{ri}^{5/4} g^{5/4} + (-E_{ss})\right]}, \tag{30}$$

where

$$\tau_{ri} = \left[\frac{2\pi\Gamma(3/4)}{\Gamma(1/4)}\right]^{4/5} \frac{1}{k^{2/5}\ \eta^{3/5}} \simeq \frac{1.8270}{k^{2/5}\ \eta^{3/5}}. \tag{31}$$

Inverse Laplace-transforming the above expression, whilst neglecting τ with respect to τ_{ri}, yields

$$\psi_0(t) = \frac{E_{sw}\Xi_0}{(-E_{ss})} \left[1 - \frac{8}{5}\ e^{-\cos(\pi/5)\ \tilde{t}}\ \cos\left(\sin(\pi/5)\ \tilde{t}\right) + I(\ \tilde{t}\ )\right], \tag{32}$$

where

$$\tilde{t} = (-E_{ss})^{4/5}\ t/\tau_{ri}, \tag{33}$$

and

$$I(\ \tilde{t}\ ) = \frac{4}{5\sqrt{2}\ \pi} \int_0^\infty \frac{e^{-y^{4/5}\ \tilde{t}}\ dy}{1 - \sqrt{2}\ y + y^2}. \tag{34}$$

The reconnection rate takes the form

$$J_{ri} = J_0 \left[\frac{8}{5} \cos(\pi/5)\ e^{-\cos(\pi/5)\ \tilde{t}}\ \cos\left(\sin(\pi/5)\tilde{t}\right) + \frac{8}{5} \sin(\pi/5)\ e^{-\cos(\pi/5)\ \tilde{t}}\ \sin\left(\sin(\pi/5)\tilde{t}\right) - K(\tilde{t}) \right] \tag{35}$$

where

$$J_0 = \frac{E_{sw}\ \Xi_0}{(-E_{ss})} \frac{(-E_{ss})^{4/5}}{\eta\ \tau_{ri}}, \tag{36}$$

and

$$K(\tilde{t}) = \frac{4}{5\sqrt{2}\ \pi} \int_0^\infty \frac{y^{4/5}\ e^{-y^{4/5}\ \tilde{t}}\ dy}{1 - \sqrt{2}\ y + y^2}. \tag{37}$$

Our "resistive-inertial" regime corresponds to regime D of Hahm & Kulsrud. (Regime C is valid for $t \sim \tau_1$, in which case we can find no closed-form analytic expression for the reconnection rate.)

IV. NUMERICAL RESULTS

A. Introduction

We have numerically advanced the linearized forms of Eqs. (3)-(6), with the boundary conditions (9)-(12), in a 1-D explicit finite-difference code that is second-order in time and space. An insignificant amount of viscosity, $\mu \sim 10^{-7}$, was added to the equations in order to control numerical instability. The computational grid is non-uniform in the x-direction, with more points being packed around the vicinity of the magnetic null (at $x = 0$), in order to improve resolution within the "inner region." The Courant-Freidrichs-Lewy condition on the Alfvén wave requires that the uniform timestep remain less than the minimum step-size in the x-direction. All numerical results discussed in this paper use a common plasma equilibrium characterized by $L = 8$, $\Xi_0 = 1 \times 10^{-4}$, $\tau = 1$, $k = \pi/4$, and $\mu = 1 \times 10^{-7}$.

B. Inertial Regime

Figure 1 shows the typical reconnection rate $J(t)$ obtained from our code at early times. The numerical reconnection rate follows the inertial rate, J_I, for times less than τ_1, in agreement with Hahm & Kulsrud's analysis. When $t \sim \tau_1$, the numerical reconnection rate undergoes a transition from the "inertial" to the "resistive inertial" rate, agreeing well with the latter for all later times. The large over-shoot seen during the transition phase is only observed when viscosity is negligible.

C. Resistive Inertial Regime

Figure 2 reveals the complete inviscid reconnection behaviour in the limit of small wall perturbation. For times larger than τ_1, the numerical reconnection rate and the "resistive inertial" rate, J_{ri}, are in excellent agreement. Two common features of the linear, inviscid reconnection rates seen in our simulations are: (1) a global maximum reached on a timescale $t \sim \tau_1$, and (2) a change in sign (corresponding to a maximum in the reconnected flux) on a timescale $t \sim \tau_{ri}$ (see Fig. 2).

D. Scaling Study

To validate Hahm & Kulsrud's theory over a wide range of resistivity values, we have derived approximate resistive scalings for the maximum reconnection rate, J_M, and the time at which the reconnection rate changes sign, t_0. Looking back to Fig. 1, we expect $J_M \sim J_I(\tau_1)$. Assuming the switch-on time, τ, is sufficiently short, we obtain

$$J_M \sim J_I(\tau_1) \sim \eta^{-1/3}. \tag{38}$$

As previously mentioned, we predict $t_0 \sim \tau_{ri}$, or $t_0 \sim \eta^{-3/5}$ [see Eq. (31)].

Figure 3 shows the numerical scaling of both t_0 and J_M with resistivity, and compares these with the theoretical predictions given above. Fewer t_0 values are calculated than J_M values (as resistivity decreases), because τ_{ri} quickly approaches several tens-of-thousands of Alfvén times and therefore becomes numerically impractical to calculate. The results for J_M deviate from the predicted curve for $\eta > 10^{-2}$, as the layer becomes of a width comparable to the whole plasma, and hence our asymptotic methods break down. Note the excellent agreement between the numerical results and the analytically predicted scalings.

V. SUMMARY AND CONCLUSIONS

We have implemented a 1-D explicit numerical scheme to study forced magnetic reconnection in the inviscid Taylor problem, where a tearing-stable slab plasma equilibrium is subjected to a sudden boundary perturbation in such a way as to drive magnetic reconnection within the plasma. In all cases studied, the initial reconnection rate agrees closely with that predicted analytically in the "inertial" regime discussed in Sect. III D. In the limit that the wall switch on time, τ, is small ($\tau \leq 1$), the inertial reconnected magnetic flux scales as $\psi_0 \sim \eta\, t^2$ (in agreement with Hahm & Kulsrud's regimes A and B), similar to the so-called "Sweet-Parker" timescale. However, any resemblence to Sweet-Parker reconnection is purely-coincidental, as the reconnection physics in the two cases are completely unrelated. Nowhere in our derivation of the "inertial" regime have we used the well-known constant-ψ ordering to obtain our results, which are identical to those of Hahm & Kulsrud. Now, Ishizawa & Tokuda claim Hahm & Kulsrud "improperly" used constant-ψ ordering to obtain their results for the initial reconnection phases. We have shown, both analytically and numerically, that this criticism is completely unfounded. We have numerically investigated the theoretical reconnection regimes over a wide range of resistivity values, and found good agreement between the numerical results and Hahm & Kulsrud's theory. We can find no reconnection regimes corresponding to those described by Ishizawa & Tokuda. We conclude that Hahm & Kulsrud's analysis for the initial phases of the reconnection process is correct.

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FIG. 1: Plot comparing the numerical reconnection rate (solid curve) with the theoretical reconnection rates in the "inertial" (dash-dotted curve) and "resistive inertial" (dashed curve) regimes. The dotted vertical line indicates the value of τ_1. The run was performed with a uniform time-step $dt = 1 \times 10^{-4}$, 600 x-grid points, and a minimum step size of $dx_{min} = 4.2 \times 10^{-4}$.

FIG. 2: Plot of the numerical reconnection rate $J(t)$ (solid curve) versus time. The dashed curve shows the theoretical "resistive inertial" reconnection rate. The right most dotted vertical line indicates the value of τ_1, whilst the left indicates the value of τ_{ri}. The run was performed with a uniform time-step $dt = 1 \times 10^{-4}$, 600 x-grid points, and a minimum step size of $dx_{min} = 4.2 \times 10^{-4}$.

FIG. 3: Scaling study. The square points show the numerical values of t_0, the time at which the reconnection rate changes sign. The triangular points show the numerical values of J_M, the maximum reconnection rate. The solid line illustrates the theoretically predicted result $t_0 \sim \eta^{-3/5}$. The dashed line illustrates the theoretical result $J_M \sim \eta^{-1/3}$. The resistivity ranges from 5.12×10^{-2} to 1.56×10^{-6}. The runs were performed with a uniform time-step $dt = 8 \times 10^{-5}$, 1000 x-grid points, and a minimum step size of $dx_{min} = 2.5 \times 10^{-4}$.

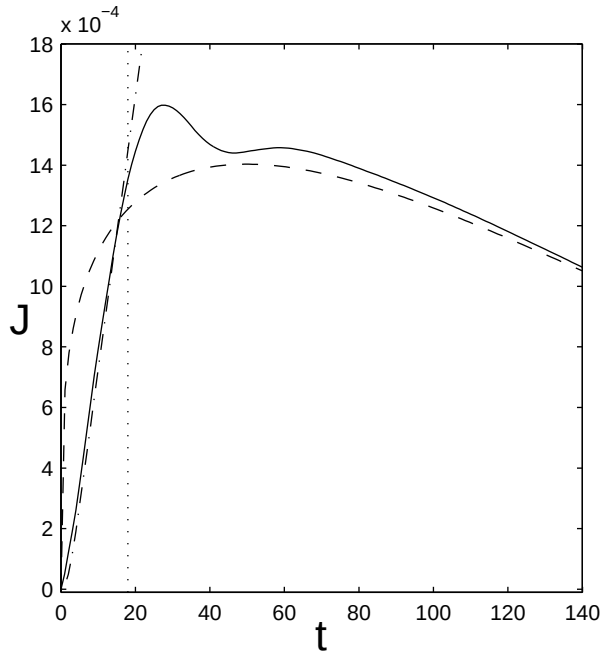
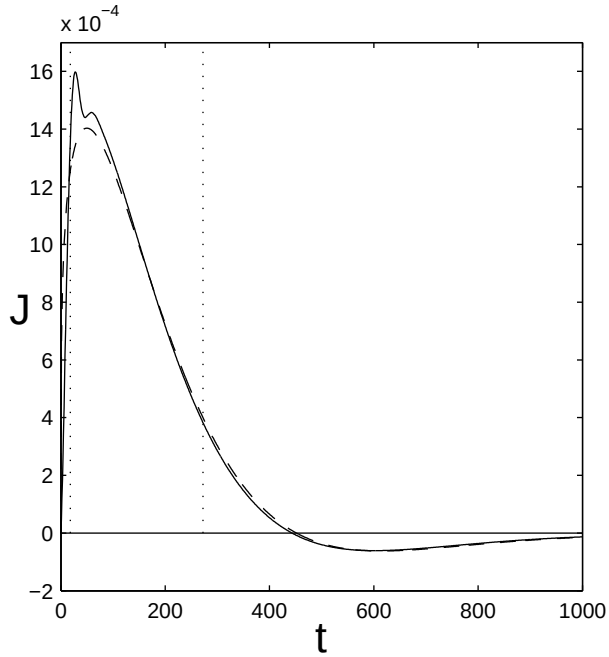


Figure 1.



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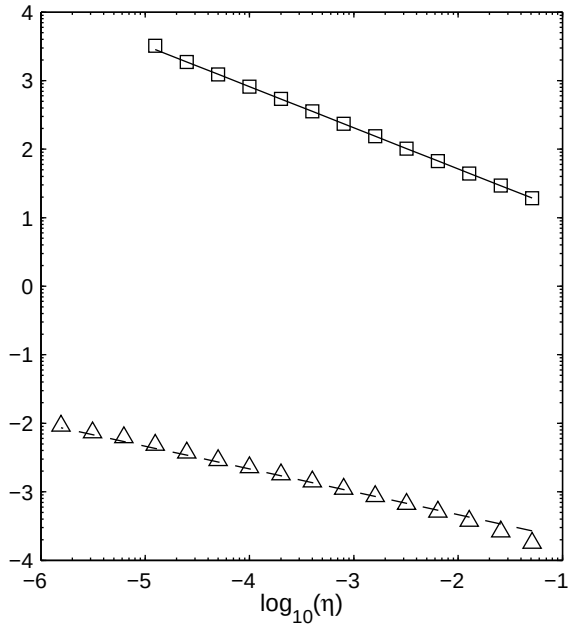

Figure 3.