Hall Current and Alfvén Wave

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Abstract

In ideal inhomogeneous magnetohydrodynamics (MHD), the Alfvén wave (the dominant low frequency mode of a magnetized plasmas) displays a continuous spectrum associated with singular eigenfunctions. It is shown that the coupling of the Hall term with the sound wave induces higher (fourth) order derivative in the Alfvén mode equation, and by resolving the singularity replaces the MHD continuum by a discrete spectrum. The mode structure resulting from the Hall resolution of the singularity is compared with the standard electron-inertia approach.

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I. INTRODUCTION

In its simplest manifestation, the Alfvén wave, one of the most important waves in magnetohydrodynamics (MHD), propagates along the magnetic field, and its frequency is determined entirely by the wavenumber in the direction parallel to the magnetic field (independent of the perpendicular wavenumbers) [1]. The spectral structure of the Alfvén wave has a close relationship with the geometric structure of the magnetic field lines. The fact that the characteristics, which are the direction of the wave energy propagation, are identical to the ambient magnetic field lines implies that the Alfvén wave equation has degenerate characteristics. Indeed, in an inhomogeneous MHD plasma, the Alfvén wave is the solution of a second-order singular differential equation implying a continuous spectrum [2, 3].

It is expected, however, that, a more realistic treatment of the plasma (inclusion of non-ideal MHD effect, for example) should yield a nonsingular system with discrete eigenmodes of the Alfvén wave. Such qualitative changes in the spectrum are relevant to the understanding of many plasma phenomena. Adding finite resistivity to the ideal MHD allows the development of the tearing mode instability represented by an imaginary frequency point spectrum occurring through a singular perturbation of zero frequency spectrum (the edge of the Alfvén continuous spectrum) [4]. Within the framework of an ideal plasma, there exist mechanisms which cure the pathology inherent in the ideal MHD and lead to nonsingular equation yielding discrete spectra with well defined non-infinite perturbations. Kinetic effects, in particular, the electron inertia have been invoked in the past to resolve the ideal MHD singularity [5, 6].

The ability of electron inertia to resolve the singularity seems to suggest that any physical effect that distinguishes between the electron and the ion motions (ideal MHD does not) may have the potential to generate a discrete spectrum. The simplest non-dissipative two fluid model in which the electron and ion velocities differ from the standard $E \times B$ drift is the Hall Magnetohydrodynamics (HMHD). The latter has been the subject of much investigation in recent years both in fusion [7] as well as in astrophysics [8]. It is of considerable importance, therefore, to investigate the structure of Alfvén waves in HMHD. In particular one would like to know if the Alfvén wave in inhomogeneous HMHD sheds its singular character, and if yes under what conditions does the singularity disappear.

Settling the question of "the continuous spectrum" in HMHD has consequences beyond

the immediate aim stated above. Since the structure of the Alfvén wave in an inhomogeneous medium is determined by the mechanism which removes the singularity, it is important to ascertain the dominant mechanism for a given plasma. We must, for example, assess the relative merits of the electron inertia and the Hall term (electron inertia is zero in HMHD) in singularity resolution and only then will we be able to predict the nature of Alfvén eigenmodes (for example, the mode width). We shall later show that the Hall term will generally dominate in standard laboratory plasmas and thus will be the primary determinant of the mode structure.

This paper is devoted to an analysis of Alfvénic perturbations in HMHD. Since the Hall term contains a higher order spatial derivative than other terms in ideal MHD, it may be expected to remove the Alfvén singularity. We shall soon see, however, that the Hall term alone is not enough; in the zero beta limit (no sound wave), it simply shifts the singularity without resolving it. It is joint effort of the combination of the Hall current with finite beta which leads to a nonsingular equation yielding a discrete spectrum.

We begin with the HMHD equations and derive a homogeneous plasma dispersion relation in Sec. II. The homogeneous plasma dispersion relation is rather revealing; it clearly indicates what we stated in the preceding paragraph that it is the coupling of sound wave and the hall current that is needed to provide higher order dispersion necessary for singularity-removal.

We then graduate to the derivation of the mode equation for inhomogeneous HMHD in Sec. III. From the mode equation we systematically draw inferences that we stated earlier:

A) Though the Hall term contributes a higher order derivative, without pressure it can not remove the singularity. In a zero-pressure plasma, it leads to the following changes; (i) wave frequency shift, (ii) polarization change from linear to elliptic and (iii) the possibility of global Alfvén eigenmodes (GAE) with discrete spectrum provided the Hall effect is strong enough.

B) The Hall term couples with finite pressure (sound wave) term to produce the higher order (fourth order) derivative that removes the singularity in the Alfvén wave equation.

In Sec. IV, we summarize our results.

II. DISPERSION RELATION OF HALL MHD

In standard Alfvén units, the equations governing HMHD are

$$n\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V} = (\nabla\times\boldsymbol{B})\times\boldsymbol{B} - \nabla p \tag{1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times\left[\left(\boldsymbol{V} - \epsilon\frac{\nabla\times\boldsymbol{B}}{n}\right)\times\boldsymbol{B}\right] \tag{2}$$

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\boldsymbol{V}) = 0, \tag{3}$$

where $\boldsymbol{V}$ is the flow velocity, $\boldsymbol{B}$ is the magnetic field, n is the number density, and p is the pressure. We have normalized $\boldsymbol{B}$ by its representative value B^*, $\boldsymbol{V}$ by the Alfvén speed $B^*/\sqrt{\mu_0 n M}$ (where M is the ion mass), p by B^{*2}/μ_0 and the length scale by its representative value L_0. The scaling coefficient $\epsilon = \lambda_i/L_0$ is a measure of the ion skin depth, $\lambda_i = \sqrt{M/\mu_0 n e^2}$. The only addition to the standard MHD is the Hall-term $\epsilon(\nabla\times\boldsymbol{B}/n)\times\boldsymbol{B}$ in Eq. (2), and it contains higher order derivatives compared to the other terms in the ideal MHD equation. One expects at the outset that this higher derivative modification will cure the singularity.

In order to gain some insight into the dispersive character of the HMHD waves, we begin with deriving the flow-less homogeneous dispersion relation. We linearize Eqs. (1)-(3) about the equilibrium $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$, $\boldsymbol{V}_0 = 0$, and $n_0 = 1$: as

$$\begin{aligned} \boldsymbol{V} &= \boldsymbol{v}, \\ \boldsymbol{B} &= \boldsymbol{B}_0 + \boldsymbol{b}, \\ n &= 1 + \widetilde{n}, \end{aligned} \tag{4}$$

with perturbations proportional to $\exp[-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})]$. The perturbed equation of state $\nabla\widetilde{p} = C_s^2\nabla\widetilde{n}$ (C_s is the sound speed), and the continuity equation, Eq. (3), into $\widetilde{n} = (\boldsymbol{k}\cdot\boldsymbol{v})/\omega$ yield the system:

$$\begin{aligned} -\omega\boldsymbol{v} &= B_0(\boldsymbol{k}\times\boldsymbol{b})\times\boldsymbol{e}_z - \frac{C_s^2}{\omega}(\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{k} \\ &= B_0(k_z\boldsymbol{b} - b_z\boldsymbol{k}) - \frac{C_s^2}{\omega}(\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{k}, \end{aligned} \tag{5}$$

$$\begin{aligned} -\omega\boldsymbol{b} &= B_0\boldsymbol{k}\times(\boldsymbol{v}\times\boldsymbol{e}_z) - i\epsilon B_0\boldsymbol{k}\times[(\boldsymbol{k}\times\boldsymbol{b})\times\boldsymbol{e}_z] \\ &= B_0[k_z\boldsymbol{v} - (\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{e}_z] - i\epsilon B_0 k_z(\boldsymbol{k}\times\boldsymbol{b}). \end{aligned} \tag{6}$$

Remembering $\boldsymbol{k}\cdot\boldsymbol{b} = 0$ (due to $\nabla\cdot\boldsymbol{B} = 0$), the inner product of $\boldsymbol{k}$ and (5) leads to

$$\boldsymbol{k}\cdot\boldsymbol{v} = \frac{B_0 k^2\omega b_z}{\omega^2 - C_s^2 k^2}, \tag{7}$$

and then, combining the z-components of Eqs. (5) and (6), we get

$$\begin{aligned} -\omega b_z &= B_0 \left(\frac{C_s^2 k_z^2}{\omega^2} - 1 \right) (\boldsymbol{k} \cdot \boldsymbol{v}) - i\epsilon B_0 k_z j_z \\ &= -B_0^2 k^2 b_z \frac{\omega^2 - C_s^2 k_z^2}{\omega(\omega^2 - C_s^2 k^2)} - i\epsilon B_0 k_z j_z, \end{aligned} \tag{8}$$

where $j_z = (\boldsymbol{k} \times \boldsymbol{b}) \cdot \boldsymbol{e}_z$. Further manipulation of Eqs. (6) and (5) gives

$$-\omega j_z = B_0 k_z (\boldsymbol{k} \times \boldsymbol{v}) \cdot \boldsymbol{e}_z + i\epsilon B_0 k_z k^2 b_z, \tag{9}$$

$$-\omega (\boldsymbol{k} \times \boldsymbol{v}) \cdot \boldsymbol{e}_z = B_0 k_z j_z \tag{10}$$

from which j_z can be expressed as

$$j_z = -i\epsilon \frac{B_0 k_z k^2 \omega b_z}{\omega^2 - B_0^2 k_z^2}. \tag{11}$$

Substituting Eq. (11) in Eq. (8), we obtain the dispersion relation of HMHD as

$$\left(\omega^2 - B_0^2 k_z^2\right) \left[\omega^4 - \left(C_s^2 + B_0^2\right) k^2 \omega^2 + C_s^2 B_0^2 k_z^2 k^2\right] = \epsilon^2 B_0^2 k_z^2 k^2 \omega^2 \left(\omega^2 - C_s^2 k^2\right). \tag{12}$$

Clearly, setting $\epsilon = 0$ reduces Eq. (12) to the MHD dispersion relation.

The HMHD dispersion relation differs from the standard MHD relation by the ϵ-dependent term on the right-hand side of Eq. (12). Notice, however, that the existence of the higher-order dispersion (proportional to k^4) requires that ϵ^2 and C_s^2 be simultaneously nonzero. The vanishing of either changes, at best, the coefficient of the k^2 term. The translation to the inhomogeneous language (to be discussed soon) is that the fourth-order derivative term needed to remove the singularity associated with the second-order differential equation is possible only when the Hall effect is augmented by finite beta: The hall term alone simply shifts the zero of the coefficient of the second order term.

III. ALFVÉN WAVE IN INHOMOGENEOUS HALL MHD

The calculation to investigate the conditions under which the Alfvén singularity in an inhomogeneous plasma is resolved, constitutes the main thrust of this work. we deal with the simplest nontrivial problem: an inhomogeneous one-dimensional slab (Cartesian) with the ambient magnetic field in the form

$$\boldsymbol{B}_0 = \frac{\boldsymbol{e}_z + f(x)\, \boldsymbol{e}_y}{\sqrt{1 + f^2(x)}}. \tag{13}$$

The proposed $\boldsymbol{B}_0$ satisfies

$$\nabla \times \boldsymbol{B}_0 = h\boldsymbol{B}_0, \qquad h(x) = \frac{f'}{1+f^2}, \tag{14}$$

where $f' = d\,f(x)/d\,x$. If we choose $f \ll 1$, $\boldsymbol{B}_0$ has much stronger z-field than y-field like the tokamak magnetic field; on the other hand, if we choose $f = \tan(\lambda x)$ with constant λ, $\boldsymbol{B}_0$ expresses the Cartesian version of a Taylor-like relaxed (RFP) magnetic field, $\nabla \times \boldsymbol{B}_0 = \lambda \boldsymbol{B}_0$ [9]. The equilibrium magnetic field, Eq. (14), allows us to define the following orthogonal unit vectors;

$$\boldsymbol{e}_x, \quad \boldsymbol{e}_\parallel = \boldsymbol{B}_0 = \frac{\boldsymbol{e}_z + f\boldsymbol{e}_y}{\sqrt{1+f^2}}, \quad \boldsymbol{e}_\perp = \frac{\boldsymbol{e}_y - f\boldsymbol{e}_z}{\sqrt{1+f^2}}, \tag{15}$$

satisfying $\boldsymbol{e}_\perp \times \boldsymbol{e}_\parallel = \boldsymbol{e}_x$ and $\nabla \times \boldsymbol{e}_{\parallel(\perp)} = h\ \boldsymbol{e}_{\parallel(\perp)}$. To derive the wave equation in the inhomogeneous variable x, we first linearize Eqs. (1)-(3) around the equilibrium, and then Fourier expand the perturbations in the ignorable directions: $\widetilde{q} = q(x)\exp[-i(\omega t - k_y y - k_z z)]$. The explicit unit vectors (15) help to define the wavenumbers;

$$k_\parallel = \frac{k_z + fk_y}{\sqrt{1+f^2}}, \quad k_\perp = \frac{k_y - fk_z}{\sqrt{1+f^2}}, \tag{16}$$

obeying

$$\begin{aligned} &k_\perp^2 + k_\parallel^2 = k_y^2 + k_z^2 = \text{const}, \\ &\tfrac{d}{dx}k_\perp = -hk_\parallel, \quad \tfrac{d}{dx}k_\parallel = hk_\perp. \end{aligned} \tag{17}$$

Following standard methods, linearized equations, Eqs. (1), and (2) can be written as

$$-i\omega\boldsymbol{v} = (\nabla \times \boldsymbol{b}) \times \boldsymbol{B}_0 + (\nabla \times \boldsymbol{B}_0) \times \boldsymbol{b} + \frac{iC_s^2}{\omega}\nabla(\nabla \cdot \boldsymbol{v}), \tag{18}$$

$$\begin{aligned} -i\omega\boldsymbol{b} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B}_0) - \epsilon\nabla \times [(\nabla \times \boldsymbol{b}) \times \boldsymbol{B}_0 + (\nabla \times \boldsymbol{B}_0) \times \boldsymbol{b}] \\ &= \nabla \times [\boldsymbol{v} \times \boldsymbol{B}_0 + i\epsilon\omega\boldsymbol{v}] \equiv \nabla \times \boldsymbol{g}. \end{aligned} \tag{19}$$

To arrive at the final form Eqs. (19) and (18) were used, and a new variable $\boldsymbol{g} \equiv \boldsymbol{v} \times \boldsymbol{B}_0 + i\epsilon\omega\boldsymbol{v}$, was introduced. Combining Eqs. (18)-(19), we obtain

$$-\omega^2\boldsymbol{v} = (\nabla \times \nabla \times \boldsymbol{g} - h\nabla \times \boldsymbol{g}) \times \boldsymbol{e}_\parallel + C_s^2\nabla(\nabla \cdot \boldsymbol{v}). \tag{20}$$

From the definition of $\boldsymbol{g}$, we have:

$$\begin{aligned} g_\parallel &= i\epsilon\omega v_\parallel, \\ g_x &= v_\perp + i\epsilon\omega v_x, \\ g_\perp &= -v_x + i\epsilon\omega v_\perp, \end{aligned} \tag{21}$$

and the resulting reciprocal relations

$$\begin{aligned} v_x &= \frac{1}{1-\epsilon^2\omega^2}\left[-g_\perp + i\epsilon\omega g_x\right], \\ v_\perp &= \frac{1}{1-\epsilon^2\omega^2}\left[g_x + i\epsilon\omega g_\perp\right]. \end{aligned} \tag{22}$$

For the proposed geometry, it is straightforward to evaluate the curl operator. In particular, we derive

$$\begin{aligned} \nabla\times\boldsymbol{g} \;=\; & i(k_\perp g_\parallel - k_\parallel g_\perp)\boldsymbol{e}_x + \left(ik_\parallel g_x - \frac{d}{dx}g_\parallel + hg_\perp\right)\boldsymbol{e}_\perp \\ & + \left(\frac{d}{dx}g_\perp - ik_\perp g_x + hg_\parallel\right)\boldsymbol{e}_\parallel. \end{aligned} \tag{23}$$

Breaking Eq. (20) into components, and using various relations that we have displayed, we arrive at the general system describing waves in an inhomogeneous HMHD plasma:

$$\left(\Omega^2 - k_\parallel^2 + \frac{d^2}{dx^2}\right)g_\perp \;=\; i\left[\frac{d}{dx}(k_\perp g_x) + \epsilon\omega\Omega^2 g_x\right] + \frac{d}{dx}\left[\left(\frac{\omega}{\epsilon k_\parallel} - h\right)g_\parallel\right] - k_\parallel k_\perp g_\parallel, \tag{24}$$

$$\left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)g_x \;=\; i\left(k_\perp\frac{d}{dx} - \epsilon\omega\Omega^2\right)g_\perp + i\left(k_\parallel\frac{d}{dx} - k_\perp\frac{\omega}{\epsilon k_\parallel}\right)g_\parallel, \tag{25}$$

and

$$g_\parallel = \frac{\epsilon^2 C_s^2\Omega^2 k_\parallel}{\omega^2 - k_\parallel^2 C_s^2}\left[i\left(\frac{d}{dx} + \frac{k_\perp}{\epsilon\omega}\right)g_x - \left(\frac{1}{\epsilon\omega}\frac{d}{dx} + k_\perp\right)g_\perp\right], \tag{26}$$

where $\Omega^2 \equiv \omega^2/1-\epsilon^2\omega^2$.

A. Zero-pressure case

In this paper we restrict ourselves to a purely analytical investigation of the system. We shall proceed in incremental steps to illustrate the nature of HMHD modes. First we consider the zero-pressure $C_s^2 = 0$ limit, for which the parallel components of the perturbed velocity and of the auxiliary variable $\boldsymbol{g}$ vanish ($g_\parallel = 0$, $v_\parallel = 0$); this follows from Eqs. (20) and (26). Under this condition, Eqs. (24) and (25) are easily combined to yield

$$\frac{d}{dx}\left(\Omega^2 - k_\parallel^2\right)\frac{d}{dx}g_\perp + \left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)\left(\Omega^2 - k_\parallel^2\right)g_\perp = \epsilon\omega\Omega^2\left(\epsilon\omega\Omega^2 - k_\parallel h\right)g_\perp, \tag{27}$$

where the relation, Eq. (17), has been used. Equation (27), in the limit $\epsilon \to 0$ ($\Omega^2 \to \omega^2$, $g_\perp \to -v_x$) reduces Eq. (27) to the second-order singular differential equation of ideal MHD. The singularity is at $\omega^2 = k_\parallel^2(x)$ and the general solution is of the Frobenius-type

corresponding to the continuous spectrum. The effect of the finite Hall current on the Alfvén wave, therefore, can be summarized as follows; (i) the singularity shifts from $\omega^2 = k_\parallel^2(x)$ to $\Omega^2 = k_\parallel^2(x)$, (ii) the polarization of Alfvén wave changes from linear to elliptic, which is clearly expressed by $\boldsymbol{g}$ [see Eq. (21)], (iii) the extra term on the right-hand side of Eq. (27) (equivalent to the modifications of the basic MHD equation by the magnetic shear and other effects [5]) has the potential to alow a discrete spectrum of global Alfvén eigenmodes (GAE) below the MHD continuum (Appendix A). However, as indicated by the dispersion relation, Eq. (12), the Hall current without pressure can not remove the basic singularity; it is simply shifted to $\Omega^2 = k_\parallel^2(x)$.

B. Finite-pressure case

Going back to the finite pressure HMHD system, we formally eliminate g_x in Eqs. (24)-(26) to obtain

$$\begin{aligned}
&\frac{d}{dx}\left(\Omega^2 - k_\parallel^2\right)\frac{d}{dx}g_\perp + \left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)\left(\Omega^2 - k_\parallel^2\right)g_\perp - \epsilon\omega\Omega^2\left(\epsilon\omega\Omega^2 - k_\parallel h\right)g_\perp \\
&\quad = -\frac{d}{dx}\left(k_\perp k_\parallel \frac{d}{dx}g_\parallel\right) + \frac{d}{dx}\left(k_\perp^2\frac{\omega}{\epsilon k_\parallel}g_\parallel\right) - \epsilon\omega\Omega^2\left(k_\parallel\frac{d}{dx}g_\parallel + k_\perp\frac{\omega}{\epsilon k_\parallel}g_\parallel\right) \\
&\qquad + \left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)\left\{\frac{d}{dx}\left[\left(\frac{\omega}{\epsilon k_\parallel} - h\right)g_\parallel\right] - k_\parallel k_\perp g_\parallel\right\}. \qquad (28)
\end{aligned}$$

The right-hand side of Eq. (28) shows the finite pressure effects induced by finite $g_\parallel$. The system is rather complicated (and in reality, it has both $g_\perp$, and g_x (through $g_\parallel$)) and a detailed analysis should be done numerically. An approximate analytic calculation, guided by the fact that the fourth order derivative term requires both ϵ^2 and C_s^2 indicated by the dispersion relation, Eq. (12), is, however, rather instructive. Let us assume that the pressure, though finite, is small (small C_s^2). In this limit the terms on the right-hand side may be ordered small.

The highest-order derivative operating on $g_\perp$ comes from the first term of the right-hand side of Eq. (28). Since this term is the precise object of our attention, we neglect all other terms. In order to find an expression for the highest derivative term, we first substitute

$$g_x \simeq \frac{i}{\left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)}k_\perp\frac{d}{dx}g_\perp, \qquad (29)$$

in (26) and estimate

$$g_\parallel \simeq -\epsilon^2\delta^2 \frac{k_\perp k_\parallel \Omega^2}{\left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)} \frac{d^2}{dx^2} g_\perp, \tag{30}$$

where $\delta^2 \equiv C_s^2/(\omega^2 - k_\parallel^2 C_s^2) \simeq C_s^2/\omega^2$. Finally we approximate the fourth order derivative term:

$$-\frac{d}{dx}\left(k_\perp k_\parallel \frac{d}{dx} g_\parallel\right) \simeq \epsilon^2\delta^2 \frac{k_\perp^2 k_\parallel^2 \Omega^2}{\left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)} \frac{d^4}{dx^4} g_\perp, \tag{31}$$

and substitute it in Eq. (28) to derive

$$\begin{aligned} &\frac{d}{dx}\left(\Omega^2 - k_\parallel^2\right)\frac{d}{dx} g_\perp + \left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)\left(\Omega^2 - k_\parallel^2\right) g_\perp - \epsilon\omega\Omega^2\left(\epsilon\omega\Omega^2 - k_\parallel h\right) g_\perp \\ &\quad \simeq \epsilon^2\delta^2 \frac{k_\perp^2 k_\parallel^2 \Omega^2}{\left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)} \frac{d^4}{dx^4} g_\perp, \end{aligned} \tag{32}$$

simplified as

$$\frac{d}{dx}\left(\Omega^2 - k_\parallel^2\right)\frac{d}{dx} g_\perp - k_\perp^2\left(\Omega^2 - k_\parallel^2\right) g_\perp - \epsilon\omega\Omega^2\left(\epsilon\omega\Omega^2 - k_\parallel h\right) g_\perp \simeq -\epsilon^2\delta^2 k_\parallel^2 \Omega^2 \frac{d^4}{dx^4} g_\perp. \tag{33}$$

To arrive at Eq. (33), the standard assumption $\Omega^2 - k_\parallel^2 \ll k_\perp^2$ has been invoked. It is clear that $\Omega^2 = k_\parallel^2$ is no more a singular point of the differential equation, Eq. (33); the singular solutions, which constitute the MHD continuum, are no more there in finite pressure HMHD. The Hall current coupling with the sound wave, formally represented by a singular perturbation, contributes a fourth-order derivative term, and this perturbation converts the continuous Alfvén spectrum into a point spectrum. We can see this modification clearly in the case when $k_\parallel^2$ has a minimum (Appendix A).

We remind the reader that an entirely similar effect is wrought by finite electron inertia on the ideal MHD spectrum. In a real plasma both these effects are simultaneously present. The question is which one of these will be dominant in a given plasma.

Finally, in order to determine the relative merits of the Hall term and the electron inertia, we calculate the ratio of the widths of the modes created when either one of these mechanism is assumed to be dominant. For the hall dominance scenario, assuming magnetic shear to be small ($h \ll 1$), the width Δ_x of the eigenmode described by (32) or (33) can be estimated as

$$\Delta_x^4 = \epsilon\delta^2 \frac{k_\parallel^2}{\omega(\epsilon\omega\Omega^2 - k_\parallel h)} \simeq \frac{C_s^2}{\omega^4} \simeq \frac{\rho_s^2}{k_\parallel^2}\frac{\omega_{ci}^2}{\omega^2}, \tag{34}$$

where $C_s = \rho_s \omega_{ci}$, ρ_s is the ion gyroradius and ω_{ci} is the cyclotron frequency. On the other hand, the width of eigenmode, when electron inertia resolves the singularity, is

$$\Delta_{x,\mathrm{k}} \simeq (\rho_s a)^{1/2} \simeq \left(\frac{\rho_s}{k_\parallel}\right)^{1/2}, \tag{35}$$

where a is the length of plasma domain (radius of cylinder) [5]. The ratio

$$\frac{\Delta_x}{\Delta_{x,\mathrm{k}}} \simeq \left(\frac{\omega_{ci}}{\omega}\right)^{1/2} > 1, \tag{36}$$

tends to be considerably greater than unity since in the Alfvénic range the cyclotron frequency is considerably greater than the mode frequency. We have stipulated here that $k_\parallel$ is of the order of a^{-1}. This may not be the case in general and $k_\parallel$ could be considerably smaller. But for small parallel wave number the mode frequency tends to be even smaller and the ratio $\Delta_x/\Delta_{x,\mathrm{k}}$ remains much larger than unity. It is thus expected, then, that the mode structure will be dominated by the Hall term in the first approximation.

IV. SUMMARY

We have demonstrated that the combined action of the Hall current and the sound wave (finite beta effects) can resolve the well-known ideal MHD singularity of an inhomogeneous plasma; neither of these mechanisms, acting alone, is adequate for the job. The Alfvénic fluctuations in finite pressure Hall MHD, therefore, are characterized by a discrete spectrum associated with nonsingular eigenmodes. The Hall mechanism is also likely to dominate the electron inertia route to regularization for most plasmas of interest.

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APPENDIX A: ANALYSIS OF THE MODE EQUATION

Following the methodology of Ref. [5, 6], we will delineate the general features of the eigenvalue problem described by the equation

$$\frac{d}{dx}\left(\Omega^2 - k_\parallel^2\right)\frac{d}{dx}g_\perp - k_\perp^2\left(\Omega^2 - k_\parallel^2\right)g_\perp - \alpha g_\perp = -\beta\frac{d^4}{dx^4}g_\perp. \tag{A1}$$

A crucial and simplifying step in this method is to assume that in the domain of interest, $k_\parallel^2$ has a well-defined minimum at x_0, and the mode is localized around this minimum. Near x_0, we may write (Ω_0^2 is the minimum of $k_\parallel^2$)

$$k_\parallel^2 = \Omega_0^2 + \Delta_x^2\xi^2, \quad \Omega^2 = \Omega_0^2 + \Delta_x^2\mu, \tag{A2}$$

where Δ_x is a small parameter and μ represents the effective new spectrum. Now we replace all other parameters in the system by their values at x_0 converting Eq. (A1) into

$$\frac{d}{d\xi}\left(\mu - \xi^2\right)\frac{d}{d\xi}g_\perp - k_\perp^2\Delta_x^2\left(\mu - \xi^2\right)g_\perp - \alpha g_\perp = -\frac{\beta}{\Delta_x^4}\frac{d^4}{d\xi^4}g_\perp, \tag{A3}$$

where $\alpha, \beta, \mu \ldots$ are to be treated independent of ξ. Let us normalize the scale as $k_\perp^2\Delta_x^2 = 1$. For a sufficiently large $k_\perp^2$, such that $\Delta_x^2 \ll \Omega_0^2$, we may stretch the range of ξ to be $(-\infty, \infty)$. The Fourier transform of Eq. (A3) yields the equation obeyed by $g(\zeta) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} g_\perp \exp(-i\xi\zeta)d\xi$,

$$\frac{d}{d\zeta}\left(1 + \zeta^2\right)\frac{d}{d\zeta}g + \mu\left(1 + \zeta^2\right)g + \alpha g - \frac{\beta}{\Delta_x^4}\zeta^4 g = 0. \tag{A4}$$

We can further transform Eq. (A4) using a new dependent variable, $\psi = (1 + \zeta^2)^{1/2}g$, to yield the effective Schrödinger equation

$$-\frac{d^2}{d\zeta^2}\psi + V(\zeta)\psi = \mu\psi, \tag{A5}$$

with the effective potential

$$V(\zeta) = \frac{1}{(1 + \zeta^2)^2} - \frac{\alpha}{1 + \zeta^2} + \frac{\beta}{\Delta_x^4}\frac{\zeta^4}{1 + \zeta^2}. \tag{A6}$$

In the limit $\alpha = \beta = 0$ (no-Hall current), we have a pure scattering potential, which yields the continuous spectrum. For $\alpha \neq 0$ but $\beta = 0$ (Hall current without pressure), the continuous spectrum spans $\mu \geq 0$. However, this continuum may be augmented by a discrete spectrum of what are known in the literature as GAE (global Alfvén eigenmodes). The GAE exist for

$\mu \leq 0$ if the potential has a sufficiently deep negative dip. The depth of the potential well is controlled by $\alpha > 0$, the measure of the strength of the Hall term. It has been rigorously shown that to obtain a point spectrum, $\alpha > 1/4$ is required.

The situation is qualitatively different when both α and β are nonzero (Hall current coupling with sound wave). Then, for $\beta > 0$, $V(\zeta) \to \zeta^2$ as $|\zeta| \to \infty$. Hence the wave is trapped in the wavenumber space, and the "eigenvalue" μ is quantized; the erstwhile MHD continuum for $\mu > 0$ has changed over to a discrete spectrum. The width of mode can be estimated by Eq. (A3) as

$$\Delta_x^4 \simeq \frac{\beta}{\alpha}. \tag{A7}$$

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