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THE HEAT CONDUCTION PROCESS IN TOKAMAK HOT ION PLASMAS

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Abstract

The two-component ion distribution observed with active charge-exchange measurements on PDX are explained using the Fokker-Planck drift-kinetic equation and assuming ion self collisions are dominant for energy scattering. The energetic tail of the distribution, which is diffusing outwards in radius and down in energy, must retain an approximately constant effective temperature $T_H \equiv (-\partial \ln f_i / m\partial\varepsilon)^{-1}$. The discontinuity in the slope of $\ln f_i$ is shown to be the boundary between the inward and outward diffusion parts of f_i and is a form of contact discontinuity. Energy scattering collisions with electrons or circulating beam ions, when important, modify the constancy of T_H.

I. INTRODUCTION

Measurements of the ion distribution function (f_i) in tokamaks by analyzing the charge exchange neutrals emerging along a line of sight have often produced $\ell n f_i$ versus energy ($m\varepsilon$) plots with two straight lines.[1] The original explanation by Russian workers[2] has generally been accepted. The lower energy particles, which exhibit a low effective temperature $T_i = (-\partial \ell n f_i / m \partial \varepsilon)^{-1}$, are assumed to come from the outer regions of the discharge where neutrals are plentiful and T_i is low; the higher energy particles with higher T_i are assumed to come from the hot core. However, recent charge exchange measurements[3] on PDX using a modulated diagnostic neutral beam as the neutral source, with discharge conditions I = 495kA, B_T = 22.5kG, $\bar{n}_e = 2.9 \times 10^{13} cm^{-3}$, Z_{eff} = 2.5, T_{io} = 5keV in hydrogen with deuterium neutral beam injection, have shown that f_i, even for a single minor radius (away from the magnetic axis), exhibits the two straight lines on a $\ell n f_i$ plot. The steeper slope at lower energies gives an effective temperature decreasing with radius as expected, but the higher energy part retains an approximately constant temperature close to the central ion temperature T_{io}.

An earlier charge exchange measurement showing strong evidence for a non-Maxwellian f_i was made by Goldston[4] on ATC. Analyzing charge exhange particles emerging along lines of sight which were tangential to the magnetic surfaces in a toroidal sense, he found the temperature of ions moving antiparallel to the current, designated by $T_{\parallel\!\!\nwarrow}$, decreased with r with an approximately parabolic dependence, whereas the temperature for ions moving parallel to the current remained constant out to near the wall. This was for I = 65kA, B_T = 15kG, $\bar{n}_e = 2 \times 10^{13} cm^{-3}$, Z_{eff} = 4, T_{io} = 220eV with only ohmic heating in deuterium. Changing to I = 85kA and hydrogen, some fall off in $T_{\parallel}$ was observed but it was only by 22% at r/a = 0.75. Similar toroidally tangential measurements made by Goldston et al.[5] on ohmically heated discharges in PDX (I = 150kA, B_T = 20kG, $\overline{n}_e = 2\times 10^{13}cm^{-3}$, Z_{eff} = 1, T_{io} = 600eV) showed only a small difference between $T_{\parallel}$ and $T_{\perp}$, the difference being within the experimental error.

A common property of the above two experiments exhibiting local non-Maxwellian ion distributions is that the Larmor radius in the poloidal magnetic field (ρ_θ) for a typical thermal ion is a significant fraction ($\gtrsim$0.2) of the minor radius (a). One can trace to this fact the discrepancy between experiment and the initial argument of neoclassical theory, which is that in lowest order f_i must be Maxwellian because the terms in the drift-kinetic equation which act to make f_i non-Maxwellian are small in the parameter (ρ_θ/L) compared with the collision term, as seen in Eq. (23) in Section III. (L is the characteristic radial gradient scale length). That a substantial departure from Maxwellian must occur can be seen by considering the published ion temperature profile for a discharge in PLT with similar conditions to the 5keV ion plasma in PDX referred to above. In Fig. 9 of Ref. 6, the ion temperature measured from the Doppler broadening of impurity spectral lines, falls from 4keV at r = 8cm to 330eV at r = 32cm. Assuming f_i were Maxwellian at both radii and that n_i is proportional to $[1-(r/a)^2]$, the ratio of f_i at the two radii for the particle energy 4keV would be

$$\frac{f_i(r=32\text{cm}, \varepsilon=4\text{keV})}{f_i(r=8\text{cm}, \varepsilon=4\text{keV})} = \left(\frac{n_{32}}{n_8}\right)\left(\frac{T_8}{T_{32}}\right)^{3/2} e^{-\frac{\varepsilon}{T_{32}} + \frac{\varepsilon}{T_8}}$$

$$= 2.3 \times 10^{-4} = e^{-8.4} \qquad (1)$$

Assuming f_i varies exponentially between the two radii, i.e. $f_i \sim \exp(-r/L)$, which gives a uniform scale length L, then L must equal 2.9cm. Since ρ_θ for 4keV and I=500kA is approximately 6cm then $\rho_\theta/L \simeq 2$ which is a long way from the basic neoclassical assumption $\rho_\theta/L \ll 1$. Clearly a substantial departure from neoclassical theory can be expected under such conditions.

In this paper the problem of ion heat conduction is reconsidered without making the assumption that f_i is Maxwellian in lowest order. In Section II an equation is developed for f_o, the lowest order part of f_i which is even in $v_\parallel$ and independent of the poloidal angle, $v_\parallel$ being the component of $\underset{\sim}{v}$ parallel to the magnetic field $\underset{\sim}{B}$. The Rosenbluth potentials needed for the collision operators and the appropriate derivatives are also determined. The resultant integro-differential equation for f_o will, in general, require computational methods for solution. In Section III, the limited problem is considered in which the low energy part of f_o is assumed known - it is taken to be a Maxwellian with temperature falling off with radius as observed experimentally - and a solution is sought for the tail of f_o. Because of the assumption that most of the total ion density is contained in the known part of f_o, simple analytic approximations can be taken for the Rosenbluth potentials in the collision operators. The resulting partial differential equation is solved approximately for the case of ion heat conduction between two minor radii with the assumption that the tail particles are in the banana regime and that energy scattering collisions with electrons and circulating beam ions can be neglected; i.e. all significant heating of the ions is assumed to occur within the smaller radius. The tail of f_o, which is diffusing outwards in radius and down in energy, is found to preserve the same effective temperature $(-\partial \ell n f_o / m\partial\varepsilon)^{-1}$. This result occurs only because of the particular velocity dependence of the banana regime diffusion and the coefficients in the energy scattering collision operator for the tail particles. f_o falls off comparatively rapidly with radius but nowhere near as rapidly as would be required to satisfy Eq. (1).

The modifying effects of energy scattering collisions with electrons and beam ions is considered in Section IV. In Section V the case is made that the observed discontinuity in the slope $\partial \ell n f_i/\partial\varepsilon$ is the equivalent in the phase space r,v of the contact discontinuity known in gas dynamics in real space.[7] Finally, in Section VI, the temperatures $T_\parallel$, $T_\perp$ observed with toroidally tangential lines of sight on ATC are identified in terms of the two component f_i.

II. THE EQUATION FOR f_0

A. The Drift-Kinetic Equation

The problem to be considered is that of steady state ion heat conduction between an inner radius r_o, such as the edge of the sawtooth region where the ion temperature $T_i = T_o$ is large, and an outer radius r_1

which is taken to be somewhat smaller than the wall radius so that charge exchange collisions and ionization source terms can be neglected. The heating of the ions is assumed to occur withing the radius r_o and energy scattering collisions with electrons and beam ions are neglected in the region r_o to r_1. In other words self collisions of the hydrogen ion species are assumed to be dominant for energy scattering. The modifying effect of energy scattering collisions with electrons is considered in Section IV.

A reference frame is chosen in which the mean toroidal velocity of the ions is zero; this is the frame in which f_i would relax to a Maxwellian if processes other than self collisions were absent. Since to a good approximation in Tokamaks the mean toroidal velocity is $V_\parallel$, the mean velocity parallel to $\underset{\sim}{B}$, the radial electric field in the moving frame becomes $E_r^* = E_r - V_\parallel B_\theta$, but the magnitude of $V_\parallel$ is assumed limited such that centrifugal force terms are higher order and can be neglected. The electrostatic potential is assumed constant on a magnetic surface. Taking f_i in the form $f(\varepsilon,\mu,\underset{\sim}{r})$ where $\varepsilon = v^2/2 + e\Phi^*/m$, $\mu = v_\perp^2/2B$ and $\Phi^* = -\int E_r^* dr$, the drift-kinetic equation for f has the standard form

$$q \frac{B_\theta}{B} \frac{\partial f}{r\partial\theta} + \underset{\sim}{v}_d \cdot \underset{\sim}{\nabla} f = C(f) \tag{2}$$

where C is the collision operator, the components of $\underset{\sim}{v}_d$ are $v_{dr} = (mq/er)\partial(q/B)/\partial\theta$, $v_{d\theta} = -(mq/e)\partial(q/B)\partial r$ and $q \equiv v_\parallel = \left[2(\varepsilon - \mu B - e\Phi^*/m)\right]^{1/2}$. The usual Tokamak assumptions have been made that $B_\theta \ll B$ and $B_\phi \simeq B$.

Multiplying Eq. (1) by $Bd\mu/|q|$, integating over μ and taking the θ-average, one obtains

$$\frac{1}{r}\frac{\partial}{\partial r} r\Gamma_\varepsilon = \int_0^{2\pi}\frac{d\theta}{2\pi}\int_0^{(\varepsilon - e\Phi^*/m)/B}\frac{Bd\mu}{|q|}\, C(f) \tag{3}$$

where

$$\Gamma_\varepsilon = \int_0^{2\pi}\frac{d\theta}{2\pi}\int_0^{(\varepsilon - e\Phi^*/m)/B}\frac{Bd\mu}{|q|}\,(hv_{dr}\, f) \tag{4}$$

and $h = 1 + (r/R)\cos\theta$. Note $4\pi\Gamma_\varepsilon d\varepsilon$ is the θ-averaged radial diffusion for particles in the spherical shell $4\pi\sqrt{2\varepsilon}\, d\varepsilon$ $(=4\pi v^2 dv)$. The integral of $4\pi\Gamma_\varepsilon$ over all velocity magnitudes yields the net ion neoclassical diffusion Γ_{iNC} which, from the principle of detailed ambipolar balancing[8], must equal the neoclassical component of the electron diffusion denoted by Γ_{eNC}. Since Γ_{eNC} is small in the parameter $(m_e/m_i)^{1/2}$ compared with the component parts of Γ_{iNC}, the condition is imposed

$$4\pi\int_{e\Phi^*/m}^{\infty}\Gamma_\varepsilon\, d\varepsilon = \Gamma_{iNC} = \Gamma_{eNC} \simeq 0 \tag{5}$$

Changing from the variable ε to $w = v^2/2$, Eq. (3) becomes

$$\frac{1}{r}\frac{\partial}{\partial r} r\Gamma_w + \frac{eE_r^*}{m}\frac{\partial\Gamma_w}{\partial w} = \int\frac{d\theta}{2\pi}\int\frac{Bd\mu}{|q|}\, C(f) \tag{6}$$

with $\Gamma_w = \Gamma_\varepsilon$.

It can be noted in passing that if Eq. (6) had been derived for the laboratory frame of reference, the $V_\parallel B_\theta$ part of the E_r^* term would appear via the collision operator which yields a term $(\partial/\partial w)(F_w V_\parallel)$ where $4\pi F_w dw$ is the friction force experienced by the particles in the spherical shell dw. Since $\Gamma_w = -F_w/eB_\theta$, the same contribution is obtained. The E_r^* term when multiplied by $4\pi mw$ and integrated over all energies, yields the neoclassical energy transfer between electrons and ions when Γ_{iNC} is non-zero.[9]

B. Expansion of f_i

f_i is first separated into the parts f^+, f^-, which are respectively even and odd in $v_\parallel$ and, in f^+, the part $\overline{f}^+$ which is independent of θ is distinguished from $\tilde{f}^+(\theta)$ which is θ-dependent and has zero θ-average. The part $\overline{f}^+$ is now expanded in the even Legendre polynomials $P_\ell(\xi)$ where ξ is the cosine $v_\parallel/v$

$$\overline{f}^+ = a_o P_o + a_2 P_2 + \dots \qquad (7)$$

From considerations of the collision term in Eq. (6), since the pitch angle scattering part of C will not be zero for the P_2 and higher order terms in Eq. (7), the magnitude of the coefficients of these terms should satisfy

$$a_\ell \sim 2a_o/Z^i_{eff}\,\ell(\ell+1),$$

which even for $\ell=2$ is $a_o/3Z^i_{eff}$. Here $Z^i_{eff} \equiv (n_i+n_z Z^2)/n_i$. The approximation is therefore made

$$\overline{f}^{+} \simeq a_o P_o \equiv f_o \tag{8}$$

Since $V_{\parallel}$ is zero in the chosen frame of reference, f^{-} will be small compared with f_o in the parameter $(r/R)^{1/2}$ and the dominant contribution to the collision term in Eq. (6) will be from $C(f_o,f_o)$ with the contribution from $C(f^{-},f^{-})$ smaller by the factor (r/R). After performing the μ-integral in Eq. (6) only the energy scattering part of C remains and Eq. (6) reduces to

$$\frac{1}{r}\frac{\partial r\Gamma_w}{\partial r} + \frac{eE_r^*}{m}\frac{\partial \Gamma_w}{\partial w} = -\gamma \frac{\partial}{\partial w} v^2 \left(f_o \frac{\partial H}{\partial v} - \frac{D_{\parallel}}{2}\frac{\partial f_o}{\partial v}\right) \tag{9}$$

where $\gamma = 4\pi e^4 \ln \Lambda/m^2$, $D_{\parallel} = \partial^2 G/\partial v^2$ and G, H are the Rosenbluth potentials[10] defined by

$$\left.\begin{aligned} G &= \int d^3\underset{\sim}{v}'\, f_o(v')\,|\underset{\sim}{v}'-\underset{\sim}{v}| \\ H &= \int d^3\underset{\sim}{v}'\, \frac{f_o(v')}{|\underset{\sim}{v}'-\underset{\sim}{v}|} \end{aligned}\right\} \tag{10}$$

Since f_o is spherically symmetric, taking ϕ as the angle between $\underset{\sim}{v}'$ and $\underset{\sim}{v}$,

$$H = \int_0^\infty f_o(v')v'^2 dv' \int_0^\pi \frac{2\pi \sin\phi\, d\phi}{|(v^2+v'^2-2vv'\cos\phi)^{1/2}|}$$

$$= \frac{4\pi}{v}\int_0^v f_o(v')v'^2 dv' + 4\pi \int_v^\infty f_o(v')v'dv'$$

and

$$\frac{\partial H}{\partial v} = -\frac{4\pi}{v^2}\int_0^v f_o(v')v'^2 dv' \qquad (11)$$

Similarly,

$$G = 4\pi \int_0^v f_o(v')v'^2 dv'\left(v + \frac{v'^2}{3v}\right)$$

$$+ 4\pi \int_v^\infty f_o(v')v'^2 dv'\left(\frac{v^2}{3v'} + v'\right)$$

and

$$\frac{D_\parallel}{2} = \frac{1}{2}\,\frac{\partial^2 G}{\partial v^2} = \frac{4\pi}{3v^3}\int_0^v f_o(v')v'^4 dv' + \frac{4\pi}{3}\int_v^\infty f_o(v')v' dv' \qquad (12)$$

C. The Derivation of Γ_w

The determination of Γ_w in terms of the gradients of f_o follows the usual neoclassical procedure with the unknown function $f_o(r,w)$ replacing the usual Maxwellian form of f_o. Γ_w is derived for both the banana and plateau regimes.

(a) Banana Regime

In this regime, the collision term is small and, in lowest order, the drift-kinetic equation requires $f_i(\varepsilon,\mu,\underset{\sim}{r})$ to be constant along a particle orbit for fixed ε, μ. (In this sub-section we are temporarily reverting to ε, μ velocity space coordinates.) Hence, if $f_i = f_o + \hat{f}$,

$$\hat{f} = -\delta r \frac{\partial f_o}{\partial r} - \frac{\delta r^2}{2} \frac{\partial^2 f_o}{\partial r^2} - \frac{\delta r^3}{6} \frac{\partial^3 f_o}{\partial r^3} \ldots + g$$

where δr measures the change in the minor radius r along a particle orbit and g is constant along the orbit being a function of ε and μ. Taking $\delta r=0$ where q=0, then for trapped particles, $\delta r = mhq/eB_{\theta_o}$ and for passing particles $\delta r = (mhq/eB_\theta - \text{constant})$. Hence, for the f^- part of $\hat{f}$ only the terms with odd powers of δr will contribute. Also since the maximum magnitude for δr is $(2r/R)^{1/2}\rho_\theta$, the ratio of the cubic term to the linear term is at most $(1/3)(r/R)(\rho_\theta/L)^2$. Here we will assume that the parameter ρ_θ/L, although not small compared with unity, is limited such that this ratio is sufficiently small to permit the neglect of the cubic and higher order terms. In this case

$$f^- = -\frac{mhq}{eB_{\theta_o}} \frac{\partial f_o}{\partial r} + g \qquad (13)$$

which is the standard neoclassical form. Also from the solubility condition, g must have the standard form

$$g = \frac{m}{eB_{\theta_o}} \frac{\partial f_o}{\partial r} \int_{\mu}^{w/B_{max}} \frac{Bd\mu}{\overline{hq}} \tag{14}$$

where the overbar denotes the θ-average. Eq. (14) is obtained using only the pitch angle scattering part of $C(f^-,f_o)$. Because of the imposed condition Eq. (5), the momentum restoring approximation to the contribution $C(f_o,f^-)$ given in Ref. 12 can be shown to be zero. Equation (5) determines the unknown E_r^*. Energy scattering collisions cause only a weak friction force parallel to $\underset{\sim}{B}$ for the trapped particles and can be neglected.

Substituting the formula for v_{dr} in Eq. (4) and integrating by parts with respect to θ

$$\Gamma_\varepsilon = - \iint \frac{d\theta}{2\pi} \frac{Bd\mu}{|q|} \frac{mq^2h}{eB} \frac{\partial \tilde{f}^+}{r\partial\theta}$$

$$= - \iint \frac{mqh}{eB_\theta} C(f^-) \frac{Bd\mu}{|q|} \frac{d\theta}{2\pi} \tag{15}$$

using the drift-kinetic equation, Eq. (2). (Here a viscosity term of the form

$$\frac{1}{r} \frac{\partial}{\partial r} r \iint mq^2h^2 v_{dr} f^- \frac{Bd\mu}{|q|} \frac{d\theta}{2\pi}$$

has been neglected because it is smaller by the factor (r/R) compared with the term in Eq. (15). See Ref. 11).

In Eq. (15), once again only the pitch-angle scattering part of C makes a significant contribution. The required integrals have been determined by Rosenbluth et al.[12]; one obtains

$$\Gamma_\varepsilon = - \frac{m^2 \nu_{PA}}{e^2 B_{\theta_o}^2} \frac{\partial f_o}{\partial r} \iint \mu B \left(\frac{1}{|q|} - \frac{1}{|\overline{qh}|}\right) B d\mu \frac{d\theta}{2\pi}$$

$$= - 0.49 \left(\frac{r}{R}\right)^{1/2} \frac{m^2 \nu_{PA} v^3}{e^2 B_{\theta_o}^2} \frac{\partial f_o}{\partial r} \qquad (16)$$

and for Γ_w the gradient $\partial f_o/\partial r$ must be replaced by $(\partial f_o/\partial r + eE_r^* \partial f_o/m\partial w)$ to allow for the change of coordinate. In obtaining Eq. (16) $\overline{qh}$ has to be taken as zero for the trapped region in the first expression. ν_{PA} is the pitch-angle collision frequency in the collision operator given by

$$\nu_{PA} = \frac{\gamma}{v^3} \frac{\partial G}{\partial v} \qquad (17)$$

with the Rosenbluth potential G defined in Eq. (10).

(b) Plateau Regime

Here, since the transport is independent of the collision frequency and a collision term is needed only to interpret the singularity at $v_\parallel = 0$, the collision operator is replaced by the simple Krook model $-\nu f$. If $\hat{f}$ is expanded in powers of (r/R), from the drift-kinetic equation, Eq. (2), the first order term in coordinates w, ξ, $\underset{\sim}{r}$, where $\xi \equiv v_{\parallel}/v$, is given by

$$f_1 = - \frac{mw(1+\xi^2) r \cos\theta}{v(\xi - \hat{\nu}) e B_\theta R} \left(\frac{\partial f_o}{\partial r} + \frac{eE_r^*}{m} \frac{\partial f_o}{\partial w}\right)$$

where $\hat{\nu} = B\nu r / B_\theta v$ and

$$4\pi \Gamma_w = 2\pi \int_{-1}^{+1} \sqrt{2}\, w^{1/2} v_{dr} f_i \, d\xi$$

or

$$\Gamma_w = - \frac{m^2 w^2 r}{2e^2 B_\theta BR^2} \overline{\sin\theta \int_{-1}^{+1} \frac{(1+\xi^2)^2 \cos\theta \, d\xi}{[\xi - i\hat{\nu}]}} \left(\frac{\partial f_o}{\partial r} + \frac{eE_r^*}{m} \frac{\partial f_o}{\partial w}\right)$$

$$= - \frac{\pi}{4} \frac{m^2 w^2 r}{e^2 B_\theta BR^2} \left(\frac{\partial f_o}{\partial r} + \frac{eE_r^*}{m} \frac{\partial f_o}{\partial w}\right) \qquad (18)$$

The equation to be solved for f_o is thus Eq. (9) with Γ_w chosen to be the smaller of Eq. (16) or Eq. (18) and with the coefficients in the collision term given by Eqs. (11) and (12).

III. Approximate Solution for the Distribution Tail

A. Assumptions and Boundary Conditions

The presence of the Rosenbluth potential integrals in Eq. (9) makes it intractable except for computer solution. However, for large v, these integrals can be replaced to a good approximation by simple algebraic formulae and in this section an approximate solution of Eq. (9) will be obtained for the tail of the ion distribution function. The assumption will be made that for v less than the magnitude V, the distribution f_o is known and has the Maxwellian form f_c related to a known temperature $T_c(r)$ which decreases with r. This assumption is, of course, based on the experimentally observed distribution in PDX referred to in the Introduction. The limited problem to be solved is the diffusion of the distribution tail in radius and energy in the presence of f_c. Since the net ion diffusion has been taken to be zero [Eq. (5)], the remaining diffusion process is the ion heat conduction which involves particles above a critical velocity magnitude diffusing outwards ($\Gamma_w>0$) and those with lower velocity diffusing inwards ($\Gamma_w<0$). The unknown tail f_H will have $\Gamma_w>0$ and in fact in Section V the velocity magnitude V will be identified with the critical velocity where Γ_w changes sign.

Equation (9) will be solved for the radial range r_o to r_1. The inner radius r_o, which could be the edge of the sawtooth region, has been chosen greater than zero so that, in the simplified problem considered here, all the heating of the ions can be assumed to have occurred within r_o. Energy scattering collisions with electrons and circulating beam ions have been neglected in the range r_o to r_1. The outer radius r_1 is chosen to be somewhat less than the tube radius a, so that the neglect of charge exchange and ionization is justified.

The chosen boundary conditions are

1. For $v < V(r)$, $f_o = f_c = \left(n_c/\pi^{3/2}v_{T_c}^3\right)\exp(-v^2/v_{T_c}^2)$ where $v_{T_c}^2 = 2T_c(r)/m$.

2. At $r = r_o$, $f_o = f_c(r_o)$ for $v<V$ and for $v>V$, $f_o = f_H \sim \exp(-mv^2/2T_H)$ with T_H somewhat larger than $T_c(r_o) = T_{co}$.

3. At $v=V$, f_o is continuous.

4. f_o is small at large r and large v.

The justification for assuming that the tail of f_o has already a higher effective temperature at r_o is that this part of f_o has arrived by diffusion from an inner hotter region, whereas much of the lower energy part of f_o has arrived by diffusion from an outer colder radius.

B. The Approximate Collision Terms

The assumption is made that $(V/v_{T_c})^2$ is large so that $\exp(-V^2/v_{T_c}^2)$ and $f_c(V)$ are small and, in addition, the number of ions in the unknown tail of f_o is small compared with the number in the known part f_c. Hence, in determining the required Rosenbluth potentials for $v>V$, the contributions from the unknown tail can be neglected. (This assumption linearizes Eq. (9) in the problem considered here.) Starting from Eqs. (11), (12) and (17) and retaining only the f_c contributions in the integrals

$$\frac{\partial H}{\partial v} \simeq -\frac{4\pi}{v^2}\int_0^V f_c v^2 dv = -\frac{\hat{n}_c}{v^2}, \tag{19}$$

where

$$\hat{n}_c \equiv 4\pi \int_0^V f_c v^2 dv = \left[\mathrm{Erf}\left(\frac{V}{v_{T_c}}\right) - \frac{2V}{\pi^{1/2} v_{T_c}} \exp\left(-\frac{V^2}{v_{T_c}^2}\right) \right] n_c$$

is the particle density for the f_c part of f_o, ($\hat{n}_c \simeq n_i$).

$$\frac{D_\parallel}{2} \simeq \frac{4\pi}{3v^3} \int_0^V f_c v^4 dv$$

$$= \frac{\hat{n}_c v_{T_c}^2}{2v^3} - \frac{2n_c V^3}{3\pi^{1/2} v^3 v_{T_c}} \exp\left(-\frac{V^2}{v_{T_c}^2}\right)$$

$$\simeq \frac{\hat{n}_c v_{T_c}^2}{2v^3} \tag{20}$$

$$\nu_{PA} = \frac{\gamma}{v^3} \frac{\partial G}{\partial v}$$

$$= \frac{\gamma}{v^3} \left\{ 4\pi \int_0^v f_o(v') v'^2 dv' \left(1 - \frac{v'^2}{3v^2}\right) + \frac{8\pi v}{3} \int_v^\infty f_o(v') v' dv' \right\}$$

$$\simeq \frac{4\pi\gamma}{v^3} \int_0^V f_c(v') v'^2 dv' \left(1 - \frac{v'^2}{3v^2}\right)$$

$$= \gamma \frac{\hat{n}_c}{v^3}\left(1 - \frac{v_{T_c}^2}{2v^3}\right) + \frac{2n_c V^3}{3\pi^{1/2} v^5 v_{T_c}} \exp\left(-\frac{V^2}{v_{T_c}^2}\right)$$

$$\simeq \frac{\gamma \hat{n}_c}{v^3} \qquad (21)$$

since $v > V$. (The expression for G given above Eq. (12) has been used in determining ν_{PA}).

At this stage it is an easy matter to generalize the limited problem being considered to the case where impurity ions are present. In the energy scattering coefficients in Eqs. (19) and (20) the modification is an extra factor $\left[1 + \sum_z (n_z Z^2 m_i / \hat{n}_c m_z)\right]$ which is assumed to be sufficiently close to unity so that it can be neglected. In the case of the pitch angle scattering frequency, Eq. (21), the modified expression is

$$\nu_{PA} = \frac{\gamma \hat{n}_c}{v^3}\left(1 + \sum_Z \frac{n_z Z^2}{\hat{n}_c}\right)$$

$$\simeq \frac{\gamma \hat{n}_c Z_{eff}^i}{v^3} \qquad (22)$$

where $Z_{eff}^i \equiv \left(n_i + \sum n_z Z^2\right)/n_i$.

C. Solution of the Approximate Equation.

Making the reasonable assumption that the tail particles are in the banana regime, Eq. (22) is substituted into Eq. (16) for Γ_w and Eqs. (16), (19) and (20) are substituted into Eq. (9), yielding

$$\frac{1}{r\hat{n}_c}\frac{\partial}{\partial r}\, r\hat{n}_c\alpha^2\left(\frac{\partial f_o}{\partial r}+\frac{eE_r^*}{m}\frac{\partial f_o}{\partial w}\right)+\frac{eE_r^*}{m}\frac{\partial}{\partial w}\,\alpha^2\left(\frac{\partial f_o}{\partial r}+\frac{eE_r^*}{m}\frac{\partial f_o}{\partial w}\right)$$

$$=-\frac{\partial}{\partial w}\left(f_o+\frac{T_c}{m}\frac{\partial f_o}{\partial w}\right), \qquad (23)$$

where

$$\alpha^2 = 0.49(r/R)^{1/2}\, Z^i_{eff}\, m^2/e^2B_\theta^2 \quad . \qquad (24)$$

If $\hat{n}_c(\simeq n_i)$ is assumed to have a typical parabolic dependence on r, i.e. $\hat{n}_c \sim 1-(r^2/a^2)$, the expression $r^{3/2}[1-(r^2/a^2)]$ is found to be weakly dependent on r over the range of interest. Also, B_θ is weakly varying with r provided q_a is not too large. Hence the radial dependence of the factor $r\,\hat{n}_c\,\alpha^2$ is neglected compared with the strong dependence of f_o. At the same time, since no explicit functions of w are present in Eq. (23), it is convenient to return to the original energy coordinate $\varepsilon = w+e\Phi^*/m$. Equation (23) becomes

$$\alpha^2 \frac{\partial^2 f_o}{\partial r^2} = -\frac{\partial}{\partial \varepsilon}\left(f_o + \frac{T_c}{m}\frac{\partial f_o}{\partial \varepsilon}\right) \tag{25}$$

since $\partial/\partial w|_r = \partial/\partial \varepsilon|_r$.

To obtain an initial approximate solution of Eq. (25), the weak radial dependence of T_c is temporarily neglected, whereupon a solution by the separable coordinates method is possible. Taking k^2 as an example separation constant, f_o must satisfy

$$\alpha^2 \frac{\partial^2 f_o}{\partial r^2} - k^2 f_o = 0 \tag{26}$$

and

$$\frac{\partial}{\partial w}\left(f_o + \frac{T_{co}}{m}\frac{\partial f_o}{\partial w}\right) + k^2 f_o = 0 \tag{27}$$

The general solution for f_o is an integral over all possible values of k^2. Here it is convenient to follow a single k^2. Of the two solutions to Eq. (26), $f_o \sim e^{\pm kr/\alpha}$, the one which is small at large r is chosen to satisfy the boundary condition 4; the solutions to Eqs. (26) and (27) then combine to give

$$f_o = e^{-\frac{kr}{\alpha}}\left(C_1\, e^{-\beta_1 \varepsilon} + C_2\, e^{-\beta_2 \varepsilon}\right) \tag{28}$$

where β_1 and β_2 satisfy

$$\frac{T_{co}}{m}\beta^2 - \beta + k^2 = 0 \qquad (29)$$

However, the boundary condition at r_o requires $f_0 \sim \exp(-mw/T_H)$, so that $\beta = m/T_H$ and Eq. (29) becomes an equation determining a unique value of the separation constant k^2. The approximate solution satisfying the boundary conditions is then

$$f_o = C\, e^{-\frac{r}{\lambda_o}}\, e^{-\frac{m\varepsilon}{T_H}} \qquad (30)$$

where

$$\lambda_o = (0.49\, Z^i_{eff}/2)^{1/2} \left(\frac{r}{R}\right)^{1/4} \left(1 - \frac{T_{co}}{T_H}\right)^{-1/2} \frac{m v_{T_H}}{e B_\theta}$$

with $v_{T_H} = (2T_H/m)^{1/2}$.

A more accurate solution allowing for the radial dependence of T_c is obtained by retaining the same dependence of f_o on ε as in Eq. (30) and using the WKBJ method. On substituting $f_o \sim \exp(-m\varepsilon/T_H)$ in Eq. (25), one obtains

$$\frac{\partial^2 f_o}{\partial r^2} - \frac{1}{\lambda_o^2}(1+\delta) f_o = 0 \qquad (31)$$

where $\delta = (T_{co} - T_c)/(T_H - T_{co})$. Assuming a solution of the form $f \sim e^{-\phi(r)}$, then

$$\phi'^2 - \phi'' - \frac{1}{\lambda_o^2}(1+\delta) = 0 \tag{32}$$

where the prime denotes $\partial/\partial r$. Solving Eq. (32) iteratively, after two iterations

$$\phi = \int_0^r \frac{dr\sqrt{1+\delta}}{\lambda_o} + \frac{1}{4}\ln(1+\delta) \, .$$

Using this result and changing back to the energy coordinate w, the more accurate solution is

$$f_o = C\left(\frac{T_H - T_{co}}{T_H - T_c}\right)^{1/4} e^{-\int_{r_o}^{r} \frac{dr}{\lambda}} \; e^{-\frac{mw}{T_H}} \tag{33}$$

where C is a constant, T_{co} is the value of T_c at r_o and

$$\frac{1}{\lambda} = -\frac{e(E_r - V_\parallel B_\theta)}{T_H} + \frac{eB_\theta}{mv_{T_H}}\left\{\frac{2\left(1 - \frac{T_c}{T_H}\right)}{0.49 Z_{eff}^i}\right\}^{1/2}\left(\frac{R}{r}\right)^{1/4} \tag{34}$$

Thus the equating in Eq. (25) of the divergence of Γ_w and the divergence of the flows in velocity space under the assumed conditions, requires the tail of f_o to maintain an approximately constant effective temperature $(-\partial \ell n f_o / m \partial w)^{-1}$ as it diffuses outward in r and down in energy. The important conditions are, firstly, banana regime diffusion and, secondly, ion self-collisions dominant for energy scattering. The resulting particular energy dependence of Γ_w and of C_{ii} for the distribution tail require the constant effective temperature. Under other conditions the constant effective temperature will not occur, as discussed in the next section.

IV. Effect of Electron Collisions.

A. The Electron Collision Term

In the preceding two sections the effect of energy scattering collisions between the ions and other particle species was neglected. If these interactions are important, the extra collision terms which must be added to Eq. (9) will, in general, have velocity dependence different from the high energy approximation for the C_{ii} term. Hence the solution in Eq. (33) with constant effective temperature will no longer be valid. In this section the effect of electron collisions will be considered. The extra term which must be added to the right-hand side of Eq. (9) is

$$vC_{ie}(f_o) = \nu_{ei}\left(\frac{m_e}{m_i}\right) \frac{\partial}{\partial w} v^3\left(f_o + \frac{T_e}{m_i} \frac{\partial f_o}{\partial w} \right) \tag{35}$$

and on the right of Eq. (23), this will take the form

$$- \frac{\nu_{ei}}{\gamma \hat{n}_c} \left(\frac{m_e}{m_i}\right) \frac{\partial}{\partial w} v^3 \left(f_o + \frac{T_e}{m} \frac{\partial f_o}{\partial w} \right)$$

Assuming $T_e \simeq T_c$, the magnitude of this term is smaller than the C_{ii} term by the factor $(m_e/m_i)^{1/2}$, larger by the factors $(n_e/\hat{n}_c)$ and $(m_i v^2/2T_e)^{3/2}$, and, for the parameters of the experiments considered in the Introduction, is typically a fraction of the C_{ii} term. But the important difference from the C_{ii} term is the presence of the extra factor v^3. (As in the proceeding section, only the tail of the ion distribution is being considered.)

B. A Physical Model

Since an analytic solution has not been obtained with the C_{ie} term included, its effect will be estimated as a perturbation to the solution obtained in the preceeding section. This will be done by introducing a physical interpretation of Eq. (9), namely, the flow of the "fluid" f_o in the phase space r, v. Introducing the unit vectors $\underset{\sim}{i}_r$, $\underset{\sim}{i}_v$ parallel to the r and v axes, respectively, the "velocity" of the "fluid" f_o is

$$\underset{\sim}{v} = v_r \underset{\sim}{i}_r + a_v \underset{\sim}{i}_v \tag{36}$$

where

$$\left.\begin{aligned} v_r &= \Gamma_w / v f_o, \\ a_v &= J_v / f_o \end{aligned}\right\} \tag{37}$$

Here, J_v is to be identified with the acceleration flow in velocity space such that, after averaging over the pitch angle

$$C(f_o) - \frac{eE_r^*}{mv}\frac{\partial \Gamma_w}{\partial w} \equiv -\frac{1}{v^2}\frac{\partial}{\partial v}(v^2 J_v) \qquad (38)$$

and $4\pi(v_r f_o)v^2 dv = 4\pi\Gamma_w dw$ is the radial diffusive flow for particles. From Eqs. (16), (22), (33) and (34)

$$v_r = -\frac{\alpha^2\gamma\hat{n}_c}{vf_o}\left(\frac{\partial f_o}{\partial r} + \frac{eE_r^*}{m}\frac{\partial f_o}{\partial w}\right)$$

$$= \frac{\alpha^2\gamma\hat{n}_c}{v}\left(\frac{1}{\lambda} + \frac{eE_r^*}{T_H}\right) \qquad (39)$$

and from Eqs. (9), (35) and (38)

$$a_v = \frac{v_r eE_r^*}{mv} - \frac{\gamma\hat{n}_c}{v^2}\left(1 - \frac{T_c}{T_H}\right) - \nu_{ei}v\left(\frac{m_e}{m_i}\right)\left(1 - \frac{T_e}{T_H}\right) \qquad (40)$$

In Eq. (40) the second and third terms come from C_{ii} and C_{ie}, respectively.

In the coordinates r, v Eq. (9) is simply

$$\underset{\sim}{\nabla} \cdot f_o \underset{\sim}{v} = 0$$

or

$$\frac{df_o}{dt} \left(= \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f_o\right) = f_o \nabla \cdot \underset{\sim}{v} = f_o\left(\frac{1}{r} \frac{\partial r v_r}{\partial r} + \frac{1}{v^2} \frac{\partial v^2 a_r}{\partial v}\right) \qquad (41)$$

where d/dt is the mobile operator $\partial/\partial t + \underset{\sim}{v} \cdot \nabla$. Introducing the time Δt for the "fluid" to flow from position, r,v to $r+\Delta r$, $v+\Delta v$ then $\Delta t = \Delta r/v_r$ and $\Delta v = a_v \Delta t$. Also

$$f_o(r+\Delta r,\ v+\Delta v) = f_o(r,v) + \Delta t \frac{df_o}{dt}(r,v)$$

$$= f_o(r,v)\ (1 + \Delta t \underset{\sim}{\nabla} \cdot \underset{\sim}{v})$$

so that

$$\ell n\ f_o(r+\delta r,\ v+\Delta v) = \ell n\ f_o(r,v) + \Delta t \underset{\sim}{\nabla} \cdot \underset{\sim}{v} \qquad (42)$$

treating Δt as small.

If the energy at $r+\Delta r$ is denoted by w', such that $w' = \frac{1}{2}(v+\Delta v)^2 \simeq w + (v a_v \Delta r/v_r)$ and

$$\frac{\partial w'}{\partial w} = 1 + \frac{1}{v} \frac{\partial}{\partial v} \left(\frac{v a_v \Delta r}{v_r}\right)$$

then

$$\frac{\partial}{\partial w'} \ln f_o(r+\Delta r,\ v+\Delta v) = \left[1 + \frac{\Delta r}{v}\frac{\partial}{\partial v}\left(\frac{v a_v}{v_r}\right)\right]^{-1} \frac{\partial}{\partial w}\left(\ln f_o(r,v) + \Delta t \underset{\sim}{\nabla}\cdot \underset{\sim}{v}\right) \qquad (43)$$

Considering first the case where the C_{ie} term is absent in Eq. (40), then $a_v \sim v^{-2}$, $v_r \sim v^{-1}$, $\underset{\sim}{\nabla}\cdot a_v \underset{\sim}{i}_v = 0$ and both $(v a_v/v_r)$ and $\Delta t \underset{\sim}{\nabla}\cdot v_r \underset{\sim}{i}_r$ are independent of v. Equation (43) reduces to

$$\frac{\partial}{\partial w'} \ln f_o(r+\Delta r,\ v+\Delta v) = \frac{\partial}{\partial w} \ln f_o(r,v) = -\frac{m}{T_H} \ . \qquad (44)$$

which reproduces the property found in Section III, that f_o diffuses in r and v maintaining constant effective temperature.

With the C_{ie} term retained in Eq. (40)

$$1 + \frac{\Delta r}{v}\ \frac{\partial}{\partial v}\left(\frac{v a_v}{v_r}\right) = 1 - 3\nu_{ei}\left(\frac{\Delta r}{v_r}\right)\left(\frac{m_e}{m_i}\right)\left(1 - \frac{T_e}{T_H}\right)$$

and

$$\frac{\partial}{\partial w}\left(\Delta t \nabla\cdot a_v \underset{\sim}{i}_v\right) = -\frac{3\nu_{ei}}{v^2}\left(\frac{\Delta r}{v_r}\right)\left(\frac{m_e}{m_i}\right)\left(1 - \frac{T_e}{T_H}\right)$$

so that

$$\frac{\partial}{\partial w'} \ln f_o(r+\Delta r,\ v+\Delta v) = \frac{-\dfrac{m}{T_H} - \dfrac{3\nu_{ei}}{v^2}\left(\dfrac{\Delta r}{v_r}\right)\left(\dfrac{m_e}{m_i}\right)\left(1 - \dfrac{T_e}{T_H}\right)}{1 - 3\nu_{ei}\left(\dfrac{\Delta r}{v_r}\right)\left(\dfrac{m_e}{m_i}\right)\left(1 - \dfrac{T_e}{T_H}\right)}$$

and the effective temperature at $r+\Delta r$ is

$$T_H(r+\Delta r) \approx T_H(r)\left[1 - 3\nu_{ei}\left(\frac{\Delta r}{v_r}\right)\left(\frac{m_e}{m_i}\right)\left(1 - \frac{T_e}{T_H}\right)\left(1 + \frac{T_H}{mv^2}\right) \right]$$

or

$$\frac{\partial T_H}{\partial r} \approx - \frac{3\nu_{ei}}{v_r}\left(\frac{m_e}{m_i}\right)(T_H - T_e) \qquad (45)$$

since T_H/mv^2 is small for the tail ions. To give an average temperature gradient for the tail, v_r should be the average value of the velocity dependent v_r given by Eq. (39).

C. Comparison with Experiment

From Eq. (45) it follows that the effective temperature of the ion distribution tail should increase with r out to the radius where $T_e = T_H$ and then decrease with radius for larger r. Assuming T_e is less than T_H the fall off of T_H with radius can be compared for different experiments by considering the fractional change in T_H in a given fraction of the discharge radius a. From Eq. (45)

$$\frac{a}{T_H}\frac{\partial T_H}{\partial r} = \frac{3a\nu_{ei}m_e}{v_r m_i}\left(1 - \frac{T_e}{T_H}\right)$$

which, using Eqs. (34) and (39), is found to be proportional to be $I\ T_i\ T_e^{-3/2}(Z^i_{eff})^{-1/2}m_i^{-1}$.

Comparing first the ATC experiments in deuterium and hydrogen referred to in the Introduction, the factor $IT_i m_i^{-1}$ is found to be increased by 3.2 in changing to hydrogen and hence significantly more fall off of T_H is expected. Using the T_e profile reported for ATC by Suckewer and Hinnov[13], for the hydrogen case, $T_e = T_{io} = 273$eV at r = 8.7cm and at r = 0.75a = 12cm, $T_{io} - T_e = 173$eV. Using these values and the mean parameters for this radial range, namely $n_e = 1.6\times10^{13}\text{cm}^{-3}$, $T_e = 187$eV, $\rho_{\theta H} = 2.6$cm and a mean value for v_r from Eqs. (39) and (34), one finds T_H should decrease by 23% between r = 8.7cm and r = 0.75a for hydrogen. For the deuterium experiment the decrease by r = 0.75a is found to be 3.2% allowing for the lower value of T_{io}. The figure for the hydrogen experiment is in close agreement with the experimental observation for the decrease of $T_{\parallel}$ with radius and the small figure for deuterium is probably equivalent to zero within the experimental error ($T_{\parallel}$ is shown to be approximately equal to T_H in Section VII.)

Comparing the Ohmic heating case in PDX referred to in the Introduction with the deuterium discharge in ATC, the factor I T_i $T_e^{-3/2}(Z_{eff}^i)^{-1/2}$ is increased by 13. Thus a large fall off of T_H with radius is expected in this PDX experiment, substantially larger than the hydrogen case in ATC. This explains the observed approximate equality of $T_{\parallel}$ and $T_{\perp}$.

In the case of the beam heated PDX experiment, the parameters are such that substantial cooling of the ion distribution tail by electrons should occur but in addition there will be substantial heating of the tail ions by the circulating beam ions. Approximate numerical estimates of these two effects show that they are nearly equal, so that the observed small decrease of T_H with radius is somewhat fortuitous.

Lastly, turning attention to the inward-flowing part of f_i, namely f_c, since the particle density $\hat{n}_c$ associated with f_c is 4 or 5 times larger than the particle density of f_H, it follows that the mean value of v_r for f_c is smaller by this factor compared to v_r for f_H. From Eq. (45) the interaction with the electrons is correspondingly more important by the same factor because of the greater time taken to cover a given radial distance. In fact the electrons will play a substantial, if not major role in heating the inward flowing ions in ohmically heated discharges. This greater importance of the C_{ie} term for f_c is only one of the factors which makes the diffusion of f_c different from that of f_H. Even if these lower energy ions are in the banana regime, the velocity dependence of the various collision terms will be different from that applicable to the tail ions. In many ohmically heated discharges much of f_c will be in the plateau regime, causing a still larger change in the velocity dependence of Γ_w. Also, in general, the smallness of v_r for f_c will cause the C_{ii} term to be correspondingly more important in the drift-kinetic equation so that in lowest order, f_c must be Maxwellian.

V. The Contact Discontinuity

One of the striking features of the active charge exchange measurements on PDX[3] is the abrupt change in the slope $\partial \ell n f_o/\partial w$ between the two parts of f_o; i.e. $\partial f_o/\partial w$ is discontinuous within the accuracy of the measurements. The critical particle energy associated with the discontinuity, denoted by $W = 1/2\ V^2$, decreases with the minor radius r. Hence, for a given value of w, the gradient $\partial f_o/\partial r$ must also be disconinuous at a particular radius. The only way large values of $\partial^2 f_o/\partial w^2$ and $\partial^2 f_o/\partial r^2$ can occur and still satisfy Eq. (9), namely $\underset{\sim}{\nabla} \cdot f_o \underset{\sim}{\nu} = 0$ in the phase space r,v, where parts of $\underset{\sim}{\nu}$ are proportional to $\partial f_o/\partial w$ and $\partial f_o/\partial r$, is if the flow $f_o \underset{\sim}{\nu}$ is tangential to the locus of V(r). In Section III and IV, the high energy part of f_o, to be denoted by f_H, was found to have v_r positive (outward radial diffusion) and a_v negative (downward diffusion in energy). Hence the signs of v_r and a_v are correct for ν to be tangential to V(r) on the high energy side; only the relative magnitudes of v_r, a_v remain unchecked.

On the low energy side of V, to determine a_v one must calculate the collision operator coefficients more accurately than in Section III in order to obtain a non-zero value. From Eqs. (11) and (12) one finds for $v \simeq V$ and neglecting the C_{ie} terms

$$\left.\begin{aligned} \frac{\partial H}{\partial v} &= -\frac{\hat{n}_c}{v^2} \\ \text{and} \\ \frac{D_\parallel}{2} &= \frac{\hat{n}_c v_{T_c}^2}{2v^3} + \frac{2n_c}{3\sqrt{\pi}\; v_{T_c}} \left(\frac{T_H}{T_c} - 1\right) e^{-\left(\frac{V}{v_{T_c}}\right)^2}, \end{aligned}\right\} \qquad (46)$$

where f_o has been assumed Maxwellian on either side of V but with the different effective temperatures T_c, T_H. (The continuity of f_o at v=V determines the unknown coefficient n_H in f_H). From Eqs. (46) and (38)

$$J_v = \frac{v_{rc} e E_r^* f_o}{mv} + \gamma f_o \frac{\partial H}{\partial v} - \frac{\gamma D_\parallel}{2} \frac{\partial f_o}{\partial v}$$

$$= \frac{v_{rc} e E_r^* f_o}{mv} + \frac{4\gamma n_c v f_o}{3\sqrt{\pi} v_{T_c}^3} \left(\frac{T_H}{T_c} - 1\right) e^{-\left(\frac{V}{v_{T_c}}\right)^2}$$

$$= \left(\frac{v_{rc} e E_r^*}{mv} + \delta_c\right) f_o \quad , \qquad (47)$$

where v_{rc} is the mean radial velocity $\Gamma_W / v f_o$ on the low energy side of V. Applying the condition that $\underset{\sim}{\nu}$ must be parallel to V(r) on either side of V, for $v \simeq V$

$$\frac{a_v}{v_r}\Big|_c = \frac{a_v}{v_r}\Big|_H$$

and substituting from Eqs. (40) and (47)

$$\frac{\dfrac{v_{rc}eE_r^*}{mv} + \delta_c}{v_{rc}} = \frac{\dfrac{v_{rH}eE_r^*}{mv} - \dfrac{\gamma\hat{n}_c}{v^2}\left(1 - \dfrac{T_c}{T_H}\right)}{v_{rH}}$$

or

$$v_{rc} = -v_{rH}\, v^2\delta_c/\gamma\hat{n}_c \left(1 - \frac{T_c}{T_H}\right) \; . \qquad (48)$$

Thus for $\underset{\sim}{v}$ to be tangential to V(r) for $v<V$ requires that v_{rc} be negative. In other words, if the experimental results and the drift-kinetic equation Eq. (9), are both correct, the boundary V(r) must be the dividing line where Γ_w changes sign; f_c is the part of f diffusing radially inwards in the heat conduction and f_H is the part diffusion radially outwards. (It can be noted in passing that for v_{rc} to be negative the effective electric field E_r^* must be sufficiently negative such that the net "force" driving the diffusion of f_c is negative for w less than W; i.e. taking $w \approx W$,

$$\frac{eE_r^*}{T_c} - \frac{n_c'}{n_c} - \frac{T_c'}{T_c}\left(\frac{mW}{T} - \frac{3}{2}\right) < 0 \; . \qquad (49)$$

This net force will be only weakly negative for $w = W$, since δ_c is only weakly positive).

From analogy with gas dynamics it seems appropriate to call the disconinuity at $v = V$ a "contact discontinuity". A contact discontinuity in gas dynamics[7] is the sharp boundary surface which occurs between two regions of gas having a difference in one or more of the properties; density, temperature or tangential velocity magnitude, there being no velocity component perpendicular to the surface. To the extent that diffusion, heat conduction and viscosity are neglected, the discontinuity is infinitely steep in one or more of these three properties. The contact discontinuity in phase space discussed here is analogous to a discontinuity in tangential velocity in real space in gas dynamics.

VI. Identification of $T_{\parallel}$, $T_{\perp}$

A. Estimate for $T_{\parallel}$, $T_{\perp}$

The charge exchange measurements made with lines of sight perpendicular to the discharge tube, or at a small angle to the perpendicular, will detect ions with small $v_{\parallel}$ and large velocity perpendicular to $\underset{\sim}{B}$, i.e. $\xi \equiv v_{\parallel}/v \simeq 0$. The measured distribution function will therefore be f^{+}, with the lowest order approximation being f_{o}. Hence the two component distributions observed experimentally with such lines of sight can be identified directly with the two components predicted for f_{o} by the theory in Section III, namely f_{c} and f_{H}. In the case of the toroidally tangential lines of sight, ions will be sampled which have $\xi \simeq \pm 1$; hence $f^{+} + f^{-}$ is being measured.

In the banana regime, f is constant on a banana orbit apart from a small collisional correction: hence, considering ions which are just trapped, $f[r,w,\xi = \pm(2r/R)^{1/2}]$ will be the same as $f(r \mp \delta r_v, w \mp \delta r_v e E_r^*/m, \xi=0)$, where δr_v is half the width of the banana orbit given approximately by $\delta r_v \simeq (2r/R)^{1/2} mv/eB_\theta$. Also, if pitch-angle scattering is strong compared with energy scattering ($Z^i_{eff} >> 1$), the gradient of f from $\xi = \pm(2r/R)^{1/2}$ to $\xi = \pm 1$ will be small. Hence, from standard neoclassical theory[7]

$$f_\parallel \equiv f(r,w,\xi = +1) \simeq f_o + f_1^-[r,w,\xi = +(2r/R)^{1/2}]$$

$$= f_o - \delta r_v \frac{\partial f_o}{\partial r} - \frac{\delta r_v e E_r^*}{m} \frac{\partial f_o}{\partial w} \quad . \tag{50}$$

Also, if the particle energy for the discontinuity between f_c and f_H is W(r) for $\xi = 0$, it should be observed at radius r for $\xi = \pm 1$ at $\hat{W} \equiv 1/2\hat{V}^2$ given by

$$\hat{W} = W(r \mp \delta r_V) \pm \delta r_V e E_r^*/m \tag{51}$$

where δr_V is the value of δr_v for v = V.

Hence for $v > \hat{V}$ and $\xi = +1$, from Eqs. (33) and (34) for the case where the interaction with electrons and beam ions is weak, the observed f given by Eq. (50) will be

$$f_\parallel \simeq f_o\left[1+2\left(\frac{r}{R}\right)^{1/4}\left(\frac{v}{v_{T_H}}\right)\left(\frac{1-\frac{T_c}{T_H}}{0.49\ Z^i_{eff}}\right)^{1/2}\right] .$$

Assuming the second term in the square brackets is less than unity

$$\ell n\ f_\parallel \simeq \ell n\ f_o + 2\left(\frac{r}{R}\right)^{1/4}\left(\frac{v}{v_{T_H}}\right)\left(\frac{1-\frac{T_c}{T_H}}{0.49\ Z^i_{eff}}\right)^{1/2}$$

and

$$T_\parallel (v>\hat{V}) = \left(-\frac{\partial \ell n f_\parallel}{m\partial w}\right)^{-1} = T_H\left[1+\left(\frac{r}{R}\right)^{1/4}\left(\frac{v_{T_H}}{v}\right)\left(\frac{1-\frac{T_c}{T_H}}{0.49\ Z^i_{eff}}\right)^{1/2}\right] \qquad (52)$$

For $v < \hat{V}$, it is convenient to write $\delta r_v = \delta r_V + (2r/R)^{1/2} m(v-\hat{V})/eB_\theta$; then $f_\parallel$ for $\xi = +1$, $v < \hat{V}$ is

$$f_\parallel = f_c(r-\delta r_V)\left\{1+\left(\frac{2r}{R}\right)^{1/2}\frac{m(\hat{V}-v)}{eB_\theta}\left[\frac{n_c'}{n_c}+\frac{T_c'}{T_c}\left(\frac{mw}{T_c}-\frac{3}{2}\right)-\frac{eE_r^*}{T_c}\right]\right\}$$

and

$$T_\parallel (v<V_\parallel) = \left(-\frac{\partial \ell n f_\parallel}{m\partial w}\right)^{-1} = T_c(r-\delta r_V)\left\{1-\frac{1}{2}\left(\frac{2r}{R}\right)^{1/2}\frac{mv_{T_c}^2}{eB_\theta v}\right.$$

$$\left[\frac{n_c'}{n_c} + \frac{T_c'}{T_c}\left(\frac{mW}{T_c} - \frac{3}{2}\right) - \frac{eE_r^*}{T_c}\right] + \left(\frac{2r}{R}\right)^{1/2} \frac{m(\hat{V}-v)}{eB_\theta}\frac{T_c'}{T_c} \Big\} \tag{53}$$

Similarly for ξ =-1 one obtains

$$T_{\parallel}(v>\hat{V}) = T_H\left[1- \left(\frac{r}{R}\right)^{1/4}\left(\frac{v_{T_H}}{v}\right)\left(\frac{1-\frac{T_c}{T_H}}{0.49\ Z^i_{eff}}\right)^{1/2}\right] \tag{54}$$

$$T_{\parallel}(v<\hat{V}) = T_c(r+\delta r_v)\Big\{ 1+ \frac{1}{2}\left(\frac{2r}{R}\right)^{1/2} \frac{mv^2_{T_c}}{eB_\theta v}\left[\frac{n_c'}{n_c} + \frac{T_c'}{T_c}\left(\frac{mw}{T_c} - \frac{3}{2}\right) - \frac{eE_r^*}{T_c}\right]$$

$$- \left(\frac{2r}{R}\right)^{1/2} \frac{m(\hat{V}-v)}{eB_\theta}\frac{T_c'}{T_c}\Big\} \tag{55}$$

In summary, Eqs. (51 - 55) predict that for the case of high Z^i_{eff} the observed ion distributions for tangential observation should show two effective temperatures, the values being modified somewhat from T_c, T_H as given by these equations. The discontinuity energy W will be displaced to lower energy for parallel observation and to higher energy for anti-parallel observation. This follows from Eq. (51) and the condition of Eq. (49) which requires a large negative value for E_r^*.

In the case where Z^i_{eff} is close to unity, theory indicates that the magnitude of f^-_1 will decay appreciably between $\xi = \pm(2r/R)^{1/2}$ and $\xi = \pm 1$ due to energy scattering. This can be seen in Figure 2 of Reference 14 which shows a 66% reduction for $Z^i_{eff} = 1$ and $r/R = 0.1$. For such clean discharges the observed $f_\parallel$ and $f_\perp$ will be much closer to f_o.

B. Comparison with Experiment

The only set of experimental measurements which can be compared with the above predictions are the ATC measurements. Since Z_{eff} for electrons was 4 and oxygen was the main impurity this gives $Z^i_{eff} = 7$, and the requirement $Z^i_{eff} \gg 1$ is well satisfied. The only other tangential measurements, the observations with ohmic heating on PDX giving $T_\parallel \simeq T_\perp$, have already been explained in Section IV on the basis of the strong interactions of f_H with the electrons.

In order to compare the above predictions with the ATC measurements approximate estimates are needed for the two parameters E^*_r and W. These quantitites can be determined from the two conditions: (a) zero net diffusion for the ions, Eq. (5); and (b) the requirement that the flows in phase space on either side of the discontinuity be anti-parallel, namely Eq. (48). Substituting for v_{rH} and v_{rc} from Eq. (39) and its equivalent on the low energy side of V, Eq. (48) becomes

$$\frac{\frac{eE_r^*}{T_c} - \frac{n_c'}{n_c} - \frac{T_c'}{T_c}\left(\frac{mW}{T} - \frac{3}{2}\right)}{\frac{4}{3}\sqrt{\pi}\ \frac{v}{v_{T_c}^3}\left(\frac{T_H}{T_c} - 1\right)\ \exp\left(-\frac{mW}{T_c}\right)} = -\frac{\frac{eE_r}{T_H} + \frac{1}{\lambda}}{\frac{1}{v^2}\left(1 - \frac{T_c}{T_H}\right)} \tag{56}$$

An example solution of Eqs. (5) and (56) for the ATC parameters $\overline{n}_e = 2 \times 10^{13}\mathrm{cm}^{-3}$, $T_{io} = 220\mathrm{eV}$, $I = 65\mathrm{kA}$, $Z_{eff} = 4$ and $r = 5\mathrm{cm}$, assuming both n_c and T_c have parabolic radial profiles, (Doppler broadening measurements for ATC[12] gave such a profile for T_c) gives $T_c = 198\mathrm{eV}$, $1/\lambda = 0.34\mathrm{cm}^{-1}$, $eE_r^*/T_c = 4.5\ n_c'/n_c = 0.19\mathrm{cm}^{-1}$ or $E_r^* = 37\mathrm{V/cm}$ and $mW = 4.6T_c = 910\mathrm{eV}$. (The details of these calculations will be published elsewhere. The effect of electrostatic trapping has been included. $e\tilde{\Phi}(\theta)/T_c$ was found to be $0.6r/R$, being caused by the non-uniforminty of the impurity ions due to their poloidal rotation velocity $-E_r^*/B$.)

Since the poloidal Larmor radius for 910eV is 6cm giving $\delta r_V \simeq 2\mathrm{cm}$, the discontinuity should be observed with parallel observations at the radius $5 + \delta r_V = 7\mathrm{cm}$ at the energy $910 - \delta r_V eE_r^* = 836\mathrm{eV}$. The only example plot of ℓn f versus energy shown in Reference 4 is for parallel observation at the somewhat larger radius of 10.7cm. Although only a single straight line is plotted, an examination of the data points shows evidence for a discontinuity at 700eV. The four data points below this energy fall on a straight line with effective temperature 162eV, whereas the straight line through the data points above this energy give 230eV (It should be noted that the departure of the lowest energy point from the straight line for

230eV is quite small because of the limited energy range involved, namely 400eV to 700eV, and the departure may be within the experimental error.)

Turning to Eq. (52), the value of the quantity in the square brackets is 1.1 at 700eV decreasing to 1.05 at 2,500eV, which was the highest energy recorded. Thus the effective temperature for $v>\hat{V}$ should vary from 242eV to 231eV, which is only a small variation, the mean being close to the experimental value 230eV. For $v<\hat{V}$, the correction due to the second term in the curly bracket in Eq. (53) will be small since the effective force in the square bracket has already been seen to be small (see discussion in Section V). The third term is zero at the discontinuity and is -.09 at 400eV. Since δr_V for 700eV at r = 10.7cm is 2.5cm, $T_c(r+\delta r_V)$ from the parabolic profile is 163eV so that the predicted effective temperature for $v<\hat{V}$ is 163eV at 700eV decreasing to 148eV at 400eV. The mean of 156eV is close to the value for the slope of the lowest four data points.

With anti-parallel observations the discontinuity should be observed at the energy $\hat{W} = W(r+\delta r_V) - \delta r_V e E_r^*/m$. For $v > \hat{V}$ the correction factor for T_H given by Eq. (54) is now less than unity and varies from 1.0 at r = 0 to 0.8 at r = a. For $v < \hat{V}$ the effective temperature will be $T_c(r+\delta r_V)$ for $v \simeq \hat{V}$ and increase with decreasing v because the third term in the curly brackets in Eq. (55) is now positive. The best single straight line fit should therefore be some weighted average between the reduced T_H value and $T(r+\delta r_V)$. This could explain why the observed values for $T_{\|}$ are somewhat larger than the values for T_i from the Doppler broadening of impurity spectral lines.[12]

VII. Conclusions

The conclusions which can be drawn from the results of the preceding sections can be summarized as follows.

1. For the case of simple ion heat conduction between two minor radii, where ion self-collisions dominate for energy scattering and heating or cooling of the ion distribution tail f_H due to electron or beam ion collisions can be neglected, this tail, which is diffusing outwards in radius and downwards in energy, maintains a constant effective temperature $T_H \equiv (-\partial \ell n f_o / m\partial\varepsilon)^{-1}$. This result applies when the tail ions are in the banana regime. The magnitude of f_o decreases exponentially with radius with a scale length of the order of the ion Larmor radius using the central ion temperature; but this is not as rapid a decrease as would be required for the tail to match the decreasing temperature of f_c, the lower energy part of f_o which is diffusing inwards.

2. Collisional interaction of the tail ions with the electrons will cause a gradient of T_H with radius given approximately by

$$\frac{\partial T_H}{\partial r} \simeq - \frac{3\nu_{ei}}{v_r}\left(\frac{m_e}{m_i}\right)(T_H - T_e)$$

where v_r is the average radial velocity associated with the radial diffusion of the tail ions. The gradient of T_H will be larger in hydrogen than deuterium and in Section IV, this effect was shown to explain firstly, the observed decrease of $T_{\|}$ with radius in the hydrogen ATC experiment, $T_{\|}$ being closely related to T_H as explained in Section VI and secondly, the approximate equality of $T_{\parallel}$ and $T_{\perp}$ observed in the ohmic heating PDX experiment.

3. In Section V it was shown that the observed discontinuity in $\partial \ell n f_o / \partial \varepsilon$ which separates the two components f_c, f_H of the ion distribution, can only be reconciled with the Fokker-Planck drift-kinetic equation if this is the boundary where the radial diffusion of f_o changes sign. The discontinuity is interpreted as a contact discontinuity in the phase space r,v, which is analogous with the contact discontinuities known in gas dynamics.

4. In those discharges where the two component ion distributions exist, the ion transport properties will be modified and in particular the ion heat conduction is increased. However, the heat conduction is no longer proportional to the temperature gradient (i.e., the apparent temperature gradient $\partial T_c / \partial r$); it depends on the values of W, the ion energy where the discontinuity occurs, $E_r^* \equiv E_r - V_{\parallel} B_\theta$, the effective electric field, and the temperature difference $T_H - T_c$. There is no longer a simple ion thermal conductivity.

In addition, the ion energy per unit volume will be significantly larger than that given by the temperature T_c - e.g. in the PDX measurements of Ref. 3, at r = 26cm it is greater by a factor 2.1 - and the rate of transfer of energy between electrons and ions is not proportional to $T_c - T_e$.

5. The measurements of ion temperature from the Doppler broadening of impurity ion spectral lines will monitor the temperature T_c of the lower energy part of f_o. There are many more particles in the f_c component than in f_H and their collision frequency is much higher so that the impurity ion temperature is strongly coupled to T_c.

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