Multi–Wave Model for Plasma–Wave Interaction

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Abstract

A model is presented that describes the interaction of electrons with both longitudinal and transverse waves in a cold plasma. Starting from the Lagrangian for the system of fields, background plasma, and particles, a finite dimensional self-consistent model is derived using the envelope approximation for the waves. The (squared) wave amplitudes and phases form action-angle variables in the closed system of waves and particles. The system conserves energy and momentum, and thus is natural for solving the beam-loading problem. Numerical simulations are performed to compare with earlier electrostatic problems.

I. INTRODUCTION

In describing the interaction of accelerated charged particles in laser driven plasma accelerators it is important to consider the reaction of the relatively small number of accelerated particles on the electromagnetic fields. This problem is known as the beam-loading problem (see e.g. [1]), which in its full form includes the reshaping of the functional form of the wave fields during the acceleration process. A simpler problem, as a first step to this full problem, is to ask the question of what happens to the amplitudes and phases of fixed sinusoidal waves during the acceleration process. Here we address this simpler problem. We start from the total Lagrangian for the particle-field system and reduce to a low-dimensional Hamiltonian system that describes the change in the wave amplitudes and phases during their interaction with a sample of N charged particles. The envelope approximation is made to eliminate the fast space-time scales of the waves in the field Lagrangian, yielding a new wave-particle Lagrangian from which we obtain the Hamiltonian for the $N + N_L + N_T - 1$ degree-of-freedom system, where N_L and N_T denote the number of longitudinal and transverse waves, respectively.

The theory developed here is particularly designed for the study of particle acceleration by laser driven fields. The theory gives a measure of the self-consistent field effects including, but not limited to, the energy-momentum conservation for the system. The approximation of a fixed sinusoidal wave does not allow for the reshaping of the wave fields, but this may be a small effect compared with the decrease in amplitudes and shifts in the phase velocity because the accelerated particles are often launched in a limited phase space region of the accelerated fields. The theory describes how test particle calculations can be expanded to include these key self-consistent field effects. For example, the increase of the longitudinal emittance from the shift in the wave phase space velocity during the acceleration process can be addressed with this new model, a phenomenon that is outside the range of conventional test particle simulations.

In the absence of transverse waves the system reduces to that for the beam-plasma system considered by many authors [2–7]. That system has been useful for exploring a number of self-consistent field issues including the corrections to the quasilinear field theory of the gentle bump on tail problem [7, 8]. In the present problem the particle orbits are significantly more complicated and thus understanding the effect of the orbits on the charge and current

densities is much less developed than in the electrostatic beam-plasma system. Thus, for example, the relevance of the "rotating bar" model introduced in [4] and the "macroparticle" model introduced in [5] for phase trapping are not clear for the present problem.

In Sec. II we give the simplifying assumptions and then use them to reduce the full field-particle Lagrangian density to the finite dimensional Lagrangian for the wave amplitudes, phases, and the particles. In Sec. III we derive the Hamiltonian for the system, introduce the action-angle variables for the waves, and derive conservation laws. In Sec. IV we give some numerical examples and discuss the results.

II. DERIVATION OF THE MODEL

The model we obtain describes the interaction of an electron beam in a cold plasma with multiple longitudinal and transverse waves. We start with a Lagrangian for the system of background plasma, fields, and particles

$$\begin{aligned} L \;=\; & \int \mathrm{d}^3x \; \left\{ \frac{1}{2}\, mn|\mathbf{v}|^2 + \frac{1}{8\pi}\left(|\nabla\phi|^2 - \left|\nabla\times\mathbf{A}^2\right|\right) - \rho\phi + \frac{1}{c}\,\mathbf{j}\cdot\mathbf{A} \right\} \\ & + \sum_{i=1}^{i=N} \left\{ \frac{1}{2}\, m|\dot{\mathbf{r}}_i|^2 + e\phi(\mathbf{r}_i,t) - \frac{e}{c}\,\dot{\mathbf{r}}_i\cdot\mathbf{A}(\mathbf{r}_i,t) \right\} \end{aligned} \tag{1}$$

where $\mathbf{v}$ is the Eulerian velocity field and n the particle density for the background plasma, ϕ and $\mathbf{A}$ are the scalar and vector potentials, respectively, $\mathbf{j}$ is the current, ρ is the charge density, and $-e$ is the electron charge. Quantities with a subscript refer to beam particles whereas those without to the background plasma. The Coulomb gauge $\nabla\cdot\mathbf{A} = 0$ is assumed.

Two kinds of waves are included: N_L longitudinal waves, which are described by the scalar potential, and N_T electromagnetic (linearly polarized) transverse waves, which are described by the vector potential.

Motivated by fluid theory and previous derivations for the beam-plasma model, we assume the following linear response relations for the background plasma:

$$\begin{aligned} & \mathbf{v}_{||} = \frac{1}{4\pi ne}\frac{\partial\nabla\phi_{||}}{\partial t}, \qquad \mathbf{v}_T = \frac{e}{mc}\,\mathbf{A}, \\ & \mathbf{j}_{||} = -en\mathbf{v}_{||}, \qquad \mathbf{j}_T = -ne\mathbf{v}_T, \\ & \rho = -\frac{1}{4\pi}\nabla^2\phi, \\ & \phi = \sum_{i=1}^{N_L}\phi_i, \qquad \mathbf{A} = \sum_{i=1}^{N_T}\mathbf{A}_i\,, \end{aligned} \tag{2}$$

where $\mathbf{v}_{||}$ and $\mathbf{v}_T$ are the longitudinal and transverse velocities, respectively, in terms of which the currents $\mathbf{j}_{||}$ and $\mathbf{j}_T$ are described.

The fields are constrained by assuming the following forms:

$$\begin{aligned} \phi_i(\mathbf{r},t) &= \phi_i(t)\cos[k_{Li}x - \omega_{Li}t - \beta_i'(t)], \quad i = 1...N_L, \\ A_{yi}(\mathbf{r},t) &= a_i(t)\cos[k_i x - \omega_i t - \theta_i'(t)], \qquad i = 1...N_T, \end{aligned} \tag{3}$$

and $A_{xi} \equiv A_{zi} \equiv 0$. The longitudinal fields only have components in the x–direction. The amplitudes, $\phi_i(t)$ and $a_i(t)$, and the phases, $\beta_i'(t)$ and $\theta_i'(t)$, of (3) are assumed to be slowly varying functions of time (Whitham's envelope approximation [10]). We substitute (3) into (1) and keep only terms up to first (linear) order in the time derivatives of the phases and the amplitudes. Assuming periodic boundary conditions we perform the spatial integration (i.e. we average) of the Lagrangian (1) over a box of size l to obtain

$$\begin{aligned} L(t) =& \sum_{i=1}^{N}\frac{1}{2}m\left[\dot{x}_i^2(t) + \dot{y}_i^2(t)\right] + e\sum_{i=1}^{N}\sum_{j=1}^{N_L}\phi_j(t)\cos[k_{Lj}x_i(t) - \omega_{Lj}t - \beta_j'(t)] \\ & -\frac{e}{c}\sum_{i=1}^{N}\sum_{j=1}^{N_T}\dot{y}(t)a_j(t)\cos[k_j x_i(t) - \omega_j t - \theta_j'(t)] \\ & \frac{l^3}{8\pi}\sum_{j=1}^{N_L}\frac{k_{Lj}^2\dot{\beta'}_j(t)\phi_j^2(t)}{\omega_p} + \frac{l^3}{8\pi}\sum_{j=1}^{N_T}\frac{\omega_j\dot{\theta}_j'(t)a_j^2(t)}{c^2}. \end{aligned} \tag{4}$$

In the above calculation we have made use of the dispersion relations for electrostatic and electromagnetic waves in a cold plasma $\omega_{Li} = \omega_p$ and $\omega_i^2 = \omega_p^2 + k_i^2c^2$, respectively. We have also assumed that wave vectors for different waves are different.

In the next section we derive the Hamiltonian of the system and show that the energy and momentum are conserved.

III. HAMILTONIAN FORM AND CONSERVATION LAWS

We write the Lagrangian (4) in dimensionless variables

$$
\begin{aligned}
L \;=\; & \sum_{i=1}^{N} \frac{1}{2}\left(\dot{\xi}_i^2 + \dot{\eta}_i^2\right) + \sum_{i=1}^{N}\sum_{j=1}^{N_L} \sqrt{\frac{J_j}{\nu_j^3 N}} \cos \nu_j(\xi_i - \beta_j) \\
& - \left(\frac{2n_b}{n}\right)^{\frac{1}{3}} \sum_{i=1}^{N}\sum_{i=1}^{N_T} \left(\frac{\omega_p}{\omega_j}\right)^{\frac{1}{2}} \sqrt{\frac{I_j}{\mu_j N}}\, \dot{\eta}_i \cos \mu_j(\xi_i - \theta_j) \\
& + \sum_{i=1}^{N_L} J_j \dot{\beta}_j + \sum_{i=1}^{N_T} I_j \dot{\theta}_j \\
& + \left(\frac{2n_b}{n}\right)^{-\frac{1}{3}} \sum_{i=1}^{N_T} \left(1 - \frac{1}{\nu_j}\right) J_j + \left(\frac{2n_b}{n}\right)^{-\frac{1}{3}} \sum_{i=1}^{N_T} \left(1 - \frac{\omega_j}{\mu_j \omega_p}\right) I_j \,, \qquad (5)
\end{aligned}
$$

where a number of scalings have been performed, $\mu_j = k_j/k_{L1}$, $\nu_j = k_{Lj}/k_{L1}$, $n_b = N/l^3$, and the following substitutions have been made

$$
\begin{aligned}
& \xi_i = k_{L1} x_i - \omega_p t(\tau), \quad \eta_i = k_{L1} y_i, \quad t = \omega_p^{-1} \left(2n_b/n\right)^{-\frac{1}{3}} \tau, \\
& \beta'(t) = \nu_j \beta_j(\tau) + (\nu_j - 1)\, \omega_p t(\tau), \quad \theta'(t) = \mu_j \theta_j(\tau) + \left(\mu_j - \frac{\omega_j}{\omega_p}\right) \omega_p t(\tau), \\
& \phi_j^2 = \left(\frac{m\omega_p^2}{e k_{L1}^2}\right)^2 \left(\frac{2n_b}{n}\right)^{\frac{4}{3}} \frac{J_j}{\nu_j^3 N}, \quad a_j^2 = \left(\frac{mc\omega_p}{e k_{L1}}\right)^2 \left(\frac{2n_b}{n}\right)^{\frac{4}{3}} \left(\frac{\omega_p}{\omega_j}\right) \frac{I_j}{\mu_j N}. \qquad (6)
\end{aligned}
$$

From this Lagrangian, by the standard procedure we find the Hamiltonian of the system

$$
\begin{aligned}
H \;=\; & \sum_{i=1}^{N} \frac{p_{\xi_i}^2}{2} + \sum_{i=1}^{N} \frac{1}{2} \left[p_{\eta_i} + \left(\frac{2n_b}{n}\right)^{\frac{1}{3}} \sum_{j=1}^{N_T} \left(\frac{\omega_p}{\omega}\right)^{\frac{1}{2}} \sqrt{\frac{I_j}{\mu_j N}} \cos \mu(\xi_i - \theta_j) \right]^2 \\
& - \sum_{i=1}^{N}\sum_{j=1}^{N_T} \sqrt{\frac{J_j}{\nu_j^3 N}} \cos \nu_j \left(\xi_i - \beta_j\right) \\
& - \left(\frac{2n_b}{n}\right)^{-\frac{1}{3}} \sum_{i=1}^{N_L} \left(1 - \frac{1}{\nu_j}\right) J_j - \left(\frac{2n_b}{n}\right)^{-\frac{1}{3}} \sum_{i=1}^{N_T} \left(1 - \frac{\omega_j}{\mu_j \omega_p}\right) I_j. \qquad (7)
\end{aligned}
$$

With this Hamiltonian the Poisson brackets are of canonical form and the conservation of energy, $\mathrm{d}H/\mathrm{d}\tau = 0$, is assured. To see conservation of momentum, we write out the

equations of motion

$$
\begin{aligned}
\dot{\xi}_i &= p_{\xi_i}, \\
\dot{p}_{\xi_i} &= -\sum_{j=1}^{N_L} \sqrt{\frac{J_j}{\nu_j N}} \sin \nu_j \left(\xi_i - \beta_j\right) \\
&\quad + \left(\frac{2n_b}{n}\right)^{\frac{1}{3}} \sum_{j=1}^{N_T} \left(\frac{\omega_p}{\omega_j}\right)^{\frac{1}{2}} \sqrt{\frac{\mu_j I_j}{N}} \sin \mu_j \left(\xi_i - \theta_j\right) \times \\
&\quad \times \left[p_{\eta_i} + \left(\frac{2n_b}{n}\right)^{\frac{1}{3}} \sum_{j=1}^{N_T} \left(\frac{\omega_p}{\omega_j}\right)^{\frac{1}{2}} \sqrt{\frac{I_j}{\mu_j N}} \cos \mu_j \left(\xi_i - \theta_j\right) \right], \\
\dot{\eta}_i &= p_{\eta_i} + \left(\frac{2n_b}{n}\right)^{\frac{1}{3}} \sum_{j=1}^{N_T} \left(\frac{\omega_p}{\omega_j}\right)^{\frac{1}{2}} \sqrt{\frac{I_j}{\mu_j N}} \cos \mu_j \left(\xi_i - \theta_j\right), \\
\dot{p}_{\eta_i} &= 0, \\
\dot{\beta}_i &= -\frac{1}{2} \frac{1}{\sqrt{\nu_i^3 J_i N}} \sum_{k=1}^{N} \cos \nu_i \left(\xi_k - \beta_i\right) - \left(\frac{2n_b}{n}\right)^{-\frac{1}{3}} \left(1 - \frac{1}{\nu_i}\right) \\
\dot{J}_i &= \sqrt{\frac{J_i}{\nu_i N}} \sum_{k=1}^{N} \sin \nu_i \left(\xi_i - \beta_i\right), \\
\dot{\theta}_i &= \frac{1}{2} \frac{1}{\sqrt{\mu_i I_i N}} \left(\frac{\omega_p}{\omega_i}\right)^{\frac{1}{2}} \left(\frac{2n_b}{n}\right)^{\frac{1}{3}} \sum_{k=1}^{N} \cos \mu_i \left(\xi_k - \theta_i\right) \times \\
&\quad \times \left[p_{\eta_k} + \left(\frac{2n_b}{n}\right)^{\frac{1}{3}} \sum_{j=1}^{N_T} \left(\frac{\omega_p}{\omega_j}\right)^{\frac{1}{2}} \sqrt{\frac{I_j}{\mu_j N}} \cos \mu_j \left(\xi_k - \theta_j\right) \right] \\
&\quad - \left(\frac{2n_b}{n}\right)^{-\frac{1}{3}} \left(1 - \frac{\omega_i}{\mu_i \omega_p}\right), \\
\dot{I}_i &= -\sqrt{\frac{\mu_i I_i}{N}} \left(\frac{\omega_p}{\omega_i}\right)^{\frac{1}{2}} \left(\frac{2n_b}{n}\right)^{\frac{1}{3}} \sum_{k=1}^{N} \sin \mu_i \left(\xi_k - \theta_i\right) \times \\
&\quad \times \left[p_{\eta_k} + \left(\frac{2n_b}{n}\right)^{\frac{1}{3}} \sum_{j=1}^{N_T} \left(\frac{\omega_p}{\omega_j}\right)^{\frac{1}{2}} \sqrt{\frac{I_j}{\mu_j N}} \cos \mu_j \left(\xi_k - \theta_j\right) \right],
\end{aligned}
\tag{8}
$$

whence it is easy to see that the two components of the total momentum

$$
P_\xi := \sum_{i=1}^{N} p_{\xi_i} + \sum_{j=1}^{N_L} J_j + \sum_{j=1}^{N_T} I_j, \tag{9}
$$

$$
P_\eta := \sum_{i=1}^{N} p_{\eta_i}, \tag{10}
$$

are conserved, $dP_\xi/dt = dP_\eta/dt = 0$.

Notice that the canonical coordinates for the waves are the phases β_j and θ_j, and the corresponding conjugate momenta are J_j and I_j, respectively. The variables η_i are ignorable and so the p_{η_i} are conserved. The number of degrees of freedom is thus reduced to $N+N_L+N_T-1$. When the transverse fields are set to zero, the Hamiltonian reduces to that given in [4, 5].

From these equations we see that the coupling between waves only occurs by means of the particle dynamics. This kind of coupling suggests there will be a transfer of energy between transverse and longitudinal waves, as well as between particles and waves. Because our model is non-relativistic, the latter interaction is expected to be small. Everywhere transverse waves in the system enter with two small factors: $(\omega_p/\omega_i)^{\frac{1}{2}}$ and $(2n_b/n)^{\frac{1}{3}}$. The first small parameter appears in laser wake field accelerator physics, where the laser frequency is much larger than the longitudinal wave frequency. However, in our derivation of the model nowhere is this small parameter used and so this parameter can in general be chosen arbitrarily. The smallness of the second parameter is required to ensure that no significant changes of the shape of the waves occurs. Therefore, generally the influence of the transverse waves is expected to be small. To increase the coupling between particles (electrons) and transverse (electromagnetic) waves we have to consider relativistic velocities of the beam particles because the phase velocity of the transverse wave is always greater than c. However, as we see from the next section, such coupling exists even in the non-relativistic case and is not negligible.

IV. NUMERICAL RESULTS AND DISCUSSION

In the absence of transverse waves the system has been extensively studied in a number of papers [2–8]. In the present section we give some preliminary numerical results that demonstrate the effect of the presence of transverse waves in the system. We analyze the system of equations (8) numerically for $N = 100$ electrons, $(2n_b/n)^{\frac{1}{3}} = 0.1$, $p_{\xi_i}(0) = 0$, $\beta_i(0) = 0$, and $J_i(0) = 0$. The longitudinal waves grow up from instability. The initial velocity of the beam particles is taken equal to the phase velocity of the longitudinal wave (resonant electrons). Because the system of equations (8) is in a moving frame, the initial momenta (velocities) are zero. All momenta p_{η_i} are taken to be zero, and the frequencies

are determined according to the formula

$$\frac{\omega_p}{\omega_i} = \left(1 + \mu_i^2 \frac{c^2}{\omega_p^2/k_{L1}^2}\right)^{-\frac{1}{2}} ,$$

The ratio of the phase velocity ω_p/k_{L1} to c is taken to be equal to 0.1 to assure non-relativistic electrons. Two quantities that are used in laser-plasma experiments are $|e\mathbf{E}_T/m\omega_i c|$ and $\left|e\mathbf{E}_{||}/m\omega_p c\right|$. Using the relations $\left|\mathbf{E}_{||}\right| = |\nabla\phi|$ and $|\mathbf{E}_T| = \left|\frac{1}{c}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right|$ for the longitudinal and transverse electric fields, and the relations (6), we find the following estimates:

$$\left|\frac{e\mathbf{E}_{||}}{m\omega_p c}\right| \simeq \left(\frac{\omega_p}{k_{L1}c}\right)\left(\frac{2n_b}{n}\right)^{\frac{2}{3}}\sqrt{\frac{J_i}{\nu_i N}} , \tag{11}$$

$$\left|\frac{e\mathbf{E}_T}{m\omega_i c}\right| \simeq \left(\frac{\omega_p}{k_{L1}c}\right)^{\frac{5}{2}}\left(\frac{2n_b}{n}\right)\sqrt{\frac{I_i}{\mu_i^4 N}} . \tag{12}$$

For the parameters we use in our numerical simulations, and assuming $J_i = 10^2$, $\nu_i = 1$, $I_i = 10^5$, and $\mu_i = 2$, we get the values of 10^{-3} and 2.5×10^{-5} for (11) and (12), respectively.

Now let us consider numerical solutions of Eqs. (8). First we look at the case of one longitudinal and one transverse wave. In Fig. 1 we show results from several runs with various values of the transverse wave amplitude. We see very little influence of the transverse wave. With no transverse wave (solid curve) the longitudinal field evolves according to the equations for the longitudinal single-wave model. For example, the results agree with those shown in [5] (up to a change of variables).

In Fig. 2, we show runs with two transverse waves with equal initial amplitudes and phases, but with different wave vectors. Now comparison with the single-wave case shows that the growth of the longitudinal wave occurs at an earlier time that depends on the size of the initial amplitudes of the transverse waves. Further, we examine the behavior of the system for long times. In Fig. 3 we show the growth of the longitudinal wave for large initial amplitudes of the two transverse waves (solid line). Figure 3 gives evidence for the chaotic behavior that is to be expected: a very small difference in one of the initial transverse wave amplitudes, a difference of one part in 10^5, causes a large difference in the subsequent evolution (dashed line). Figure 4 shows the evolution of the two transverse waves. In Fig. 5 we show the case of two transverse waves with slightly different wave vectors. The character of the longitudinal wave growth is generally the same, but this demonstrates that such large growth in the longitudinal wave is not a resonant phenomenon because resonance requires the condition $\mu_2 - \mu_1 = 1$. Remembering that the length scale in our equations is given

by k_L^{-1}, we see that the resonance condition is not fulfilled in this case. In Fig. 6 we show results for two transverse waves with equal initial amplitudes, but different initial phases and different wave vectors. Again a large growth in the longitudinal wave is present. The two curves again show sensitivity to initial conditions – in this case due to round-off error. The small scale oscillation evident in these plots (and also in some of the others) appears to be a real effect and not an artifact of numerical errors because it is present in runs with different accuracy.

The accelerated and larger growth of the longitudinal wave with the addition of the transverse waves is not too surprising because our system is highly nonlinear and has many degrees of freedom. Chaotic behavior is to be expected, see also [5]. There are two kinds of electrons: those trapped in (the bottom of) the trough of the electrostatic wave, and those that can gain enough energy to overcome the electrostatic wave potential. The large transverse wave amplitudes facilitates the transfer between the two kinds of motion and as a result there is a more effective transfer of energy between the transverse and longitudinal waves giving rise to the large growth of the longitudinal wave.

V. CONCLUSIONS

We have derived a system of ordinary differential equations for describing the evolution of an electron beam in a cold plasma interacting with multiple longitudinal and transverse waves. In this Hamiltonian formulation we have shown conservation of momentum and energy. The model is nonlinear and has features of chaotic behavior. An example with one longitudinal and two transverse waves shows a large growth in the longitudinal wave amplitude that is due to energy transfer between the transverse and longitudinal waves through the beam particles.

A further modification of the model to include relativistic effects is needed for a more realistic description of experiments. In laser-plasma experiments on acceleration of electrons by a wake field (longitudinal wave) the electrons are relativistic. To make conclusions based on our model about such a physical situation, we need to extend the theory to include relativistic effects. This is work currently in progress.

Acknowledgments

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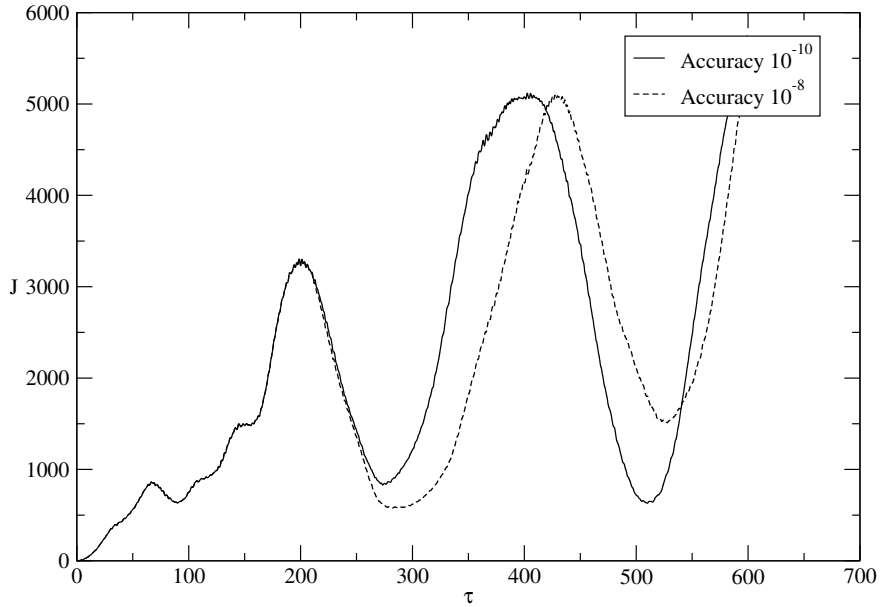

FIG. 1: Case of one transverse and one longitudinal wave. Several runs with different transverse wave initial amplitude. $\mu_1 = 2.0$.

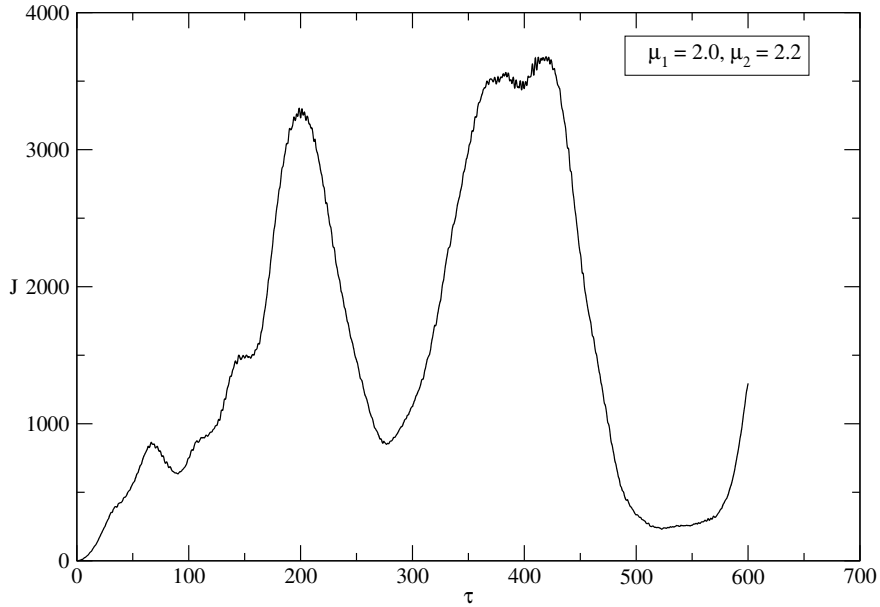

FIG. 2: Case of two transverse and one longitudinal wave. The longitudinal wave exhibits faster growth with larger initial I_i. $\mu_1 = 2.0$, $\mu_2 = 3.0$, and $\theta_1(0) = \theta_2(0) = 0.0$.

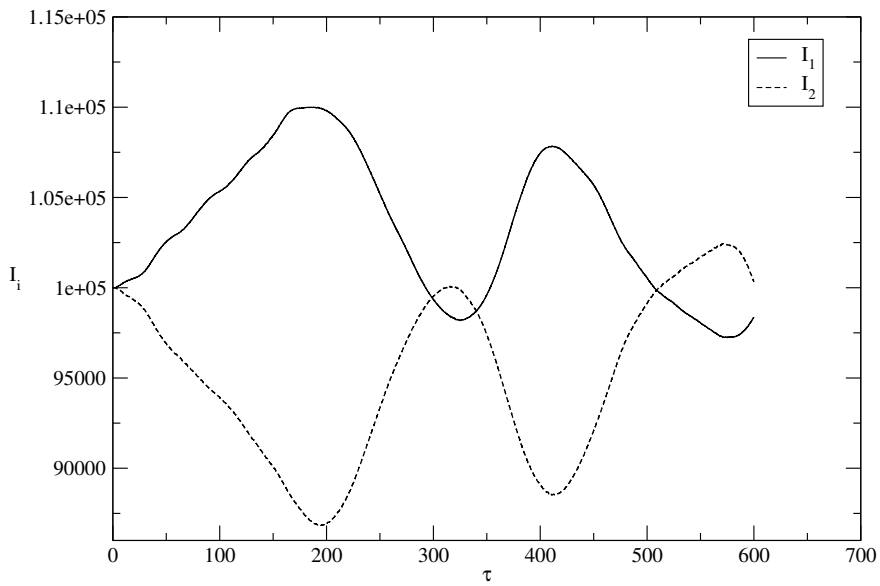

FIG. 3: Case of two transverse and one longitudinal wave. The two transverse waves have large initial amplitudes. Sensitivity to the initial conditions is evident. $\mu_1 = 2.0$, $\mu_2 = 3.0$, and $\theta_1(0) = \theta_2(0) = 0.0$.

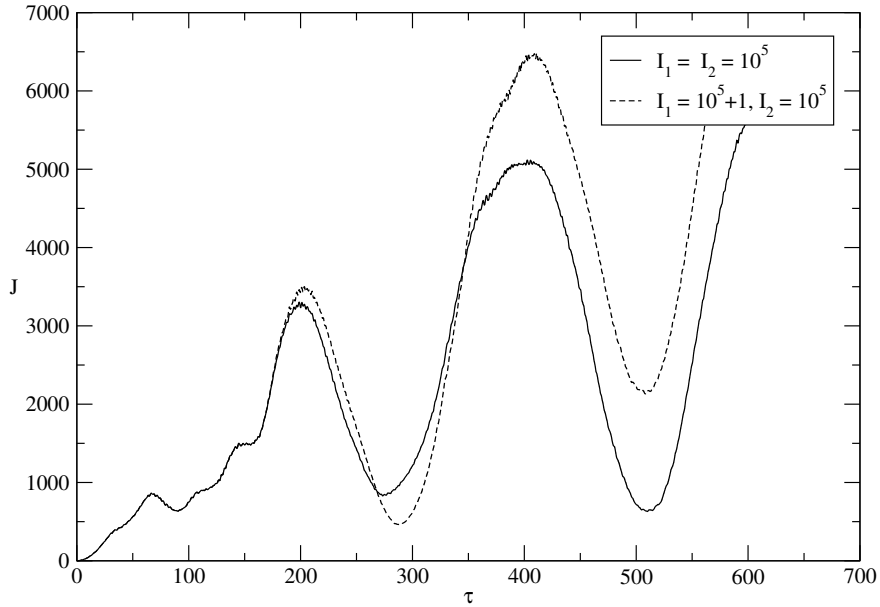

FIG. 4: Case of two transverse and one longitudinal wave. The evolution of the two transverse amplitudes is shown. $\mu_1 = 2.0$, $\mu_2 = 3.0$, and $\theta_1(0) = \theta_2(0) = 0.0$.

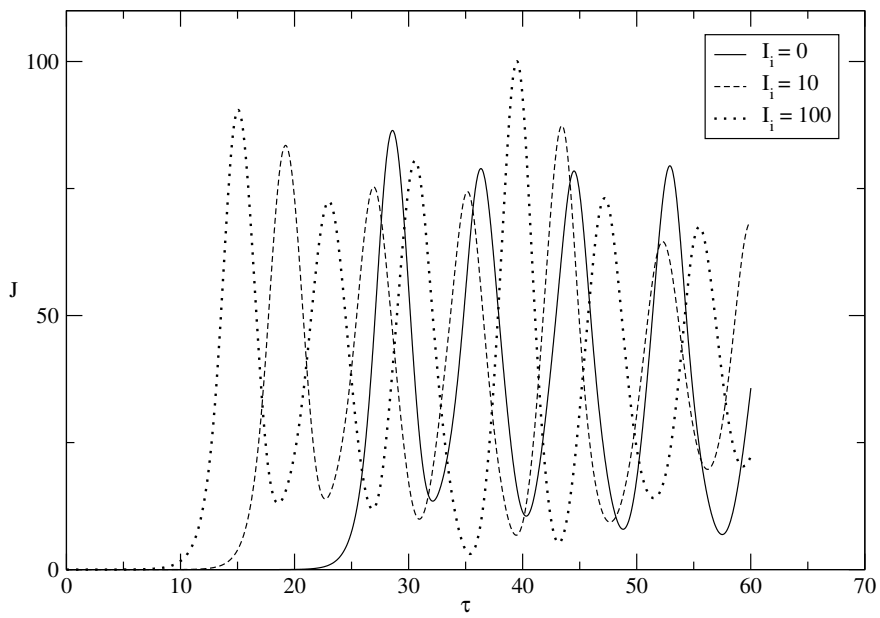

FIG. 5: Case of two transverse and one longitudinal wave. The two transverse waves have slightly different wave vectors. $\mu_1 = 2.0$, $\mu_2 = 2.2$, and $\theta_1(0) = \theta_2(0) = 0.0$.

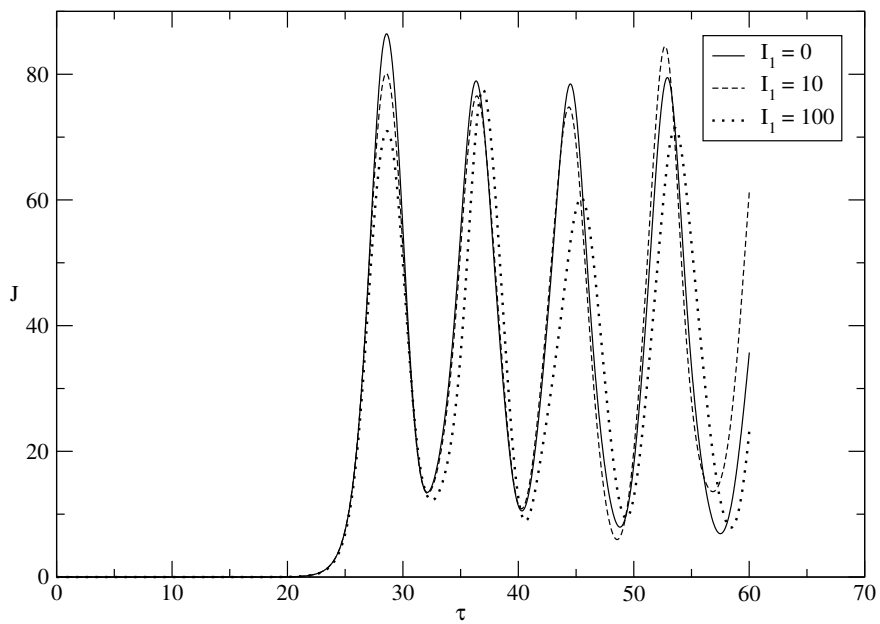

FIG. 6: Case of two transverse and one longitudinal wave. The initial phases of the transverse waves are different. Diminishing accuracy causes sensitivity to round-off error. $\mu_1 = 2.0$, $\mu_2 = 3.0$, $\theta_1(0) = 0$, and $\theta_2(0) = 3\pi/2$.