# On the Fluctuation Spectrum of Plasma


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## Abstract


The spectrum of electron phase space density fluctuations of a plasma is calculated by a novel method that parallels conventional calculations of the partition function in statistical physics. Expressions for the electric field fluctuations and the closely related form factor agree with existing results. The method clears up ambiguous statements about equipartition and provides a new expression for the spectrum of phase space density fluctuations about stable non-Maxwellian equilibria.

Several approaches can be taken for calculating the fluctuation spectrum of a homogeneous plasma. In the Klimontovich approach [1, 2] a one-point phase space density, concentrated on the phase space positions of $N$ point particles, is smoothed by ensemble averaging and then the resulting hierarchy is truncated. Alternatively, one can begin with an $N$-point Liouville equation and construct and truncate the BBGKY hierarchy (see e.g. [3, 4]), or follow a third approach which is to consider the superposition of dressed test particles (see e.g. [5–8]). Lastly, a direct statistical mechanical approach can be taken where one constructs and coarse grains the partition function for $N$ point particles interacting through the Coulomb potential (see e.g. Chap. VIII of [4]). In this paper we present a new method that is based on the partition function, where van Kampen modes [4, 9] are taken to be the basic degrees of freedom, and, consequently, transient (nonwave) dynamics is included.

Our approach parallels the specific heat calculations of Maxwell, Einstein, and others. For example, Einstein calculated the specific heat of a solid (later refined by Debye) by supposing it to be an equilibrium lattice configuration of point masses connected by springs. He then considered the statistical mechanics of the linear vibrations about the equilibrium configuration. He quantized the energy associated with the vibrational degrees of freedom and calculated a resulting discrete sum that appears in the partition function, whence he obtained the specific heat. In the classical limit his result agreed with the Dulong–Petit relation, which accounts for $k_B T$ for each degree of freedom. Our approach is philosophically the same: we begin with a homogeneous stable equilibrium solution of the Vlasov–Poisson system, which is analogous to Einstein's lattice, and then calculate the classical partition function where the basic degrees of freedom are the normal modes of the plasma, which are analogous to the lattice vibrations.

Classical partition function calculations lead to the evaluation of the following integral:

$$Z = \int d\mu \, e^{-\beta E},$$

where $\beta := 1/k_B T$. Evidently, two things are required to evaluate this integral: an expression for the energy, $E$, and a notion of invariant measure, $d\mu$. Both are provided by the Hamiltonian form of classical physics, where the energy is given by the Hamiltonian and, according to Boltzmann and Gibbs [10], the appropriate measure is given in terms of canonical variables, $d\mu = \prod dq dp$. In practice, one evaluates the integral by diagonalizing the Hamiltonian by a canonical transformation and then calculates the resulting product of

Gaussian integrals. This gives the well-known equipartition theorem, which states that in the average value of the energy one obtains $k_B T/2$ for each quantity appearing as a square in the energy. Thus, a gas which has no interaction potential energy gets half the value of the solid.

Our calculation is significantly more difficult than that described above because the 'vibrations' are governed by the linearized Vlasov equation which is a field theory. Thus we must evaluate the functional integral:

$$Z = \int \mathcal{D}p\,\mathcal{D}q\, e^{-\beta H}, \tag{1}$$

and this requires the Hamiltonian, $H$, the canonical field variables, $(q, p)$, and a means for calculating the functional integral. This calculation is hampered by the fact that the basic variable in Vlasov theory, the phase space density $f$, does not constitute a set of canonical variables, and the fact that the linear normal modes of interest, the van Kampen modes, have a continuous eigenvalue spectrum and associated singular eigenfunctions. Two advances make this calculation possible—techniques to canonize and to diagonalize. In [11] it was shown how the Vlasov equation is a Hamiltonian theory in terms of noncanonical variables, and in [12] (refined and extended in [13–15]) it was shown how to make sense out of the energy associated with van Kampen modes and how to diagonalize this energy by constructing an integral transform that is a generalization of the Hilbert transform. The diagonalization procedure turns (1) into a Gaussian functional integral that is rudimentary to evaluate.

We consider a plasma with immobile ions and electron dynamics governed by the Vlasov–Poisson system. We integrate out all but the longitudinal variables, and expand the phase space density as $f(x, v, t) = f_0(v) + \frac{1}{2}\sum_k f_k(v, t)\, e^{ikx}$. The equilibrium $f_0(v)$ is taken to be stable, which is assured if it is a monotonically decreasing function of $v^2$. The energy of the linearized dynamics, the Kruskal-Obermann energy [16], is given by $H = -\sum_k \int dv\, mv|f_k|^2/(2f_0') + \sum_k k^2|\phi_k|^2/8\pi$, where $f_0' := df_0/dv$ and $-k^2\phi_k = -4\pi e \int dv\, f_k$. In [12–15] it is shown that canonical field variables for the linearized dynamics are given by $q_k(v, t) = f_k$ and $p_k(v, t) = mf_{-k}/(ikf_0')$, where $k > 0$. We are now in a position to evaluate (1), by inserting the canonical variables into $H$ and by writing $\mathcal{D}p\,\mathcal{D}q$ as $\prod_k Dq_k Dp_k$. However, the electrostatic contribution to $H$ possesses two integrals over $v$ resulting in H not being diagonal, and this complicates the evaluation of (1).

Diagonalization is achieved by transforming from the canonical field variables $(q_k(v,t), p_k(v,t))$ to a new set of variables $(Q_k(u,t), P_k(u,t))$. We find it convenient to introduce an intermediate set of variables $(Q'_k(u,t), P'_k(u,t))$ where the new coordinate $Q'_k$ is obtained from the old by a transformation $\mathcal{G}_k$ given by

$$
\begin{aligned}
q_k(v,t) &= \mathcal{G}_k[Q'_k] \\
&= \epsilon_R(k,v)\, Q'_k(v,t) + \epsilon_I(k,v)\, H[Q'_k]\,. \qquad (2)
\end{aligned}
$$

Here $\epsilon_I := -\pi\omega_p^2 f'_0/k^2$, the Hilbert transform of a function g is $H[g] := \mathcal{P}/\pi \int du\, g(u)/(u-v)$, with $\mathcal{P}$ indicating Cauchy principal value, and $\epsilon_R = 1 + H[\epsilon_I]$. It can be shown that this transformation is invertible (on appropriate Banach spaces; see [15]). Equation (2) is the coordinate part of the canonical transformation generated by the type-two mixed variable generating functional $\mathcal{F}[q_k, P'_k] = \sum_k \int P'_k\, \mathcal{G}_k^{-1}[q_k]\, du$, which follows from $Q'_k = \delta\mathcal{F}/\delta P'_k$. The momentum part of the canonical transformation is obtained from $p_k = \delta\mathcal{F}/\delta q_k$, which gives $p_k(v,t) = (\mathcal{G}_k^{-1})^\dagger[P'_k]$, where $\mathcal{G}_k^{-1}$ is given by

$$
\begin{aligned}
Q'_k(u,t) &= \mathcal{G}_k^{-1}[q_k] \\
&= \frac{\epsilon_R(k,u)}{|\epsilon|^2(k,u)}\, q_k(u,t) - \frac{\epsilon_I(k,u)}{|\epsilon|^2(k,u)}\, H[q_k]\,. \qquad (3)
\end{aligned}
$$

We know $q_k$ is equal to $f_k$, but what is the physical meaning of the $Q_k$? It is a coordinate in which the linear dynamics is decoupled and it can be shown to be proportional to the electric field, $E_k(u,t)$, associated with a van Kampen mode. (Note, a van Kampen modes labeled by the continuum label $u$ corresponds to the oscillation frequency $ku$.) We use the relationship between $Q'_k(u,t)$ and $E_k(u,t)$ to obtain the electric field fluctuation spectrum.

Now, the diagonalizing coordinates are given by the canonical transformation $Q_k = (Q'_k - iP'_k)/\sqrt{2}$, $P_k = (P'_k - iQ'_k)/\sqrt{2}$, in terms of which the Hamiltonian becomes $H_L = \sum_k \int du\, ku\, (Q_k^2 + P_k^2)/2$. Having achieved diagonal form we can evaluate $Z$. This is done by discretizing the continuum label of $Q_k$ and $P_k$ through $u_j = j\Delta u$, for $j = -N, \ldots, N$ where $\Delta u = v_*/N$ and taking the limit $v_*$, $N \longrightarrow \infty$. Because the details of such calculations are well-known and because, as we will see, the results are compelling, we will report the details elsewhere. The ensemble average of a quantity $\mathcal{O}$ is given by $\langle\mathcal{O}\rangle = \int \mathcal{D}Q\mathcal{D}P\; \mathcal{O} e^{-\beta H}/Z$. In term of the above discretizations,

$$
\langle\mathcal{O}\rangle = \lim_{\substack{N\to\infty \\ v_*\to\infty}} \frac{1}{Z} \prod_{k=1} \prod_{j=-N}^{N} \int \mathcal{D}Q_k(u_j)\, \mathcal{D}P_k(u_j)\; \mathcal{O}\, e^{-\beta H} \qquad (4)
$$

whence we obtain after a relatively simple calculation

$$\langle E_k(u)E^*_{k'}(u')\rangle = \frac{16}{V\beta}\frac{\epsilon_I}{u|\epsilon|^2}\,\delta_{k,k'}\delta(u-u'). \tag{5}$$

Precisely this expression appears in the previous fluctuation calculations referred to above [1, 3, 5–8]. In fact, it can be argued that (5) is actually what is meant by the statistical part of the fluctuation-dissipation theorem.

In [1–8] it is noted that the right hand side of (5) approaches $k_BT/2$ in the limit $k\lambda_D << 1$, which suggests a failure of the equipartition theorem when this limit is not taken. However, one should not expect $E_k$ to be in equipartition because it is not a canonical variable. In statistical mechanics, equipartition is a property defined in terms of the canonical variables in which the Hamiltonian is diagonal. For each quadratic term in the Hamiltonian one obtains a contribution of $k_BT/2$ to the expectation value of the energy. Thus, the statement of equipartition in the present plasma physics context is

$$\langle ku\,Q_k(u)Q_{k'}(u')\rangle = \frac{k_BT}{2}\,\delta_{k,k'}\delta(u-u')\,. \tag{6}$$

Equation (6) can be written in terms of $f_k$, but we will not do so here.

Using (5), we can compute both the dynamic and static form factors. The dynamic form factor, $S(k,\omega)$, is defined (e.g. [3]) in terms of the density fluctuations, $\rho_k(\omega)$, by $\langle\rho_k(\omega)\,\rho^*_k(\omega')\rangle = 4\,\pi^2\,\delta(\omega-\omega')S(k,\omega)$. Some algebra yields the well-known result

$$S(k,\omega) = -\frac{N}{\pi\,\omega}\,\frac{k^2}{k_D^2}\,\mathrm{Im}\left(\frac{1}{\epsilon(k,\omega)}\right), \tag{7}$$

where $N$ is the total number of particles. The static form factor, $S(k)$, is defined by the sum-rule $\int d\omega\, S(k,\omega) = N\,S(k)$, hence

$$S(k) = \frac{k^2}{k_D^2}\,H\left[\mathrm{Im}\left(\frac{1}{\epsilon(k,\omega)}\right)\right](0)\,. \tag{8}$$

Without loss of generality we choose a frame where $f_0$ has a maximum at $v=0$, thus $\epsilon_I(0)=0$. We define $\epsilon_R(0) = 1 + k_\theta^2/k^2$, where $k_\theta$ is a measure of the width of $f_0$. (For a Maxwellian equilibrium, $k_\theta = k_D$.) Using, $H[\epsilon_I](0) = \epsilon_R(0)-1$, we obtain

$$S(k) = \frac{k^2}{k^2+k_\theta^2} \tag{9}$$

which is the standard expression that describes both self-correlation and Debye shielding (see e.g. [3]).

Equation (5) can be used to obtain $\langle f_k(v) f^*_{k'}(v')\rangle$ by mapping back from $(Q_k, P_k)$ to $(q_k, p_k)$ and writing the result in terms of $f_k$. Accounting for the scalings in the definitions of the various variables we obtain

$$\langle f_k(v) f^*_{k'}(v')\rangle = \frac{kk'}{16\pi^2 e^2}\langle \mathcal{G}_k[E_k](v)\mathcal{G}_{k'}[E^*_{k'}](v')\rangle.$$

Making use of identities associated with the transform, which are derived in [12–15], we obtain in a straightforward way

$$\langle f_k(v) f^*_k(v')\rangle = \delta_{k,k'}\,\frac{k^2}{\pi^2 e^2 V\beta}\times \Big\{\frac{\epsilon_I(v)}{v}\delta(v-v') - \frac{1}{\pi}\frac{\epsilon_R(0)}{|\epsilon(0)|^2}\frac{\epsilon_I(v')\epsilon_I(v)}{vv'}\Big\}. \tag{10}$$

This is a rigorous calculation, the details of which will be presented elsewhere. We note that this general result is not in [1–8], although the special case where the equilibrium is Maxwellian appears in [1].

Our approach is perhaps most akin to that of classical $N$-particle statistical mechanics of an electron gas neutralized by a positive charge background (see e.g. [4]), where the classical partition function is constructed for $N$ interacting electrons. Calculation of the partition function in the $N$-particle approach is difficult because of the Coulomb interaction. Consequently, the partition function is expanded and coarse grained, and eventually written as a product of one-particle partition functions. En route, a diagonalization of a discrete Hamiltonian is effected. Thus the $N$-particle approach like ours involves diagonalization within a Hamiltonian context. The $N$-particle dynamics is most basic in plasma physics, but the partition function obtained after approximation is not that for any known dynamics. In contrast, the linearized dynamics of our approach is limited, but our partition function calculations are exact. The $N$-particle approach produces the static form factor of (9) with $k_\theta = k_D$, the Maxwellian special case of our result (8).

The Liouville, Klimontovich, and test particle approaches contain discrete particle dynamics in some form, which is then smoothed and truncated, and they all give more or less the same answers. Equation (5) appears in these calculations, and fluctuation information needed for calculation of the Lenard–Balescu collision operator is calculated. However, none except Klimontovich explicitly obtain an expression for $\langle f_k(v) f^*_{k'}(v')\rangle$. If we insert a Maxwellian distribution function into our result (10) it reduces to his expression (cf. Eq. (10.38) of [1]).

Our generalization to non-Maxwellian equilibria is important because hot plasmas can exist in states different from thermodynamic equilibrium for substantial lengths of time. The temperature associated with these equilibrium states need not be the same as the heat bath temperature $1/\beta$ that characterizes the fluctuations. The thermal nature of the heat bath arises from the large number of degrees of freedom that couple to the plasma, which can be distinct from the temperature associated with a prepared equilibrium state.

One may wonder how Vlasov theory, being ostensibly collisionless, can produce results about correlations in agreement with the above calculations. Essentially this is possible because the Vlasov equation is identical to the Klimontovich equation, and thus contains the correct dynamics on small velocity scales. The distinction between the two equations amounts to a choice of initial conditions. Physically it is the interaction with the heat bath that communicates the interaction in the plasma, and the heat bath does not distinguish between Vlasov and Klimontovich theory. (Recall, in Einstein's calculation the Hamiltonian is the sum over independent linear oscillators.) That Vlasov theory can correctly produce correlations is not a new idea, for it was used in [5] and has also been used to obtain Lenard-Balescu type collision operators (see e.g. [17, 18]). However, for this to work it is essential that the linear operator equation be solved exactly. Calculations that expand to obtain the plasma dispersion function and Landau damping eliminate the essential transient effects.

The approach given here is of general utility. It is straightforward to include multiple species, and the essential ingredients, the Hamiltonian structure and the diagonalizing integral transform, exist for electromagnetic fluctuations [14]. In fluid mechanics functional integral calculations of partition functions for homogeneous turbulence have existed since the work of Onsager [19] and Lee [20] (see also [21–25]). The methods described here can be adapted to describe the fluctuations about inhomogeneous fluid states such as those that occur in shear flow and Rossby (or Drift) wave dynamics, because the diagonalizing transform has been worked out [26, 27]. Approximate fluid and plasma systems, such as the single wave model (e.g. [30]) and vorticity defect dynamics [29] are also amenable. Basically, for any stable equilibrium of a general class of Hamiltonian systems with a continuous spectrum [28] an analogous calculation can be performed.

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