Minimal Coupling and the Magnetofluid Unification

S. M. Mahajan

Institute for Fusion Studies, The University of Texas at Austin, Austin, TX 78712

The dynamics of a relativistic, hot charged fluid is expressed in terms of a hybrid magnetofluid field which unifies the electromagnetic field with an appropriately defined but analogous flow-field. In this unified field, the fluid experiences no net force. Suitably modified (due to temperature) minimal coupling prescription for particle dynamics may be invoked to affect the transformation from the flow-field (or the electromagnetic) to the magnetofluid field for the homentropic fluids. An appropriate prescription for the general isentropic fluids is also derived. A few consequences of the unification are worked out.

The minimal coupling prescription, epitomized in the substitution,

$$p_\mu \to p_\mu + qA_\mu, \quad mU_\mu \to mU_\mu + qA_\mu \tag{1}$$

reproduces the Lorentz force correctly, and is therefore routinely used for incorporating the electromagnetic field in particle dynamics. In Eq. (1), $p_\mu(U_\mu)$ and A_μ, are respectively the particle four momentum (four velocity) and the electromagnetic four potential, q is the electric charge and m is the mass of the particle. This prescription, however, does not tell us anything about the dynamics of the field; Maxwell's equation for the e.m. field have to be added for the full picture.

In this letter I examine the following questions:

(1) Is there an appropriate and transparent translation of the minimal coupling when one is dealing with a many-body system like a hot plasma for which closed fluid equations may be derived for many cases of interest.

(2) Is it possible to cast the velocity field in clothing designed to fit the electromagnetic field and does such a casting lead to a "unification" of the flow-field with electromagnetism? And will the new structure be mathematically simpler and more revealing than the standard formalism?

For this enquiry, I deal with a standard two-species plasma (one positively and one negatively charged), although generalization to many species is straightforward. A local Maxwellian closure will ensure that in the ensuing description, the scalar pressure, along with the four flux (constructed from density and three components of the velocity) constitutes the entire set of fluid variables. In addition to Maxwell's equations

$$\partial_\nu F^{\mu\nu} = J^\mu, \tag{2}$$

$$\partial_\nu \mathcal{F}^{\mu\nu} = 0, \tag{3}$$

the system consists of the following fluid equations valid for each species (α is the species index): the continuity equation

$$\partial_\nu \Gamma^\nu_{(\alpha)} = 0, \tag{4}$$

and the equation of motion

$$\partial_\nu T^{\mu\nu}_{(\alpha)} = q_\alpha F^{\mu\nu}\Gamma_{\nu(\alpha)} \tag{5}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the fully antisymmetric electromagnetic field tensor related to the electric and magnetic fields through

$$\begin{aligned} F^{0i} &= E^i \\ F^{ij} &= B^k,\ \ i,j,k \text{ cyclic,} \end{aligned} \tag{6}$$

and $\mathcal{F}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ is its dual obtained by the substitution $\mathcal{F}^{\mu\nu} = F^{\mu\nu}(\boldsymbol{E} \to \boldsymbol{B},\ \boldsymbol{B} \to -\boldsymbol{E})$. The fully relativistic energy-momentum tensor is given by [1, 2]

$$T^{\mu\nu}_{(\alpha)} = p\eta^{\mu\nu} + h_\alpha U^\mu_{(\alpha)} U^\nu_{(\alpha)} \tag{7}$$

where p_α is the pressure, h_α is the enthalpy density and $U^\mu_{(\alpha)} = \{\gamma_\alpha, \gamma_\alpha \boldsymbol{V}\} \equiv \{\gamma_\alpha, \boldsymbol{U}_\alpha\}$ is the four-velocity of the α-th component of the fluid and $\gamma = (1 - V^2)^{-1/2}$ is the relativistic factor. The flux

$$\Gamma^\mu_{(\alpha)} = n_{R(\alpha)} U^\mu_{(\alpha)} \tag{8}$$

where n_R is a scalar measuring the density in the rest-frame (local) of the given fluid. The current

$$J^\mu = \sum_\alpha q_\alpha \Gamma^\mu_{(\alpha)} = \sum_\alpha q_\alpha n_{R\alpha} U^\mu_{(\alpha)} \tag{9}$$

is the input needed to couple the fluid to Maxwell's equations. Notice that summing over species converts (5) into $[\mathcal{T}^{\mu\nu} = \sum_\alpha T^{\mu\nu}_{(\alpha)}]$

$$\partial_\nu \mathcal{T}^{\mu\nu} = F^{\mu\nu} J_\nu \tag{10}$$

which, in principle, can be solved to find J_ν. From now on, I will drop the species index, and much of what follows will hold for each species.

As stated earlier, I will use the standard, local Maxwellian closure, which, for the relativistic plasmas, yields the enthalpy density [2],

$$h = mn_R \frac{K_3(m/T)}{K_2(m/T)} \equiv mn_R f(T), \tag{11}$$

where m is the rest mass of the particles constituting the αth component of the fluid and K_n's are modified Bessel functions of the second kind with the argument $\zeta = m/T$. The function $f(T) \equiv f$ is a function only of the temperature. The theory being developed is fully relativistic, both in temperature (arbitrary $T/mc^2 \equiv T/m$) and in the directed speed.

After these necessary preliminaries, I now introduce, in complete analogy to the e.m. field tensor, the fully antisymmetric second-rank "flow" tensor

$$S^{\mu\nu} = \partial^\mu f U^\nu - \partial^\nu f U^\mu \tag{12}$$

constructed from the kinematic (U_μ) and the statistical $(f(T))$ attributes of the fluid. Its nonzero components [Greek (Latin) indices go from 0-3(1-3)] are

$$S^{0i} = Q^i \tag{13}$$

$$S^{ij} = R^k, \qquad i, j, k \text{ cyclic.} \tag{14}$$

The three-vectors $\boldsymbol{Q}$ and $\boldsymbol{R}$ are the fluid equivalents of the electric and the magnetic fields $[\boldsymbol{U} = \gamma \boldsymbol{V}]$

$$\boldsymbol{Q} = -\left[\frac{\partial}{\partial t} f\boldsymbol{U} + \nabla f\gamma\right], \tag{15}$$

$$\boldsymbol{R} = \nabla \times f\boldsymbol{U}. \tag{16}$$

The reason for introducing the weight factor$f$ in the definitions will become clear later. By construction $\boldsymbol{Q}$ and $\boldsymbol{R}$ satisfy the equivalent of the homogeneous Maxwell's equations

[Eq. (3)] $\Rightarrow \boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\ \partial\boldsymbol{B}/\partial t+\boldsymbol{\nabla}\times\boldsymbol{E}=0$],

$$\boldsymbol{\nabla}\cdot\boldsymbol{R}=0 \tag{17}$$

$$\frac{\partial\boldsymbol{R}}{\partial t}+\boldsymbol{\nabla}\times\boldsymbol{Q}=0, \tag{18}$$

which could be expressed explicitly covariantly

$$\partial_\nu\Sigma^{\mu\nu}=0, \tag{19}$$

where $\Sigma^{\mu\nu}=(1/2)\epsilon^{\mu\nu\alpha\beta}S_{\alpha\beta}$ is the dual of $S^{\mu\nu}$. Until now I have just given a set of self-consistent definitions. Now I will work on the equation of motion (5) to express it in terms of $S^{\mu\nu}$.

Using Eq. (11) for h, we could spell out Eq. (5) as

$$\partial^\nu p+mn_RU^\mu\partial_\mu fU^\nu=qn_RF^{\nu\mu}U_\mu, \tag{20}$$

where we have invoked $\partial_\mu\Gamma^\mu=\partial_\mu n_RU^\mu=0$. Contracting (12) with U_μ, we obtain

$$U^\mu\partial_\mu fU^\nu=U_\mu S^{\mu\nu}-\partial^\nu f. \tag{21}$$

Substituting (21) into (20) and combining terms containing p and f,

$$T\ \partial^\nu\sigma\ =\ qU_\mu M^{\nu\mu} \tag{22}$$

with

$$M^{\nu\mu}\ =\ F^{\nu\mu}+(m/q)S^{\nu\mu} \tag{23}$$

representing the effective field tensor combining the electromagnetic and fluid 'forces'. The entropy-like quantity $\sigma=\ell n\,[(p/K_2)(m/T)^2\exp[-(m/T)(K_3/K_2]]$ is an expression of purely fluid (thermal) attributes. From the indicial asymmetry of M (22), we find that $TU_\nu\partial^\nu\sigma=\gamma T\,d\sigma/dt=0$ ($d/dt=\partial/\partial t+\boldsymbol{V}\cdot\boldsymbol{\nabla}$) is the total time derivative) leading to the well-known isentropic equation of state $\sigma=$ const [2]. If σ were further assumed to be a global constant (homentropic fluid), the ensuing formalism becomes rather simple. The special case of the homentropic fluid is of great importance in the theory of plasma self-organization; it is the most general system (others have either constant density or constant temperature) in which the pressure force in the nonrelativistic equation of motion becomes a full gradient. This, in turn, allows the equation to be cast in a vortex dynamics form essential for the existence of the constant of motion which lie at the foundation of self-organization [3-4]. For the homentropic fluid, then, the equation of motion becomes

$$U_\mu M^{\nu\mu} = 0, \tag{24}$$

a remarkably simple and revealing form. Remembering that $U_\mu F^{\nu\mu}$ is the expression for the electromagnetic force, (24) tells us that the "Magnetofluid" field $M^{\nu\mu}$, obtained from the unification of the flow-field and the electromagnetic field, exerts no net force on the fluid; the entire complicated fluid dynamics of relativistic charged particles in an electromagnetic field is contained in this seemingly trivial statement.

The construction of $M^{\mu\nu}$ is the centerpiece of this effort. It is through $M^{\mu\nu}$ that the flow-field and the electromagnetic field are put on the same footing. Needless to say that it is an inherent and fundamental property of the equation of motion; the formalism simply reveals the unity.

Let us further examine the character of $M^{\nu\mu}$. Notice that this tensor is obtained from $S^{\mu\nu}$ by the transformation $fU^\nu \to fU^\nu + (q/m)A^\nu$, which for $f = 1$, would have been the minimum coupling prescription. For nonrelativistic temperatures, f does tend to unity and our prescription does approximately reduce to minimum coupling. As long as there is finite temperature, however, the statistical nature of the system (temperature being a statistical notion) becomes manifest and a purely inertial notion like the minimum coupling has to be suitably modified. The effective momentum $\boldsymbol{p} = mf\gamma\boldsymbol{V}$ could be interpreted either in terms of an effective mass (mf) or an effective relativistic factor (γf); either of these can be very different form its original value for plasmas with relativistic temperatures.

Before working out some of the consequences of Eq. (24), let me go back to the more general case of the isentropic fluid. After straightforward algebra (including using $\gamma d/dt = U_\mu \partial^\mu$), we could rewrite Eq. (22) as

$$U_\mu H^{\nu\mu} = 0, \tag{25}$$

where the general magnetofluid tensor $H = M + N$ has an additional term $N^{\nu\mu} = (m/q)[[\partial^\nu(T\sigma U^\mu) - \partial^\mu(T\sigma U^\nu)] - \sigma[\partial^\nu(TU^\mu) - \partial^\mu(T^\mu TU^\nu)]]$ which disappears when σ is a global constant. Equation (25), though identical in form to Eq. (24), is qualitatively different. Because of the existence of σ outside the derivatives in N, there is no unification at the potential (minimal) level, i.e., no prescription of the form $A_\mu \to A_\mu + (m/q)g\,U_\mu$ (where g is some function of pressure and temperature) can be devised.

However, even for this general case, the unification does exist at the field level; the prescription $F \to F + (m/q)[M + N]$ defines the generalized electromagnetic field which encompasses the fluid forces. This feature bridges the gap between the esentropic fluid and its special case; their behavior will be similar for those aspects of plasma dynamics that do not depend explicitly upon the potential (A_μ), and depend only on the fields ($\boldsymbol{E}$ and $\boldsymbol{B}$).

Just to make sure that the compact form (24) [or Eq. (25)] has all the familiar physics, let us take its nonrelativistic limit. This is easily accomplished by letting $\gamma \to 1$, $f \to 1$ everywhere except in the term $\nabla f\gamma$ in $\boldsymbol{Q}$, where the first order terms are needed (leading order terms vanish). Using the large (m/T) expansion of $f \simeq 1 + 5/2(T/m)$, and $\gamma \simeq 1 + V^2/2$, we find

$$\nabla f\gamma \to \nabla\left[\frac{V^2}{2} + \frac{5}{2}\frac{T}{m}\right], \tag{26}$$

which leads to the NR limit

$$m\frac{\partial \boldsymbol{V}}{\partial t} + \nabla\left[m\frac{V^2}{2} + \frac{5}{2}T\right] = q\left[\boldsymbol{E} + \boldsymbol{V} \times (\boldsymbol{B} + (m/q)\boldsymbol{\nabla} \times \boldsymbol{V})\right] \tag{27}$$

precisely the standard equation of motion for a homontropic fluid with the equation of state $p \propto n^{5/3}$, the nonrelativistic limit of $\sigma =$ const. It is interesting that the gradient forces seem to have been completely subsumed in the definition of the flow tensor $S^{\mu\nu}$.

To give a glimpse of the conceptual as well as calculational potential of this approach which assigns co-primacy to the flow and the electromagnetic field I will present a few consequences.

I will first delineate a procedure for deriving the bilinear constants of motion of the system. It is well known that for the electromagnetic field in vacuum, the helicity

$$h = \int A \cdot B \; d^3x \tag{28}$$

is a constant of the motion; it follows from the general notion in the field theories that the total "charge" $\int K^0 d^3x$ associated with a conserved four-vector

$$\partial_\mu K^\mu = 0 \tag{29}$$

is a constant of the motion, i.e., $(d/dt)\int K^0 d^3x = 0$. It is easy to identify that the four-vector leading to helicity conservation is $A_\mu \mathcal{F}^{\mu\nu}$. Our unification naturally tells us that the equivalent invariants for the hot fluid will be obtained by replacing $\mathcal{F}^{\mu\nu} \to \mathcal{M}^{\mu\nu}$ (the dual of M) and A_μ by $A_\mu + (m/q)fU_\mu$, the minimal coupling peculiar to hot fluids. Thus the vector

$$K^\mu = \left(A_\mu + \frac{m}{q} f U_\mu\right) \mathcal{M}^{\mu\nu} \tag{30}$$

will yield the constant of motion

$$G = \int d^3x \ K^0 \tag{31}$$

where

$$K^0 = (\boldsymbol{A} + (m/q)f\boldsymbol{U}) \cdot (\boldsymbol{B} + (m/q)\nabla \times f\boldsymbol{U}), \tag{32}$$

if

$$\partial_\mu K^\mu = 2\left[\boldsymbol{E} + \frac{m}{q}\boldsymbol{Q}\right] \cdot \left[\boldsymbol{B} + \frac{m}{q}\boldsymbol{R}\right] = 0. \tag{33}$$

This is indeed the case, and can be readily verified by using the vector part of (24).

There thus exists an invariant for each dynamical species of the plasma; there is one for electrons and one for the ions for a two-component electron-ion system. The "helicity" invariant found in Eqs. (30) and (31) is extremely general; it pertains for arbitrary temperatures and flow speeds. To the best of my knowledge, this result has not been derived before although its limiting case for nonrelativistic temperatures ($f = 1$) is well known [3]. Using the conventional techniques, it would take much effort and algebra to derive a result of this generality; the new formalism makes it accessible in a few well-defined steps.

We would like to remark here that for the general isentropic fluid $\partial_\mu K^\mu$ is not zero. Barring the special cases of either uniform temperature or density it is not even a full three divergence. Thus but for the three exceptional cases enumerated earlier, G is not a constant of the motion.

If $m/q \to 0$ for a given species, then the invariant associated with its motion simply reverts to the standard helicity h; this is often done in two-fluid theories where the electron inertia is neglected. In these theories, the ionic motion, does, indeed, create the additional generalized helicity defined by (31) [3].

I will now derive a nonrelativstic result of considerable value and significance by exploiting the unified magnetofluid field. The equation of motion (24) can be broken into its scalar and vector parts as [$\boldsymbol{U} = \gamma \boldsymbol{V}$],

$$\boldsymbol{V} \cdot \hat{\boldsymbol{E}} = 0 \tag{34}$$

$$\hat{\boldsymbol{E}} + \boldsymbol{V} \times \hat{\boldsymbol{B}} = 0 \tag{35}$$

where $\hat{\boldsymbol{E}} = \boldsymbol{E} + (m/q)\boldsymbol{Q}$, and $\hat{\boldsymbol{B}} = \boldsymbol{B} + (m/q)\boldsymbol{R}$ are the effective electric and magnetic fields. The fluid velocity takes the form

$$\boldsymbol{V} \;=\; \hat{\boldsymbol{e}}_{||} V_{||} + \boldsymbol{V}_{\perp} \tag{36}$$

with

$$\boldsymbol{V}_{\perp} \;=\; \frac{\hat{\boldsymbol{E}} \times \hat{\boldsymbol{B}}}{|\hat{\boldsymbol{B}}|^2}. \tag{37}$$

where $\boldsymbol{e}_{||}$ is the unit vector along $\hat{\boldsymbol{B}}$. For a nonrelativistic plasma, i.e, for $V_{||} \ll 1$, and $|\boldsymbol{V}_{\perp}| \ll 1, \hat{\boldsymbol{E}}| \ll |\hat{\boldsymbol{B}}|$. In this approximation, the effective Lagrangian density for the magnetofluid field,

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} M^{\mu\nu} M_{\mu\nu} = -\frac{1}{2}\left[\hat{\boldsymbol{B}}^2 - \hat{\boldsymbol{E}}^2\right], \tag{38}$$

reduces to

$$\mathcal{L}_{\text{eff}}^{\text{N.R}} \simeq -\frac{1}{2}\hat{\boldsymbol{B}}^2 \simeq -\frac{1}{2}\left(\boldsymbol{B} + \frac{m}{q}\nabla \times V)\right)^2, \tag{39}$$

from which it is easy to deduce that the total effective Hamiltonian (in the same approximation) is simply

$$H = \frac{1}{2}\int d^3x \;\; (\boldsymbol{B} + \frac{m}{q}\nabla \times V)^2 \tag{40}$$

i.e., it is proportional to the effective ion-enstrophy or the effective energy of the magnetic field seen by the ions. The association of the ion enstrophy with an effective Hamiltonian strongly fortifies the case for using it as a minimizing functional for the derivation of self-organized magnetofluid states [4].

This formalism and Eq. (24) are likely to provide new insights into the dynamics of relativistic high-temperature charged fluids. It is also hoped this compact and ecompassing formalism may help guide the formulation of a similar theory for the interaction of fluids with nonabelian gauge fields.

The author is grateful to P. Valanju, R. D. Hazeltine, and Z. Yoshida for discussions. This work was supported by the U.S. Dept. of Energy Contract No. DE-FG03-96ER-54346.

[1] See, for example, S. Weinberg, *Gravitation and Cosmology*, (John Wiley and Sons, Inc., New York, 1972).

[2] L. D. Landau and E. M. Lifshitz, *Hydrodynamics*, (Science, Moscow, 1986); D. I. Dzhavakhrishvili and N. L. Tsintsadze, Sov. Phys. JETP **37**, 666 (1973).

[3] V. I. Berezhiani and S. M. Mahajan, Phys. Rev. E **52**, 1968 (1995); L. C. Steinhauer and A. Ishida, Phys. Rev. Lett. **79**, 3423 (1997); S. M. Mahajan and Z. Yoshida, Phys. Rev. Lett. **81**, 4863 (1998).

[4] Z. Yoshida and S. M. Mahajan, Variation Principle and Self-Organization in Plasmas, Phys. Rev. Lett. **88**(9), 095-001-1 (2002).