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STATISTICAL DESCRIPTION OF DRIFT WAVE TURBULENCE

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Abstract

Dissipative drift wave fluctuations are studied with the Terry-Horton nonlinear drift wave model. The $\underset{\sim}{k}\ \omega$ spectral characteristics of the fluctuations are parameterized in terms of the nonlinear frequency $\omega_{\underset{\sim}{k}}$ and line-width $\nu_{\underset{\sim}{k}}$ from computer simulations and the renormalized wave-kinetic equation. The probability distributions of the fluctuations are analyzed to assess the validity of the quasi-normal approximation made in the closure of the hierarchy of correlations in statistical turbulence theory.

I. Introduction and Physical Model

The drift wave instability of the inhomogeneous magnetized plasma and the transition of the nonlinear oscillations to turbulence has been the subject of numerous experimental and theoretical studies in plasma physics. Only recently, however, has it been possible to explain the $\underset{\sim}{k}\omega$ spectral features of the fluctuation spectrum in terms of theoretical models. In this chapter we show how the broad frequency spectra observed in drift wave turbulence can be understood in terms of a chaotic attractor in the phase space of the dissipative nonlinear dynamics. The randomness produced by the chaotic attractor as measured by the probability distributions of the fluctuating fields is shown to be sufficient to suggest that closure of the correlation hierarchy by the quasi-normal approximation is a reasonable first approximation.

The drift wave instability arises from quasi-neutral collective oscillations with $\omega \sim k_y v_{de}$ in magnetized plasmas with density and temperature gradients across the magnetic field.[1] Referring to Fig. 1 which shows the mechanism of the drift wave instability we define the gradient scale lengths by

$$\frac{1}{r_n} = \frac{-1}{N}\frac{dN}{dx} \quad , \quad \frac{1}{r_T} = \frac{-1}{T_e}\frac{dT_e}{dx} \quad \text{and} \quad \eta_e = \frac{r_n}{r_T} \, . \tag{1}$$

Long wave oscillations propagate at the electron diagmagnetic drift velocity v_{de} where $v_{de} = cT_e/eBr_n = (\rho/r_n)c_s$ where $c_s = (T_e/m_i)^{1/2}$, $\rho = c_s/\omega_{ci}$, and $\omega_{ci} = eB/m_i c$.

Since the drift wave oscillations $\omega \simeq k_y v_{de}$ are slow compared with the ion cyclotron gyrations ω_{ci}, the cross-field velocity $\underset{\sim}{v}(\underset{\sim}{x},t)$ of the ions can be derived from the momentum balance equation by an expansion in $1/\omega_{ci}$. In the presence of an electrostatic field $\underset{\sim}{E}(\underset{\sim}{x},t) = -\nabla\phi(\underset{\sim}{x},t)$ the ion velocity is

$$\underset{\sim}{v}(\underset{\sim}{x},t) = \underset{\sim}{v}_E + \underset{\sim}{v}_p + \underset{\sim}{v}_\mu \tag{2}$$

where

$$\begin{aligned} \underset{\sim}{v}_E &= c_s \rho \; \hat{z}\times\nabla\phi \\ \underset{\sim}{v}_p &= -\rho^2\nabla_\perp\left(\frac{\partial\phi}{\partial t} + \underset{\sim}{v}_E\cdot\nabla\phi\right) \\ \underset{\sim}{v}_\mu &= \mu\rho^2\nabla^2\nabla\phi \end{aligned} \tag{3}$$

where in Eq. (3) the potential is in units of T_e/e and $\mu \sim \nu_{ii}$ is the cross-field viscosity coefficient due to ion-ion collisions.[2]

The drift motions of the plasma are quasi-neutral with the equation $\partial_t\rho_Q = -\nabla\cdot\underset{\sim}{j} = 0$ determining the evolution of the potential $\phi(\underset{\sim}{x},t)$. The electron $\underset{\sim}{E}\times\underset{\sim}{B}$ drift cancels the ion $\underset{\sim}{E}\times\underset{\sim}{B}$ current so that quasi-neutrality reduces to $\nabla\cdot[n(\underset{\sim}{v}_p+\underset{\sim}{v}_\mu)] = -\nabla_\parallel(j_\parallel^e/e)$ where $j_\parallel^e$ is the parallel electron current.

The divergence of the parallel electron current and the parallel electron momentum equation

$$\nabla_\parallel(j_\parallel^e/e) = \frac{\partial n}{\partial t} + \underset{\sim}{v}_E\cdot\nabla n$$

$$j_{\parallel}^{e} = \left(\frac{ne^2}{m_e\nu_{ei}}\right)\left(E_{\parallel} + \frac{1}{ne}\nabla_{\parallel} p_e\right) \tag{4}$$

complete the system of equations for $\nu_e > k_{\parallel} v_e$. The remaining details of the derivation are given in Hinton and Horton[2] for the collisional regime $\nu_e > k_{\parallel} v_e$ and in Terry and Horton[3,4] for the collisionless regime $\nu_e < k_{\parallel} v_e$. Finally, all derivatives parallel to the magnetic field are approximated by $\nabla_{\parallel} \sim 1/L_c$ where L_c is the characteristic length along $\vec{B}$ of the system under consideration.

Recently, a kinetic theory derivation of the Terry-Horton equation, and its higher order generalizations, is given by Dubin et al.[5] based on the drift wave ordering of kω.

In terms of the natural dimensionless variables of the dynamics $t[r_n/c_s]$, $(x,y)[\rho]$, and $\phi[\rho/r_n]$ the nonlinear dynamics of the drift wave fluctuations is given by

$$(1+\hat{\mathscr{L}})\frac{\partial\phi}{\partial t}(x,y,t) = -\frac{\partial\phi}{\partial y} - \left[\frac{\partial\phi}{\partial x}\frac{\partial}{\partial y}\hat{\mathscr{L}}\phi - \frac{\partial\phi}{\partial y}\frac{\partial}{\partial y}\hat{\mathscr{L}}\phi\right] - \mu\nabla^4\phi \tag{5}$$

as given earlier by Terry and Horton.[3,4] The linear operator is $\hat{\mathscr{L}} = \mathscr{L}^h + \mathscr{L}^{ah}$ where $\mathscr{L}^h$ and $\mathscr{L}^{ah}$ are the Hermitian and anti-Hermitian parts of $\hat{\mathscr{L}}$, repectively. The Hermitian operator $\mathscr{L}^h$ determines the wave dispersion, and the anti-Hermitian $\mathscr{L}^{ah}$ the dissipation in the oscillations.

The exact forms of $\mathscr{L}^h$ and $\mathscr{L}^{ah}$ depend on the geometry and collisionality regime of the system. In this study we take

$$\mathscr{L} = -\nabla_\perp^2 + \delta\left(\frac{1}{2}\eta_e + \nabla_\perp^2\right)\frac{\partial}{\partial y}. \tag{6}$$

For higher collisionality regimes $\eta_e \to 3\eta_e$ and for the trapped electron regime $\eta_e \to -\eta_e$. From Eqs. (5) and (6) it is evident that η_e controls the dissipation at small $|\underset{\sim}{k}|$ and μ the dissipation at large $|\underset{\sim}{k}|$. For the dissipationless system $\delta = \mu = 0$ the equation becomes Hasegawa-Mima equation[6] or the Rossby-Wave equation[6,8] and has exact 2D solitary wave solutions.[8]

For a plasma system with frozen background gradients (r_n, r_T = const.) we may consider the local two dimensional turbulence in a box of volume $L_x \times L_y$ and write

$$\phi(\underset{\sim}{x},t) = \sum_{\underset{\sim}{k}} \phi_{\underset{\sim}{k}}(t)\exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}) \tag{7}$$

with $\underset{\sim}{k} = (2\pi n/L_x, 2\pi m/L_y)$. Truncating the $\underset{\sim}{k}$ space to $|\underset{\sim}{k}| \leqslant K$ we derive from Eqs. (5), (6), and (7) the dynamics in the truncated $\underset{\sim}{k}$ space

$$(1+\chi_{\underset{\sim}{k}})\frac{d\phi_{\underset{\sim}{k}}}{dt} = -(ik_y+\mu k_\perp^4)\phi_{\underset{\sim}{k}} + \sum_{\underset{\sim}{k}_1+\underset{\sim}{k}_2=\underset{\sim}{k}} \frac{\underset{\sim}{k}_1\times\underset{\sim}{k}_2\cdot\hat{z}}{2}(\chi_{\underset{\sim}{k}_2}-\chi_{\underset{\sim}{k}_1})\phi_{\underset{\sim}{k}_1}\phi_{\underset{\sim}{k}_2} \tag{8}$$

where

$$\chi(\underset{\sim}{k}) = \chi'_{\underset{\sim}{k}} + i\chi''_{\underset{\sim}{k}} = k_\perp^2 + i\delta\, k_y\left(\frac{1}{2}\eta_e - k_\perp^2\right).$$

The linear modes of Eq. (8) are $\omega_0=(k_y-i\mu k_\perp^4)/(1+\chi_{\underset{\sim}{k}})$ which defines the linear frequency and growth-damping rate by

$$\omega^\ell_{\underset{\sim}{k}} = \frac{k_y(1+\chi'_{\underset{\sim}{k}})-\mu k_\perp^4\chi''_{\underset{\sim}{k}}}{(1+\chi'_{\underset{\sim}{k}})^2+(\chi''_{\underset{\sim}{k}})^2} \quad \text{and} \quad \gamma^\ell_{\underset{\sim}{k}} = -\frac{k_y\chi''_k+\mu k_\perp^4(1+\chi'_k)}{(1+\chi'_k)^2+(\chi''_k)^2} \tag{9}$$

The stability parameters of the problem are μ, δ and η_e.

II. Phase Space Definitions and Properties

For a truncation with N independent $\underset{\sim}{k}_\ell$ vectors $\ell=1,2,\ldots N$ (excluding $-\underset{\sim}{k}_\ell$ where $\phi_{-\underset{\sim}{k}}=\phi^*_{\underset{\sim}{k}}$) there are 2N first order differential equations giving a deterministic trajectory for $\underset{\sim}{Y}^{2N}(t) = \{y_i(t)\}_{i=i}^{2N}$ in a particular realization of the system. The most convenient definition of the phase space coordinates $y_i(t)$ is

$$(1+k_{\underset{\sim}{\ell}}^2)^{1/2}\phi_{\underset{\sim}{k}_\ell}(t) = y_{2\ell-1}(t)+i\, y_{2\ell}(t), \qquad \ell=1,2,\ldots N. \tag{10}$$

For an ensemble of systems the associated probability density $\rho^{2N}(\underset{\sim}{Y},t)$ in Γ^{2N} evolves according to the conservation equation is

$$\frac{\partial\rho^{2N}}{\partial t} + \sum_{i=1}^{2N} \dot{y}_i \frac{\partial\rho^{2N}}{\partial y_i} = -\rho^{2N}\sum_{i=1}^{2N}\frac{\partial\dot{y}_i}{\partial y_i}\,.$$

where $\dot{y}_i = dy_i/dt$.

A. Phase Space Metric and Volume Contraction

The physical energy density W in the fluctuation field as a fraction of the electron thermal energy density is given by $W/n_eT_e = \frac{1}{2}\int [\phi^2+\nabla_\perp\phi\cdot\nabla_\perp\phi]\,d\underset{\sim}{x}$, and it is easily shown from Eqs. (5) and (6) that W = const. for $\mu=\delta=0$. In the dissipationless limit there is an additional invariant of the motion[9,10] called the potential enstrophy $U = \frac{1}{2}\int [(\nabla_\perp\phi)^2+(\nabla_\perp^2\phi)^2]\,d\underset{\sim}{x}$ which is also easily shown from Eqs. (5) and (6) to be constant for $\mu = \delta = 0$.

In the phase space of the truncated system the energy becomes

$$E(t) = \frac{1}{2}\sum_{\underset{\sim}{k}} (1+k_\perp^2)\,|\phi_{\underset{\sim}{k}}(t)|^2 \equiv \frac{1}{2}\sum_{i=i}^{2N} y_i^2(t) \tag{11}$$

and the potential enstrophy is

$$U(t) = \frac{1}{2}\sum_{\underset{\sim}{k}} k_\perp^2(1+k_\perp^2)|\phi_{\underset{\sim}{k}}|^2 \equiv \frac{1}{2}\sum_{i=i}^{2N} k_i^2 y_i^2(t). \tag{12}$$

The microcanonical ensemble for the system has entropy $S=k_B\,\ell n\,\Sigma^{2N}(E)$ where $\Sigma^{2N}(E)dE = d\Omega^{2N}(E)$ with the volume of the 2N sphere in phase space is $\Omega^{2N}(E) = \pi^N(2E)^N/N!$. The statistical physics and the evolution to equipartition at constant E and U for the dissipationless system are studied by Montgomery et al.[9,10] The system is ergodic with Gibbs canonical ensemble $\rho^{2N} = \exp(-\alpha W-\beta U)/Z$ predicting the time averaged $|\phi_{\underset{\sim}{k}}|^2$.

For finite values of E the positive definite quadratic form (11) defines the rectangular Cartesian metric for the phase space. The square of the radius vector $\underset{\sim}{Y}(t)$ to the system's phase state is twice the energy of the system.

The volume of the region R in the phase space with this metric is

$$\Omega_R^{2N} = \int_R dy_1 dy_2 \cdots dy_{2N} \tag{13}$$

and the rate of change of the volume defined by the flow in Eq. (8) is

$$\frac{1}{\Omega^{2N}} \frac{d\Omega^{2N}}{dt} = \sum_{i=1}^{2N} \frac{\partial}{\partial y_i}\left(\frac{dy_i}{dt}\right) = 2 \sum_{\underset{\sim}{k}} \gamma_{\underset{\sim}{k}}^{\ell} \equiv 2\gamma_T(\delta, \mu, \eta_e) \tag{14}$$

with $\gamma_{\underset{\sim}{k}}^{\ell}$ given by Eq. (9). Since the $\gamma_{\underset{\sim}{k}}^{\ell}$'s are constants and functions of the stability parameters there is

$$\text{uniform rate of volume contraction} \Leftrightarrow \sum_{\underset{\sim}{k}} \gamma_{\underset{\sim}{k}}^{\ell} < 0.$$

When $\gamma_T \equiv \sum_{\underset{\sim}{k}} \gamma_{k}^{\ell} < 0$ every phase volume $\Omega_R^{2N}(t)$ contracts to zero at $t\to\infty$ according to

$$\Omega_R^{2N}(t) = \Omega_R^{2N}(t_o) \exp[-2\gamma_T(t-t_o)]. \tag{15}$$

Equation (15) shows that after a transient, the steady state fluctuations occur on a lower dimensional d<2N manifold in the phase space. The situation is shown schematically in Fig. 2. We conjecture that d is at most $\approx$ N and may be as small as the number of linearly unstable modes in the system.

B. Ergodic Behavior: Phase and Time Averages

Computer experiments indicate that to a good approximation the time averages of the quantities such as E(t), U(t) and the anomalous flux $\Gamma(t) = \sum_k k_y \chi_k'' |\phi_{\underset{\sim}{k}}(t)|^2$ are equal to the ensemble average over random initial phases. The random phase ensemble has initial data $\phi_{\underset{\sim}{k}}(t_o) = |\phi_{\underset{\sim}{k}}(t_o)| \exp(i\alpha_{\underset{\sim}{k}})$ with $P(\underset{\sim}{\alpha}_k)\, d\underset{\sim}{\alpha}_k = (1/2\pi)^N d\alpha_1 d\alpha_2 \ldots d\alpha_N$.

Due to the phase independence of the physical observables such as energy and flux, it is useful to consider the reduced dynamics in the random-phase-approximation (RPA). We define the reduced state space of N dimensions with positive definite coordinates $I_{\underset{\sim}{k}_\ell}(t) = y^2_{2\ell-1}(t) + y^2_{2\ell}(t) = |\phi_{\underset{\sim}{k}}(t)|^2$. The system energy is $E = \frac{1}{2} \sum_k I_k$. Near the origin of the RPA phase space the volume contaction $\sum_k \partial \dot{I}_k / \partial I_k$ is given by $-2\gamma_T$; however, for large $E = \frac{1}{2} \sum_k I_k$ the rate of change of volume depends on $\{I_\ell\}$ and can be positive or negative.

C. Divergence of Neighboring Trajectories

Computations for the Lyapunov exponents for neighboring trajectories shows that the largest Lyapunov exponent λ is positive and typically of order unity in the saturated state. Details of the calculation for a three wave problem in a 4D phase space are given in Ref. 3 and also apply here.

The exponential divergence of neighboring orbits accounts for the intrinsic stochasticity observed in the time signals. In the presence of an attracting region of phase space, the exponential divergence of neighboring trajectories produces the well-defined fluctuating steady states with properties independent of the initial data $\underset{\sim}{Y}(t_0)$.

III. Characteristics of the Fluctuation Spectrum

With extensive computer experiments we have studied the characteristics of the fluctuation spectrum. Saturation or a state with bounded fluctuations in $E(t)$ occurs when the average growth-damping rate per mode $\langle\gamma\rangle = \gamma_T/N = \frac{1}{N}\sum_{i=1}^{N} \gamma^{\ell}_{k_i}$ is slightly negative. For some $\gamma^{\ell}_{k}>0$ and $\langle\gamma\rangle < 0$ the origin $\|\underset{\sim}{Y}\| \to 0$ and $\|\underset{\sim}{Y}\| \to \infty$ are repelling regions of phase space and there appears a chaotic attracting region at finite $\|Y\|$.

We find that the basin of attraction to the chaotic attractor is large by observing that for numerous experiments with initial data of the form

$$\{\phi_{\underset{\sim}{k}_\ell}(t_o)\} = \{10^n e^{i\alpha_\ell}\} \quad \text{with } -3 \leq n \leq +2 \quad \text{and} \quad \text{random } \alpha_\ell$$

appear to evolve to the same chaotic attractor as identified by the equivalence of the time averaged $\underset{\sim}{k}$ spectrum of E, U and Γ.

That the condition $\langle\gamma\rangle = \gamma_T/N < 0$ is approximately a necessary and sufficient condition for saturation follows from the conservation of energy by the nonlinear coupling and the tendency of the system to evolve to equipartition for $\gamma^{\ell}_{\underset{\sim}{k}}=0$. The equipartition tendency is checked with computer experiments by switching off $\gamma^{\ell}_{\underset{\sim}{k}}$ for $t > t_1$ in a saturated nonlinear state.

The total energy $E(t>t_1)$ is constant but the spectrum $E(\underset{\sim}{k},t)$ evolves to the distribution $E(\underset{\sim}{k},\ t\to\infty) = \text{const.} = E(t_1)/N$ as shown in Figs. 5 and 6 of Ref. 4 for $N = 10$. The result that $E(\underset{\sim}{k},t)$ evolves to equipartition is also a well known property of the wave-kinetic equation for $\partial_t I_{\underset{\sim}{k}}$ in the RPA.

To calculate the $dE(t)/dt$ from Eq. (8) we note that for every triplet $\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3=0$ $(\underset{\sim}{k}=-\underset{\sim}{k}_3)$ interaction the nonlinear contribution to $\partial_t^{n\ell}(E_{k_1}+E_{k_2}+E_{k_3})$ is proportional to $\phi_{k_1}\phi_{k_2}\phi_{k_3}$ times

$$(\underset{\sim}{k}_1\times\underset{\sim}{k}_2\cdot\hat{z})(\chi_{k_2}-\chi_{k_1})+(\underset{\sim}{k}_2\times\underset{\sim}{k}_3\cdot\hat{z})(\chi_{k_3}-\chi_{k_2})+(\underset{\sim}{k}_3\times\underset{\sim}{k}_1\cdot\hat{z})(\chi_{k_1}-\chi_{k_3}) \equiv 0.$$

Thus, the rate of change of $E(t)$ is given by

$$\frac{dE}{dt} = \sum_{\underset{\sim}{k}} 2\gamma^{\ell}_{\underset{\sim}{k}}E_{\underset{\sim}{k}}(t) = \Big(2\sum_{k}\gamma^{\ell}_{\underset{\sim}{k}}\Big)\,E^*(t) \qquad (16)$$

where the mean-value theorem is used in Eq. (16) to evaluate $E_{\underset{\sim}{k}}(t)$ at a mean-value $E^*(t)$ over the spectrum. Due to energy conservation the rate of change $\partial_t E$ is bounded $2\gamma_{min}E \leq dE/dt \leq 2\gamma_{max}E$ where the limiting values occur when the energy spectrum is concentrated on the $\underset{\sim}{k}$ with γ_{min} or γ_{max}. Suppose the system point tends to infinity $\|Y(t)\| \to \infty$, then the nonlinear transfer dominates and the system evolves to equipartition of the modal energy. For equipartition the mean value of E^* becomes

$$E^* = \frac{1}{N}E = \frac{1}{N}\sum_{\ell=1}^{N} E_{k_\ell} \quad \text{for } \|\underset{\sim}{Y}\| \to \infty\ . \qquad (17)$$

Substituting Eq. (17) in Eq. (18) shows that for sufficiently large $\|\underset{\sim}{Y}\|$ the total energy decays

$$\frac{dE}{dt} = \frac{2}{N}\sum_{\underset{\sim}{k}} \gamma^{\ell}_{\underset{\sim}{k}} E(t) = 2\langle\gamma\rangle E(t) \tag{19}$$

provided $\langle\dot{\gamma}\rangle < 0$.

A typical simulation experiment satisfying $\langle\gamma\rangle < 0$ is shown in Fig. 3. The parameters are $\delta = 1.0$, $\mu = 0.2$ and $\eta_e = 0.5$ with $\Delta k = 2\pi/L = 0.14$ on a 15×15 grid, and thus there are $2N = 224$ first order equations for $\underset{\sim}{Y}(t)$. For these parameters $\gamma_{min} = -.0941$, $\gamma_{max} = .0834$ and $\langle\gamma\rangle = -.012$.

The center of Fig. 3 shows the time averaged energy spectrum $E(\underset{\sim}{k})$. The guide lines to the four surrounding graphs show the time signals $\phi_{\underset{\sim}{k}}(t)$ and the frequency spectra $|\phi_{\underset{\sim}{k}}(\omega)|$ of four typical modes from the 112 in $\{E_{\underset{\sim}{k}}\}$. The frequency spectrum $|\phi_{\underset{\sim}{k}}(\omega)|$ is computed from

$$\phi_{\underset{\sim}{k}}(\omega) = \frac{1}{T}\int_{t_o}^{t_o+T} dt\, \phi_{\underset{\sim}{k}}(t)e^{i\omega t} = \frac{1}{M}\sum_{m=0}^{M-1} \phi_{\underset{\sim}{k}}(t_o+m\Delta t)\exp\left(\frac{i2\pi m\Delta t}{T}\right) \tag{20}$$

where $T = 1200[r_n/c_s]$ and $M = 1024$. A typical correlation function $\langle\phi_{\underset{\sim}{k}}(t)\phi_{\underset{\sim}{k}}(t+\tau)\rangle$ for one of these modes is shown in Fig. 4 and has $\tau_{1/2} \simeq 5[r_n/c_s]$. The saturated amplitude of the turbulence follows from Fig. 3 with $y_{rms} = (2E)^{1/2} = 2.6$ or $E = 3.38$.

As reported in earlier studies[3,4] the time series for each mode contains a wide range of frequencies due to the chaotic attractor. For each mode we parameterize the frequency spectrum by the characteristic nonlinear frequency $\omega_{\underset{\sim}{k}}$, line width $\nu_{\underset{\sim}{k}}$ and the frequency integrated spectral intensity $I_{\underset{\sim}{k}}$. These are the quantities defined in renormalized turbulence theory, e.g. Horton and Choi.[11,12] Two alternative parameterizations have been studied: (1) the gaussian frequency distribution $G_{\nu_k}(\omega-\omega_k) = I_k(2\pi/\nu_k)^{-1/2}\exp[-(\omega-\omega_{\underset{\sim}{k}})^2/2\nu_k^2]$ and (2) the Lorentzian parameterization

$$L_{\nu_{\underset{\sim}{k}}}(\omega-\omega_{\underset{\sim}{k}}) = I_{\underset{\sim}{k}} \frac{2\nu_{\underset{\sim}{k}}}{(\omega-\omega_{\underset{\sim}{k}})^2+\nu_{\underset{\sim}{k}}^2} . \tag{21}$$

where $(2\pi)^{-1}\int d\omega\, G_{\nu_k}(\omega) = (2\pi)^{-1}\int d\omega\, L_{\nu_k}(\omega) = I_{\underset{\sim}{k}}$. Generally, we find that the Lorentzian, or perhaps the square of a Lorentzian, represents the frequency spectrum better than the Gaussian.

The four frequency spectra in Fig. 3 show both the observed values $\omega_{\underset{\sim}{k}}$, $\nu_{\underset{\sim}{k}}$ derived from the nonlinear regression fit to the Lorentzian (21) and the linear frequency and growth-damping rate $\omega_{\underset{\sim}{k}}^{\ell}$, $\gamma_{\underset{\sim}{k}}^{\ell}$ from Eq. (9). Weak turbulence theory[13] is based on $\omega_{\underset{\sim}{k}}^{\ell}$, $\gamma_{\underset{\sim}{k}}^{\ell}$ and renormalized turbulence theory[11,12,14,15] gives $\omega_{\underset{\sim}{k}}$, $\gamma_{\underset{\sim}{k}}$ from $\omega_{\underset{\sim}{k}}^{\ell}$, $\gamma_{\underset{\sim}{k}}^{\ell}$ and $I_{\underset{\sim}{k}}$.

Finally, to assess the applicability of the basic assumption in the statistical theories of plasma turbulence we introduce several measures of the correlation functions and the statistics of the interacting modes. The principal assumption common to different forms of statistical turbulence theory is that the statistics of the modes $\phi_{\underset{\sim}{k}}(t) = y_{2\ell-1}+iy_{2\ell}$ are near to those of a gaussian or a normal probability distribution. It is this assumption, expressed in various forms, that allows the hierarchy of multi-field correlations $\langle\phi_1\phi_2\cdots\phi_k\rangle$ to be closed in terms of a few low order moments.

As a typical example we consider the closure of the hierarchy for the quadratic problem $\varepsilon_k\phi_k + \sum_{k_1}\varepsilon^{(2)}_{k_1,k_2}\phi_{k_1}\phi_{k_2} = 0$ where $k_2 = k - k_1$. Assuming that the skewness or three-field correlations are weak, one calculates perturbatively the correlations

$$\langle\phi^*_k\phi_{k_1}\phi_{k-k_1}\rangle = \langle\phi^{(1)*}_{\underset{\sim}{k}}\phi^{(0)}_{\underset{\sim}{k}_1}\phi^{(0)}_{\underset{\sim}{k}-\underset{\sim}{k}_1}\rangle + \langle\phi^{(0)*}_{\underset{\sim}{k}}\phi^{(1)}_{\underset{\sim}{k}_1}\phi^{(0)}_{\underset{\sim}{k}-\underset{\sim}{k}_1}\rangle + \langle\phi^{(0)*}_{\underset{\sim}{k}}\phi^{(0)}_{\underset{\sim}{k}_1}\phi^{(1)}_{\underset{\sim}{k}-\underset{\sim}{k}_1}\rangle \tag{22}$$

and closes the four field correlation function

$$\langle\phi^*_k\phi_{k_1}\phi_{k_2}\phi_{k_3}\rangle = \langle\phi_k{}^*\phi_{k_1}\rangle\langle\phi_{k_2}\phi_{k_3}\rangle + \langle\phi^*{}_k\phi_{k_2}\rangle\langle\phi_{k_1}\phi_{k_3}\rangle + \langle\phi_k{}^*\phi_{k_3}\rangle\langle\phi_{k_1}\phi_{k_2}\rangle \,. \tag{23}$$

assuming gaussian statistics at this order. These assumptions are often stated as the "quasi-normal approximation" or the "assumption of maximal randomness".

For an individual mode $\phi_{\underset{\sim}{k}}(t) = y_{2\ell-1}(t) + iy_{2\ell}(t)$ we compute the probability distribution from the cumulative distribution of $\{y_\ell(t+m\Delta t), m=1,2...M\}$. A typical result is shown in Fig. 5 for $M=10^3$ sample points and 50 intervals. The distribution $P(y_\ell)$ has the first four moments given in the Fig. 5. From this and other samples we conclude that the experiments show that for the mean and the variance are $\langle y_\ell\rangle \simeq \langle y_{2\ell-1}\rangle \simeq 10^{-3} - 10^{-2}$ and $\langle y^2_{2\ell-1}\rangle \simeq 10^{-2} - 10^{-1}$ for typical modes and that the measures of non-gaussianity are typically

$C^3_\ell = \langle y^3_\ell\rangle/\langle y^2_\ell\rangle^{3/2} \sim$ few tenths

$$E_\ell = \langle y_\ell^4\rangle/\langle y_\ell^2\rangle^2 - 3 \sim \text{few tenths.} \tag{24}$$

In terms of the amplitude $a_\ell = (y_{2\ell-1}^2 + y_{2\ell}^2)^{1/2}$ and phase $\theta_\ell = \tan^{-1}(y_{2\ell}/y_{2\ell-1})$ the measured probability distributions are $P(\theta) = 1/2\pi$ and $P(a) = a\exp[-a^2/2\langle y^2\rangle]/\langle y^2\rangle$. Time sample points plotted in the $y_{2\ell-1}$-$y_{2\ell}$ phase plane of oscillator $\underset{\sim}{k}_\ell$ give a gaussian cloud.

We conclude that the non-gaussianity is a sub-dominant, although non-negligible, feature of the fluctuations in a given mode -- at least for the experiments with $\delta \sim \eta_e \sim 1$ and $\mu << 1$. We suspect that this implies that for the parameters of this experiment, statistical turbulence theories based on small C_ℓ^3 and C_ℓ^4 should yield qualitatively, but not quantitatively, correct predictions for quantities such as $E(\underset{\sim}{k})$ and $\nu(\underset{\sim}{k})$.

The statistics of the total signal $\phi(\underset{\sim}{x},t) = \sum_{\underset{\sim}{k}} \phi_{\underset{\sim}{k}}(t) \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x})$ are also considered at a given space point $\underset{\sim}{x}$. To the extent that the modes $\phi_{\underset{\sim}{k}}(t)$ are statistically independent we would expect that the skewness of $\phi(\underset{\sim}{x},t)$ would decrease as $C_\phi^3 \sim NS_\ell/N^{3/2} \sim S_\ell/N^{1/2}$ and that $C_\phi^4 \sim C_\ell^4/N$ by the central limit theorem. We observe from the experiments, however, that C_ϕ^3 and C_ϕ^4 are rather comparable to C_ℓ^3 and C_ℓ^4 indicating that there are strong, or at least significant, correlations among the $\{\phi_{\underset{\sim}{k}}\}$ in the experiment. We have not been able to test the scaling with N above $N \sim 224$ at this time.

Finally, as another measure of the correlations we consider the spatial average $\langle\ \rangle_x \equiv (L_xL_y)^{-1}\int dx\,dy$ of $\phi^3(\underset{\sim}{x},t)$. The average of ϕ^3 is

$$\langle\phi^3(\underset{\sim}{x},t)\rangle_x = \sum_{\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3=0} \phi_{\underset{\sim}{k}_1}(t)\phi_{\underset{\sim}{k}_2}(t)\phi_{\underset{\sim}{k}_3}(t)$$

$$= \sum_{\underset{\sim}{k}} \phi^*_{\underset{\sim}{k}}(t) \sum_{k_1} \phi_{\underset{\sim}{k}_1}(t)\, \phi_{\underset{\sim}{k}-\underset{\sim}{k}_1}(t)$$

which, relative to $\langle\phi^2\rangle_x^{3/2}$, is defined as

$$T(t) = \frac{\langle\phi^3\rangle_x}{\langle\phi^2\rangle_x^{3/2}} = \frac{\sum_{\underset{\sim}{k}} \phi^*_{\underset{\sim}{k}} \sum_{k_1} \phi_{\underset{\sim}{k}_1} \phi_{\underset{\sim}{k}-\underset{\sim}{k}_1}}{\left(\sum_{\underset{\sim}{k}} |\phi_{\underset{\sim}{k}}|^2\right)^{3/2}} \quad . \tag{25}$$

The mean value of T(t) is approximately zero and its root-mean-square value is $\langle T^2\rangle_t^{1/2} = 0.17$. Again the finite value of T(t) suggest that sub-dominant but significant correlations are present in the system.

IV. Renormalized Wave Kinetic Equation

Various statistical theories of plasma turbulence have been developed in works too numerous to reference in detail here. The works develop reduced turbulence equations from the primitive equations at various levels of rigor or with _ad hoc_ phenomenological assumptions. In each case, however, the underlying assumption is that the turbulent fields are sufficiently random that only a few lower order moments of the fluctuations are independent. In such theories the hierarchy of correlations that arise from the nonlinearity of the equations are truncated by neglecting higher order intrinsic or irreducible correlations. In the usual procedure the fourth order cumulant (approximation (23)) is neglected in the _renormalized_ perturbation expansion and the three field correlations are computed perturbatively (approximation (22)). A recent review of the various theoretical formulations is given by Krommes.[13]

The numerical experiments presented in Sec. III suggest that the dissipative drift wave turbulence may be a problem for which the principal assumption of statistical turbulence theory is satisfied. In this section we apply turbulence theory to this problem with the objective of testing its prediction with the results of the simulation experiments. In $\underset{\sim}{k}\,\omega$ space the mode coupling equation for Eq. (8) is

$$(1+k_\perp^2)(\omega-\omega_{\underset{\sim}{k}}^\ell-i\gamma_{\underset{\sim}{k}}^\ell)\phi_k - \sum_{k_1+k_2=k} \frac{i\underset{\sim}{k}_1\times\underset{\sim}{k}_2\cdot\hat{z}}{2}(\chi_{\underset{\sim}{k}_2}-\chi_{\underset{\sim}{k}_1})\phi_{k_1}\phi_{k_2}=0 \qquad (26)$$

where the short-hand notation is $k=(\underset{\sim}{k}_1,\omega)$ and

$$\sum_{k_1+k_2=k} = (2\pi)^{-2}\int d\omega_1\, d\omega_2\, \delta(\omega-\omega_1-\omega_2)\sum_{\underset{\sim}{k}_1,\underset{\sim}{k}_2}\delta_{\underset{\sim}{k}_1+\underset{\sim}{k}_2,\underset{\sim}{k}}$$

are introduced.

The mode coupling Eq. (26) is of the form

$$\varepsilon_k^\ell\,\phi_k + \sum_{k_1+k_2=k}\varepsilon_{k_1,k_2}^{(2)}\,\phi_{k_1}\phi_{k_2} = 0$$

which has been studied extensively in the plasma physics literature. The weak turbulence equation are given by Galeev and Sagdeev[13] in Eqs. (1.63) and (3.9) .

The renormalized turbulence equations are given by Kadomtsev[15] in Eqs. (III.6) and (III.7); in Ref. 11 in Eqs. (3.40) and (3.41); in Ref.12 in Eqs. (14) and (15);in Ref. 14 in Eqs. (118) and (123) ; in Ref. 16 in Eqs. (2.5) and (2.18) or (2.28); and in Ref.17 in Eqs. (2.121) and 2.113). The basic result of these works is two reduced equations for the unknown nonlinear response function $\varepsilon_{\underset{\sim}{k}}(\omega)$ and the spectral distribution $I_{\underset{\sim}{k}}(\omega)$ of the potential ϕ_k fluctuation spectrum in terms of the linear response function ε^{ℓ}_{k} and the coupling function $\varepsilon^{(2)}_{k_1,k_2}$.

The two turbulence theory equations are

$$\varepsilon_{\underset{\sim}{k}}(\omega) = \varepsilon^{\ell}_{\underset{\sim}{k}}(\omega) - \sum_{\underset{\sim}{k}_1} \int_{-\infty}^{+\infty} \frac{d\omega_1}{2\pi} \frac{(\underset{\sim}{k}\times\underset{\sim}{k}_1\cdot\hat{z})^2(\chi_{\underset{\sim}{k}-\underset{\sim}{k}_1}-\chi_{\underset{\sim}{k}_1})(\chi_{-\underset{\sim}{k}_1}-\chi_{\underset{\sim}{k}})I(\underset{\sim}{k}_1,\omega_1)}{\varepsilon_{\underset{\sim}{k}-\underset{\sim}{k}_1}(\omega-\omega_1)} \tag{27}$$

$$|\varepsilon_{\underset{\sim}{k}}(\omega)|^2 I(\underset{\sim}{k},\omega) = \frac{1}{2}\sum_{\underset{\sim}{k}_1} \int_{-\infty}^{+\infty} \frac{d\omega_1}{2\pi} (\underset{\sim}{k}\times\underset{\sim}{k}_1\cdot\hat{z})^2 \, |\chi_{\underset{\sim}{k}-\underset{\sim}{k}_1}-\chi_{\underset{\sim}{k}_1}|^2 \, I(\underset{\sim}{k}_1,\omega_1)I(\underset{\sim}{k}-\underset{\sim}{k}_1,\, \omega-\omega_1) \tag{28}$$

Equations (27) and (28) determine I_k and ε_k. The equations are non-Markovian depending on the time history of the fluctuations. Physically, Eqs. (27) and (28) follow from the selective summation to all orders of the most secular contributions in the small ϕ_k perturbation expansion of the mode coupling equation. These equations are often called the DIA (Direct Interaction Approximation) equation from the early fluid turbulence work of Kraichnan.[18]

A. Markovian Reduction

The simulation study in Sec. III shows that the turbulent fluctuations have a short correlation time $\tau_c = 1/\nu_k$ due to the chaotic attractor. To describe the rapid fluctuations and the nonlinear evolution of the fields we introduce the relative time $\tau = t-t'$ and the centered time $T = \frac{1}{2}(t+t')$ variables in the two-time correlation function

$$\langle \phi^*_{\underset{\sim}{k}}(t)\phi_{\underset{\sim}{k}}(t')\rangle = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, I(\underset{\sim}{k},\omega,T)\, \exp(-i\omega\tau) \qquad (29)$$

and observe that

$$\omega_k \sim \nu_k \sim \frac{1}{\tau_c(k)} \gg \frac{d}{dT} \;. \qquad (30)$$

from Figs.)3 and 4. Thus, we consider approximate solutions of Eqs. (27) and (28) of the form

$$\varepsilon_{\underset{\sim}{k}}(\omega) = (1 + k_\perp^2)\,(\omega - \omega_{\underset{\sim}{k}} + i\nu_{\underset{\sim}{k}}) \qquad (31)$$

$$I_{\underset{\sim}{k}}(\omega) = I(\underset{\sim}{k})\,\frac{2\nu_{\underset{\sim}{k}}}{(\omega-\omega_{\underset{\sim}{k}})^2 + \nu_{\underset{\sim}{k}}^2} \qquad (32)$$

with $\omega_{\underset{\sim}{k}}$, $\nu_{\underset{\sim}{k}}$, $I(\underset{\sim}{k})$ having a local dependence on the slow time variable T. This ω parameterization is called the "pole approximation" by Dubois and Rose[19] who study its validity for the Zakaharov fluid model of coupled Langmuir-ion acoustic turbulence.

Eqs. (31) and (32) are then a particularly simple frequency parameterization of the unknown functions $\varepsilon_{\underset{\sim}{k}}(\omega)$ and $I_{\underset{\sim}{k}}(\omega)$. In demanding that approximations (31) and (32) satisfy Eqs. (27) and (28) to the dominant order we obtain three reduced Markovian equations for the unknowns $\omega_{\underset{\sim}{k}}(T)$, $\nu_{\underset{\sim}{k}}(T)$ and $I_{\underset{\sim}{k}}(T)$.

For the Lorentzian frequency distribution $L_{\nu_k}(\omega-\omega_{\underset{\sim}{k}})$ defined in Eq. (21) the ω_1 integrals in Eqs. (27) and (28) can be performed by contour integration to obtain

$$\int_{-\infty}^{+\infty} \frac{d\omega_1}{2\pi} \frac{L_{\nu_{\underset{\sim}{k}_1}}(\omega_1-\omega_{\underset{\sim}{k}_1})}{\omega-\omega_1-\omega_{\underset{\sim}{k}_2}+i\nu_{\underset{\sim}{k}_2}} = \frac{1}{\omega-\omega_{\underset{\sim}{k}_1}-\omega_{\underset{\sim}{k}_2}+i(\nu_{\underset{\sim}{k}_1}+\nu_{\underset{\sim}{k}_2})} \tag{33}$$

and

$$\int_{-\infty}^{+\infty} \frac{d\omega_1}{2\pi} L_{\nu_{\underset{\sim}{k}_1}}(\omega_1-\omega_{\underset{\sim}{k}_1}) L_{\nu_{\underset{\sim}{k}_2}}(\omega-\omega_1-\omega_{\underset{\sim}{k}_2}) = L_{\nu_{\underset{\sim}{k}_1}+\nu_{\underset{\sim}{k}_2}}(\omega-\omega_{\underset{\sim}{k}_1}-\omega_{\underset{\sim}{k}_2}) \tag{34}$$

We substitute Eqs. (31), (32), (33) and (34) into Eqs. (27) and (28) and evaluate the right-hand members at $\omega=\omega_{\underset{\sim}{k}}+i\nu_{\underset{\sim}{k}}$ the frequency about which the equation is peaked. The resulting equations for $\omega_{\underset{\sim}{k}}$, $\nu_{\underset{\sim}{k}}$ and $I_{\underset{\sim}{k}}$ are

$$\omega_{\underset{\sim}{k}} = \omega_{\underset{\sim}{k}}^{\ell} + \mathrm{Re}\sum_{\underset{\sim}{k}_1} \frac{(\underset{\sim}{k}\times\underset{\sim}{k}_1\cdot\hat{z})^2(\chi_{\underset{\sim}{k}-\underset{\sim}{k}_1}-\chi_{\underset{\sim}{k}_1})(\chi_{\underset{\sim}{k}}-\chi_{-\underset{\sim}{k}_1})I_{\underset{\sim}{k}_1}}{(1+k^2)(1+k_2^2)[\omega_{\underset{\sim}{k}}-\omega_{\underset{\sim}{k}_1}-\omega_{\underset{\sim}{k}_2}+i(\nu_{\underset{\sim}{k}}+\nu_{\underset{\sim}{k}_1}+\nu_{\underset{\sim}{k}_2})]} \tag{35}$$

$$\nu_{\underset{\sim}{k}} = -\gamma^{\ell}_{\underset{\sim}{k}} - \text{Im} \sum_{\underset{\sim}{k}_1} \frac{(\underset{\sim}{k}\times\underset{\sim}{k}_1\cdot\hat{z})^2(\chi_{\underset{\sim}{k}_2}-\chi_{\underset{\sim}{k}_1})(\chi_{\underset{\sim}{k}}-\chi_{-\underset{\sim}{k}_1})I_{\underset{\sim}{k}_1}}{(1+k^2)(1+k_2^2)[\omega_{\underset{\sim}{k}}-\omega_{\underset{\sim}{k}_1}-\omega_{\underset{\sim}{k}_2}+i(\nu_{\underset{\sim}{k}}+\nu_{\underset{\sim}{k}_1}+\nu_{\underset{\sim}{k}_2})]} \tag{36}$$

and

$$2\nu_{\underset{\sim}{k}}(1+k^2)^2 I_{\underset{\sim}{k}} = -\text{Im} \sum_{\underset{\sim}{k}_1} \frac{(\underset{\sim}{k}\times\underset{\sim}{k}_1\cdot\hat{z})^2|\chi_{\underset{\sim}{k}_1}-\chi_{\underset{\sim}{k}_2}|^2 I_{\underset{\sim}{k}_1} I_{\underset{\sim}{k}_2}}{\omega_{\underset{\sim}{k}}-\omega_{\underset{\sim}{k}_1}-\omega_{\underset{\sim}{k}_2}+i(\nu_{\underset{\sim}{k}}+\nu_{\underset{\sim}{k}_1}+\nu_{\underset{\sim}{k}_2})} \tag{37}$$

where $\underset{\sim}{k}_2$ is short-hand for $\underset{\sim}{k}-\underset{\sim}{k}_1 = \underset{\sim}{k}_2$ under the $\underset{\sim}{k}_1$ summation.

Equations (35), (36) and (37) describe the steady state turbulence. The evaluation and stability of the system on the slow time variable $T = \frac{1}{2}(t+t')$ defined in Eq. (29) is given by the evaluation of the frequency integral of $\text{Im}\,\varepsilon_k\, I_k$. We introduce the phase averaged modal energy from Eq. (11)

$$E(\underset{\sim}{k}, t = T) = (1+k_\perp^2) I(\underset{\sim}{k}, T) \tag{38}$$

and define the response function

$$R_{\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}_2} = \frac{1}{\omega_{\underset{\sim}{k}}-\omega_{\underset{\sim}{k}_1}-\omega_{\underset{\sim}{k}_2} + i(\nu_{\underset{\sim}{k}}+\nu_{\underset{\sim}{k}_1}+\nu_{\underset{\sim}{k}_2})} = \frac{1}{-\omega_{\underset{\sim}{k}_1}-\omega_{\underset{\sim}{k}_2}-\omega_{\underset{\sim}{k}_3}+i(\nu_{\underset{\sim}{k}_1}+\nu_{\underset{\sim}{k}_2}+\nu_{\underset{\sim}{k}_3})} \tag{39}$$

which is symmetric in $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$, $\underset{\sim}{k}_3$ where $\underset{\sim}{k}_3 = -\underset{\sim}{k}$.

From Eq. (37) the steady state balance occurs when the source $S_{\underset{\sim}{k}}$ function balances the decay rate $\nu_{\underset{\sim}{k}}$

$$2\nu_{\underset{\sim}{k}} E_{\underset{\sim}{k}} = S_{\underset{\sim}{k}} .$$

where

$$S_{\underset{\sim}{k}} = -\mathrm{Im} \sum_{\underset{\sim}{k}_1} \frac{(\underset{\sim}{k}\times\underset{\sim}{k}_1\cdot\hat{z})^2 |\chi_{\underset{\sim}{k}_2}-\chi_{\underset{\sim}{k}_1}|^2 R_{\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}_2} E_{\underset{\sim}{k}_1} E_{\underset{\sim}{k}_2}}{(1+k^2)(1+k_1^2)(1+k_2^2)} \tag{40}$$

where $\nu_{\underset{\sim}{k}}$ is given by Eq. (36). Out of equilibrium, the slow time scale evolution of the system is given by

$$\frac{dE_{\underset{\sim}{k}}}{dt} = -2\nu_{\underset{\sim}{k}} E_{\underset{\sim}{k}} + S_{\underset{\sim}{k}}$$

$$= 2\gamma^{\ell}_{\underset{\sim}{k}} E_{\underset{\sim}{k}} + C^{n\ell}_{\underset{\sim}{k}} (\{E_{\underset{\sim}{k}}\}) \tag{41}$$

where the nonlinear modal interaction or "collision" operator $C^{n\ell}_{\underset{\sim}{k}}(\{E_{\underset{\sim}{k}}\})$ is given by

$$C^{n\ell}_{\underset{\sim}{k}}(\{E_{\underset{\sim}{k}}\}) = \mathrm{Im} \sum_{\underset{\sim}{k}_1} \frac{(\underset{\sim}{k}\times\underset{\sim}{k}_1\cdot\hat{z})^2 (\chi_{\underset{\sim}{k}_2}-\chi_{\underset{\sim}{k}_1})}{(1+k^2)(1+k_1^2)(1+k_2^2)} R_{\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}_2}$$

(eq. continued next page)

$$[-(\chi^*_{\underset{\sim}{k}_2}-\chi^*_{\underset{\sim}{k}_1})E_{\underset{\sim}{k}_1}E_{\underset{\sim}{k}_2} + (\chi_{\underset{\sim}{k}}-\chi^*_{\underset{\sim}{k}_1})E_{\underset{\sim}{k}_1}E_{\underset{\sim}{k}} - (\chi_{\underset{\sim}{k}}-\chi^*_{\underset{\sim}{k}_2})E_{\underset{\sim}{k}_2}E_{\underset{\sim}{k}}] \qquad (42)$$

where we have used Eqs. (36) and (40) in Eq. (41) to write out $C^{n\ell}_{\underset{\sim}{k}}$.

Equation (41) is a Markovian description of the phase averaged spectral dynamics where the evolution $C^{n\ell}_{\underset{\sim}{k}}$ depends on the local nonlinear frequency $\omega_{\underset{\sim}{k}}(T)$ and decay rate $\nu_{\underset{\sim}{k}}(T)$. Equations (35), (36) and (41) are called the renormalized wave kinetic equation.

B. Properties of the Renormalized Wave Kinetic Equation

The renormalized wave kinetic equation describes the nonlinear system in an N dimensional state space with the following properties:

Equipartition of Modal Energy

The equilibrium solution of the undriven problem $\gamma^{\ell}_{\underset{\sim}{k}} = 0$ is

$$E_{\underset{\sim}{k}}(t) = \frac{1}{N} E_T = \text{const.} \qquad (43)$$

The equipartition solution is a stable fixed point in the N dimensional state space with $\dot{E}_{\underset{\sim}{k}} = C^{n\ell}(\{E_{\underset{\sim}{k}}\}) = 0$. The stability of the fixed point (43) can be shown by introducing the entropy production functional $\sigma(\{E_{\underset{\sim}{k}}\}) \geqslant 0$ as the Lyapunov stability functional and showing that all perturbations from (43) result in $d\sigma/dt < 0$.

Conservation of Energy in the Nonlinear Transfer

The nonlinear interaction $C_{\underset{\sim}{k}}^{n\ell}(\{E_{\underset{\sim}{k}}\})$ acts on each triplet in such a way as to conserve wave energy. Consider the summation (with $\underset{\sim}{k}_3 = -\underset{\sim}{k}$) of Eq. (42)

$$\sum_{\underset{\sim}{k}} C_{\underset{\sim}{k}}^{n\ell}(\{E_{\underset{\sim}{k}}\}) = \mathrm{Im} \sum_{\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3=0} \frac{(\underset{\sim}{k}_1\times\underset{\sim}{k}_2\cdot\hat{z})^2 R_{1,2,3}}{(1+k_1^2)(1+k_2^2)(1+k_3^2)}$$

$$\times \left[-(\chi_2-\chi_1)(\chi_2^*-\chi_1^*)E_1E_2+(\chi_2-\chi_1)(\chi_3^*-\chi_1)E_1E_3-(\chi_2-\chi_1)(\chi_3^*-\chi_2)E_2E_3\right] .$$

Interchanging 3 and 2 in the second term and 3 and 1 in the third term and using the symmetry of $R_{123} = (-\sum_i \omega_{\underset{\sim}{k}_i} + i\sum_i \nu_{k_i})^{-1}$ we obtain that

$$\sum_{\underset{\sim}{k}} C_{\underset{\sim}{k}}^{n\ell}(\{E_{\underset{\sim}{k}}\}) = \mathrm{Im} \sum_{\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3=0} \frac{(\underset{\sim}{k}_1\times\underset{\sim}{k}_2\cdot\hat{z})^2 R_{1,2,3}}{(1+k_1^2)(1+k_2^2)(1+k_3^2)}$$

$$\times E_1E_2(\chi_2^*-\chi_1^*)[-\chi_2+\chi_1+\chi_3-\chi_1+\chi_2-\chi_3] = 0.$$

If $\chi_{\underset{\sim}{k}}$ is purely real (or purely imaginary) then we can repeat the analysis for $\sum_{\underset{\sim}{k}} \chi'_{\underset{\sim}{k}} C_{\underset{\sim}{k}}^{n\ell}$ (or $\sum_{\underset{\sim}{k}} \chi''_{\underset{\sim}{k}} C_{\underset{\sim}{k}}^{n\ell}$) to show that the summation reduces to

$$E_1E_2(\chi_2-\chi_1)[-\chi_3(\chi_2-\chi_1)+\chi_2(\chi_3-\chi_1)+\chi_1(\chi_2-\chi_3)] \equiv 0 .$$

Thus, for example, for $\chi''_{\underset{\sim}{k}} = 0$ and $\chi'_{\underset{\sim}{k}} = k_\perp^2$ the potential enstrophy $U = \sum_{\underset{\sim}{k}} k_\perp^2 E_{\underset{\sim}{k}}$ is also a constant of the motion of the renormalized wave-kinetic equation..

Decoupling of $k = 0$

In the limit $\underset{\sim}{k} \to 0$ the quantities $\omega_{\underset{\sim}{k}} \to 0$, $\nu_{\underset{\sim}{k}} \to 0$ and $S_{\underset{\sim}{k}} \to 0$ so that the fluctuations completely decouple from the $\underset{\sim}{k} = 0$ equations.

Small $|k|$ Limit

For small $|\underset{\sim}{k}|$ compared with the average wavenumber $\bar{k}$ in the spectrum, defined by $\bar{k} = (\sum_k k^2 I_k / \sum_k I_k)^{1/2}$, the induced turbulent damping ν_k reduces to a nonlinear eddy viscosity varying as $\nu_n k^n$ where ν_n varies as E for $E<1$ or $E^{1/2}$ for $E > 1$ as shown in Fig. 9 of Ref. 4. For small k the resonance function reduces to $R_{\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}-\underset{\sim}{k}_1} \simeq (\omega_{\underset{\sim}{k}} - \underset{\sim}{k} \cdot \underset{\sim}{v}_{k_1} + 2i\nu_{\underset{\sim}{k}_1})^{-1}$ where we define $\underset{\sim}{v}_{k_1} = \partial\omega_{\underset{\sim}{k}_1}/\partial \underset{\sim}{k}_1$.

For small k compared with $\langle k_1^2 \rangle^{1/2} = \bar{k}$ and an isotropic spectrum for $E_{\underset{\sim}{k}_1}$ (for simplicity) we obtain two regimes for $\nu_{\underset{\sim}{k}}$ and $S_{\underset{\sim}{k}}$ depending on the strength of dissipation. For the nearly conservative system the small k limit becomes

$$\nu_k = 2\pi k^4 \int_{k^2}^{\infty} \frac{dk_1^2 k_1^4}{(1+k_1^2)^2} \, \frac{2\nu_{k_1} \frac{\partial}{\partial k_1^2}(k_1^2 E_{k_1})}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}_{k_1})^2 + 4\nu_{k_1}^2} \simeq \nu_c(E) k^4 \quad .$$

The $\nu_k = \nu_c k^4$ limit applies when $|\omega - \underset{\sim}{k} \cdot v_{\underset{\sim}{k}_1}| < 2\nu_{\bar{k}}$ where $\nu_c \propto E^{1/2}$. The result of $\nu_c k^4$ has been given earlier by Horton[18].

In contrast for dissipative systems the leading contribution is

$$\nu_k = 4\pi k^2 \int_{k^2}^{\infty} \frac{dk_1^2 k_1^2 (\chi''_{\underset{\sim}{k}_1})^2 2\nu_{k_1} E_{\underset{\sim}{k}_1}}{(1+k_1^2)^2 [(\omega - \underset{\sim}{k}\cdot v_{k_1})^2 + 4\nu_{k_1}^2]} \cong \nu_d(E)k^2$$

where the $\nu_k = \nu_d k^2$ limit applies for $|\omega - \underset{\sim}{k}\cdot v_{\underset{\sim}{k}_1}| < 2\nu_{\bar{k}}$ and $\nu_d \propto E^{1/2}$. This analysis for the eddy viscosity shows that the dissipation changes the wavenumber scaling of the decorrelation or decay rate of the turbulent fluctuations.

V. Conclusions

The dissipative drift wave equations have stochastic solutions with broad frequency spectra for each $\underset{\sim}{k}$ mode. The simplest explanation for this behavior is the presence of a chaotic attractor in the phase space of the system. The chaotic attractor exists in each (typical) triplet of in the $\underset{\sim}{k}$ spectrum as shown in the three wave study of Terry and Horton.[3] The statistical properties of the signals on the attractor are independent of initial data due to the exponential separation of neighboring orbits, a property also present in each triplet. The flow is mixing on the attractor.

The basin of attraction defined as the domain of initial data $\{\phi_{\underset{\sim}{k}}(t_o)\}$ which are pulled into the attractor after a transient may be very large as indicated from a few randomly selected initial data sets. No attempt has been made at determining the dimensionality or the boundary of the attractor.

On the attractor the time average appears equal to the phase average over the random-phase ensemble. Thus, the attractor gives rise to a well defined turbulent steady state with a unique spectral distribution and a well defined anomalous transport flux. The turbulent state has no obvious inconsistencies with the $\underset{\sim}{k}\omega$ spectra and anomalous transport of the so-called microturbulence measured by electromagnetic wave scattering experiments in tokamaks such as in the Mazzuccto experiment.[21]

The statistical properties of the solutions in the steady state are examined for deviations from gaussian statistics. The cumulative distribution of the time series $\phi_{\underset{\sim}{k}}(t)$ shows an approximately normal distribution. The deviations from gaussianity are finite and characterized by the values given in Eqs. (24). The composite signal $\phi(\underset{\sim}{x},t)$ has non-gaussian features, Fig. 3, of the same order of magnitude as the individual $\phi_{\underset{\sim}{k}}(t)$'s indicating significant correlations between the $\underset{\sim}{k}$ modes. We conclude that the signals are sufficiently close to gaussian to allow turbulence theories based on the quasi-normal approximation (23) to be qualitatively, but perhaps not quantitatively, valid for these simulations.

Although we cannot rule out the possibility that these non-gaussian features will scale out with increasing N to justify statistical turbulence theory, we find no evidence for this trend within present experiments with N $\leq$ 224.

Statistical turbulence theory applied to these experiments is reduced to a Markovian description for $\omega_{\underset{\sim}{k}}(T)$, $\nu_{\underset{\sim}{k}}(T)$ and $I_{\underset{\sim}{k}}(T)$ based on the good parameterization of the frequency spectra $\phi_{\underset{\sim}{k}}(\omega)$ by a Lorentzian frequency distribution. The resulting system of Eqs. (35), (36) and (41) when solved numerically show stable steady states with properties in reasonable qualitative agreement with the time-averaged features of the phase dependent equations. A similar demonstration and conclusion is given by Waltz[22] in his study of the dissipative drift wave equations.

The renormalized wave kinetic equation predicts broad frequency spectra with the nonlinear fluctuation decay rate $\nu_k \sim \omega_k$. The renormalized equations show that the scaling of ν_k with k at small k is approximately $k^2E^{1/2}$ where E is the total turbulent energy density. The turbulence equation also predicts that the root-mean-square fluctuating amplitude $\tilde{\phi}$ increases with weak instability and saturates at the mixing length level $\langle(\underset{\sim}{k}\cdot \underset{\sim}{v}_E)^2\rangle^{1/2} \sim |k_y v_{de}| \sim \nu_k \gg \gamma_k$ as also shown by Waltz.[21] The scaling of $\tilde{\phi}$ and ν_k predicted by the Terry-Horton model for dissipative drift wave turbulence is in marked contrast to the prediction of the dissipationless Hasegawa-Mima model where $\nu_k \propto k^4\tilde{\phi}$ and $\tilde{\phi}$ is an arbitrary constant of the motion for $\gamma_k = 0$.

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Figure Captions

1. Geometry and mechanism of the dissipative drift wave instability.
2. Schematic diagram of the chaotic attractor in the Γ^{2N} phase space.
3. Energy spectrum and four typical frequency spectra in a simulation experiment.
4. Correlation function for a typical $\phi_{\underset{\sim}{k}}(t)$.
5. Probability distribution of a typical $\phi_{\underset{\sim}{k}}(t)$.

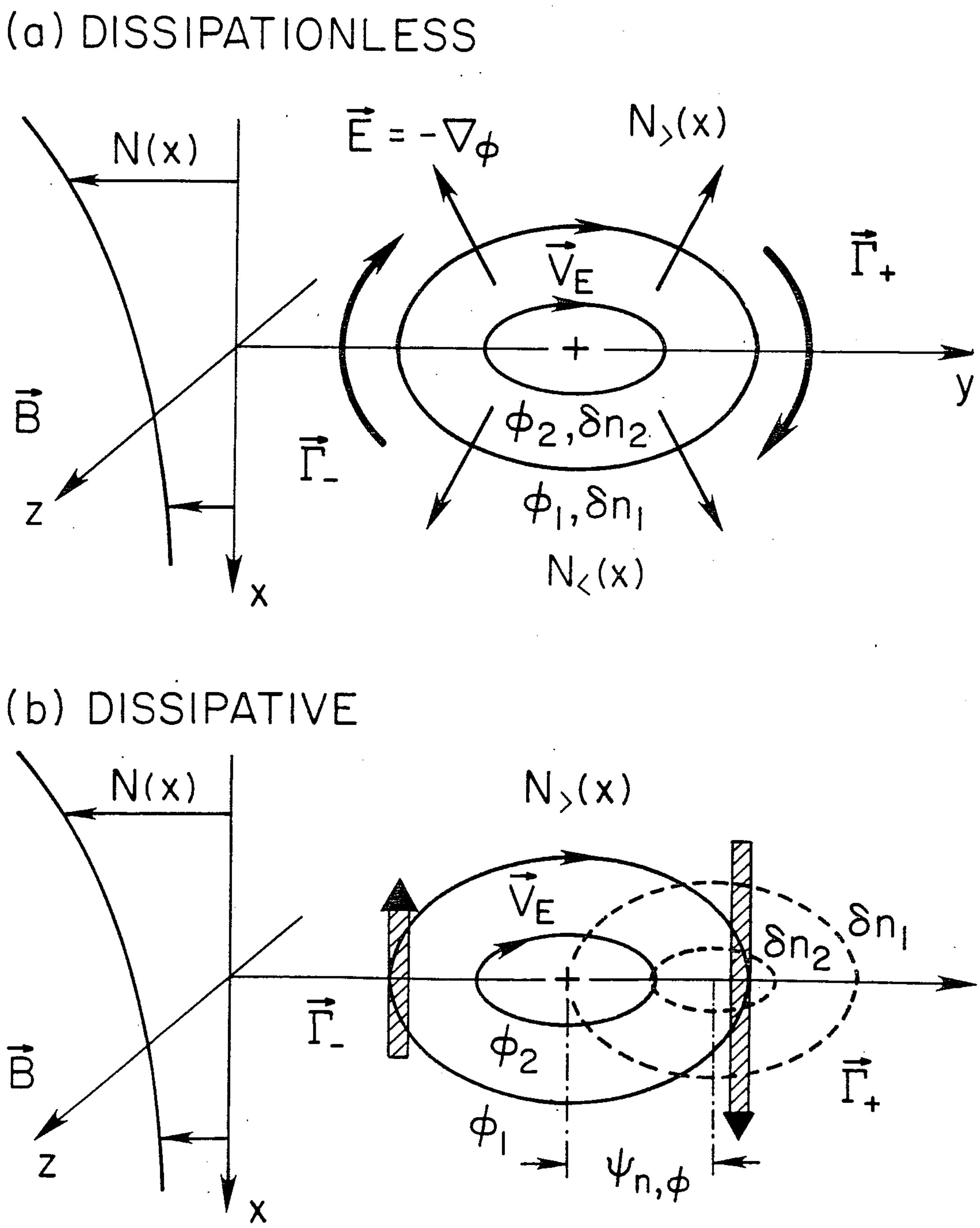

(a) DISSIPATIONLESS
N(x)
$\vec{E} = -\nabla\phi$
$N_>(x)$
$\vec{V}_E$
$\vec{\Gamma}_+$
$\vec{B}$
$\phi_2, \delta n_2$
$\vec{\Gamma}_-$
$\phi_1, \delta n_1$
z
$N_<(x)$
x
y
(b) DISSIPATIVE
N(x)
$N_>(x)$
$\vec{V}_E$
δn_1
δn_2
$\vec{\Gamma}_-$
ϕ_2
$\vec{B}$
$\vec{\Gamma}_+$
ϕ_1
$\psi_{n,\phi}$
z
x

FIG. 1

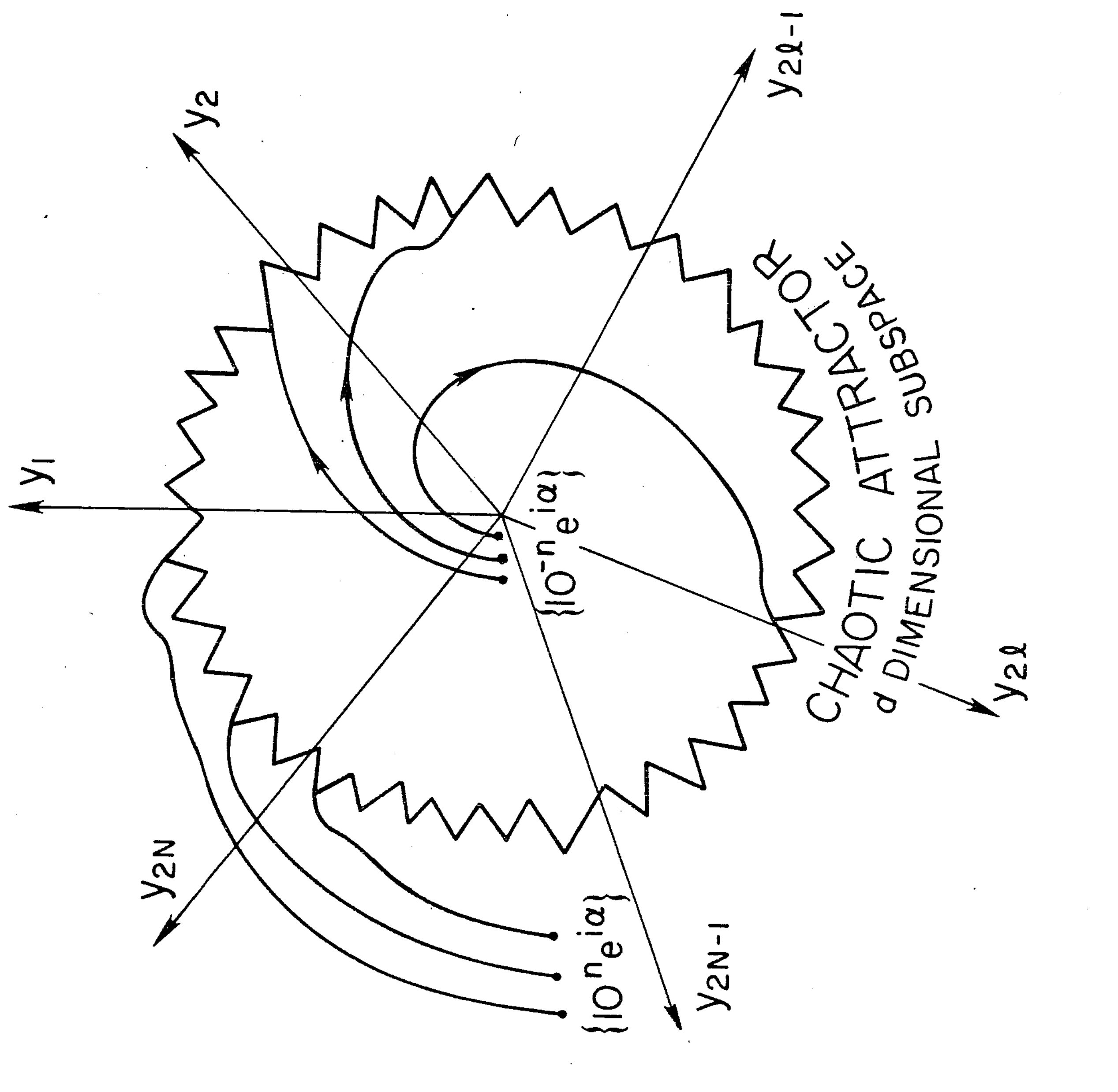

y_1
y_2
$y_{2\ell-1}$
$y_{2\ell}$
y_{2N}
y_{2N-1}
$\{10^{-n} e^{i\alpha}\}$
$\{10^{n} e^{i\alpha}\}$
CHAOTIC ATTRACTOR
d DIMENSIONAL SUBSPACE

FIG. 2

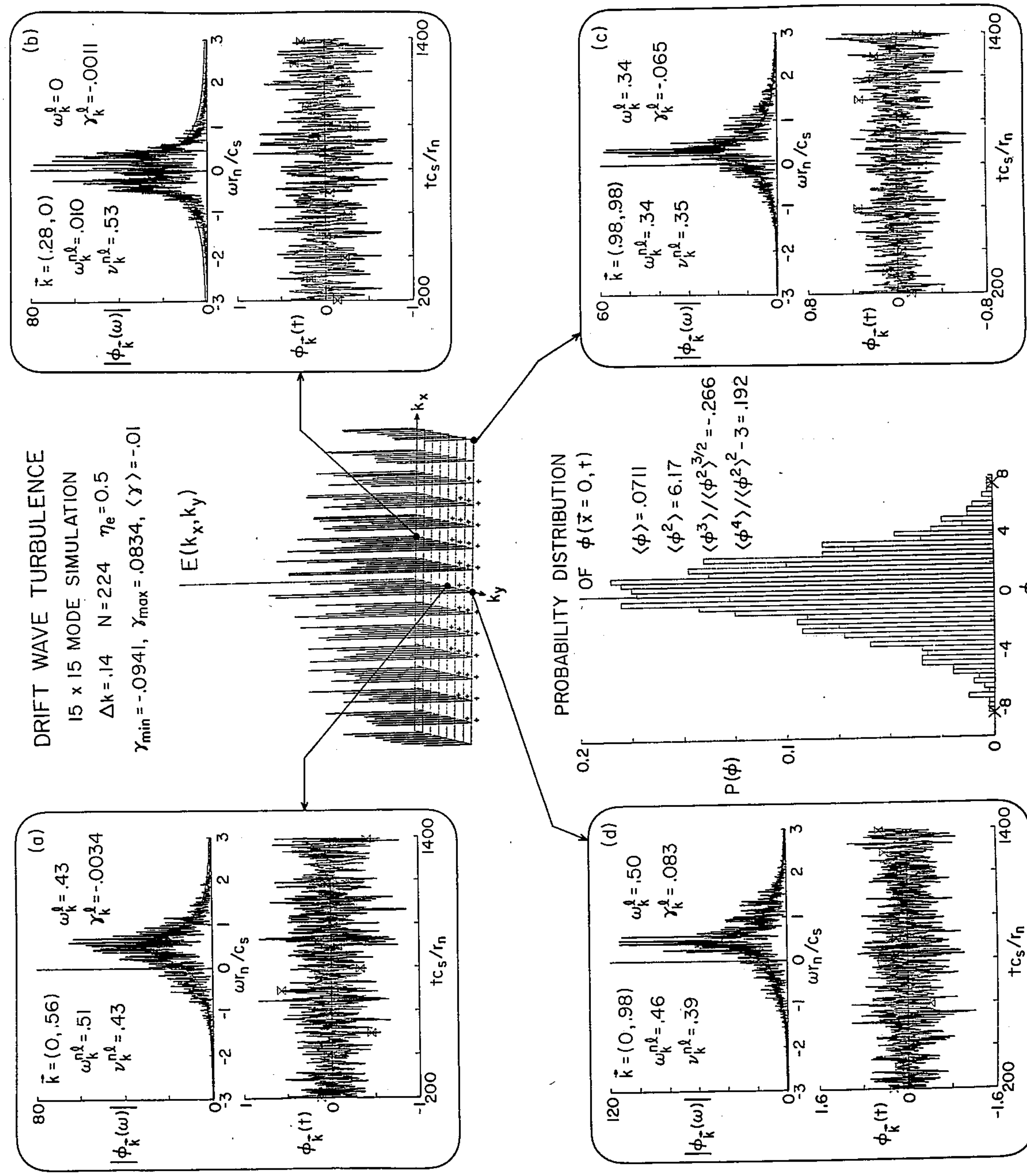

DRIFT WAVE TURBULENCE
15 x 15 MODE SIMULATION
Δk = .14 N = 224 ηe = 0.5
γmin = -.0941, γmax = .0834, ⟨γ⟩ = -.01
E(kx, ky)
kx
ky
(a)
k = (0, .56)
ωk^nl = .51
νk^nl = .43
ωk^l = .43
γk^l = -.0034
(b)
k = (.28, 0)
ωk^nl = .010
νk^nl = .53
ωk^l = 0
γk^l = -.0011
(c)
k = (.98, .98)
ωk^nl = .34
νk^nl = .35
ωk^l = .34
γk^l = -.065
(d)
k = (0, .98)
ωk^nl = .46
νk^nl = .39
ωk^l = .50
γk^l = .083
|φk(ω)|
ωrn/cs
φk(t)
tcs/rn
PROBABILITY DISTRIBUTION
OF φ(x = 0, t)
⟨φ⟩ = .0711
⟨φ²⟩ = 6.17
⟨φ³⟩/⟨φ²⟩^3/2 = -.266
⟨φ⁴⟩/⟨φ²⟩² - 3 = .192
P(φ)
φ

FIG. 3

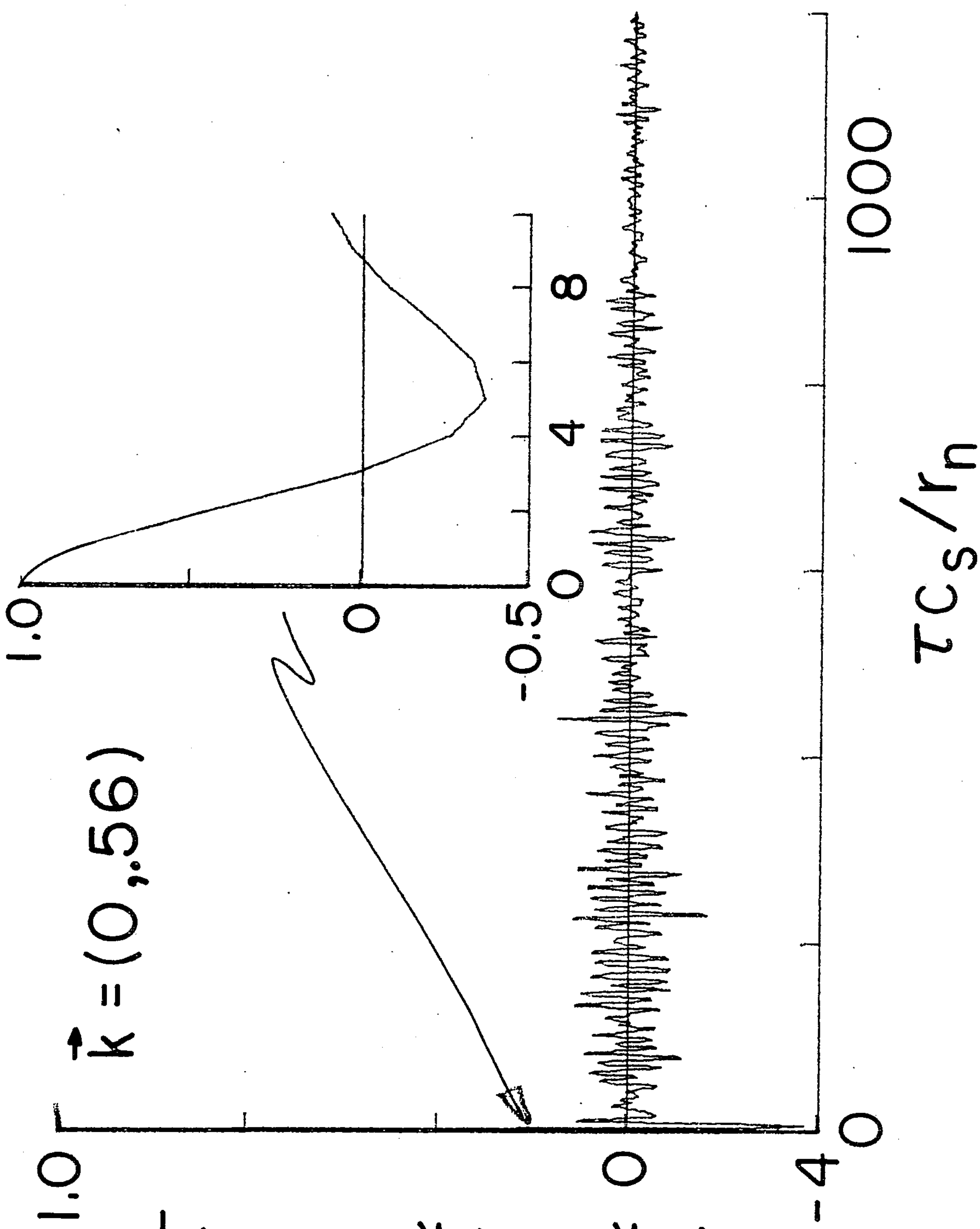
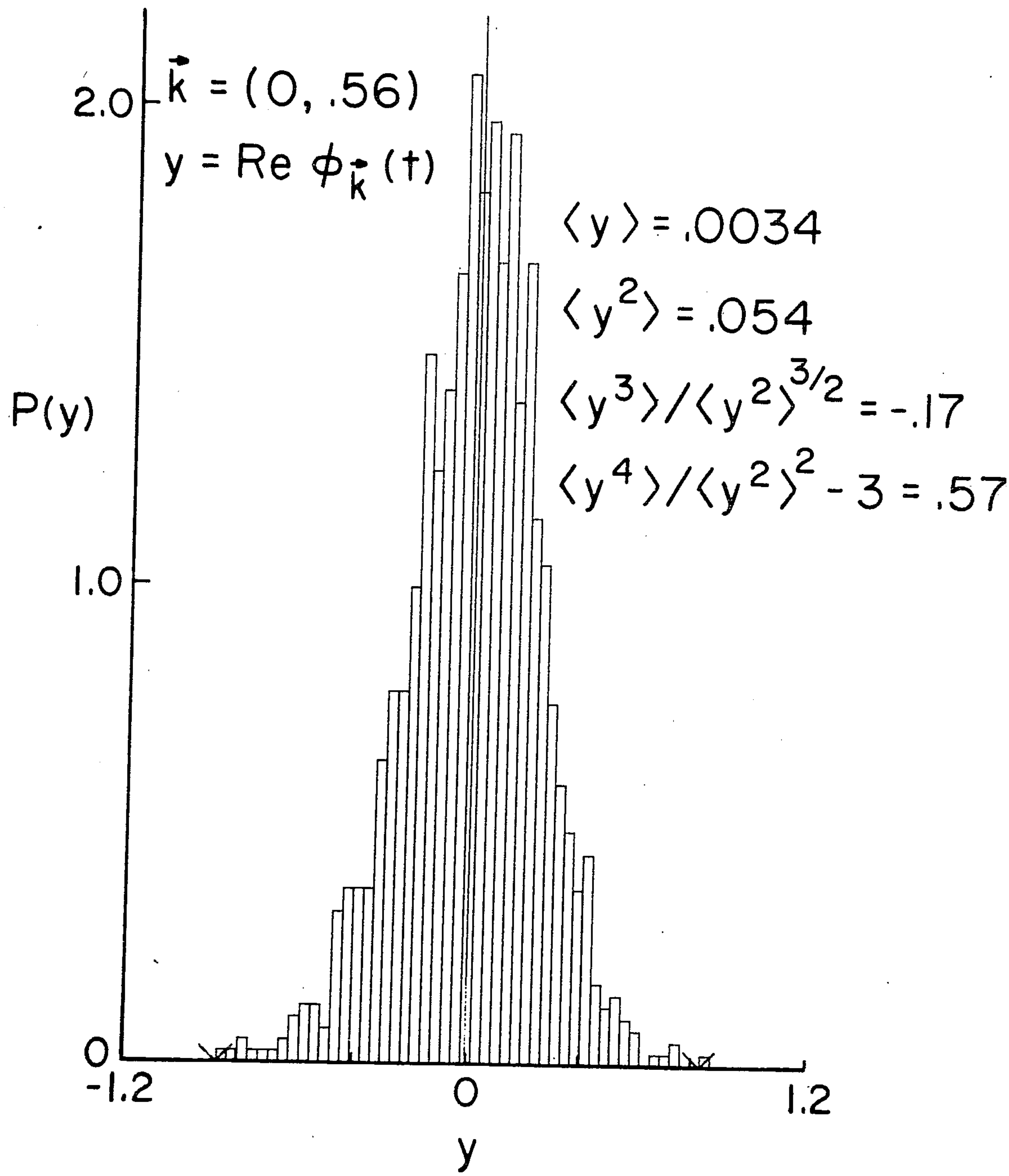


FIG. 4



2.0
1.0
0
P(y)
$\vec{k} = (0, .56)$
$y = \mathrm{Re}\,\phi_{\vec{k}}(t)$
$\langle y \rangle = .0034$
$\langle y^2 \rangle = .054$
$\langle y^3 \rangle / \langle y^2 \rangle^{3/2} = -.17$
$\langle y^4 \rangle / \langle y^2 \rangle^2 - 3 = .57$
-1.2
0
1.2
y

FIG. 5