Fluid description of relativistic, magnetized plasma

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ABSTRACT

Many astrophysical plasmas and some laboratory plasmas are relativistic: either the thermal speed or the local flow speed (in a convenient frame) approaches the speed of light. Many such plasmas are also magnetized, in the sense that the thermal Larmor radius is smaller than gradient scale lengths. Relativistic MHD, conventionally used to describe such systems, requires the collision time to be shorter than any other time–scale in the system. On this assumption it uses the thermodynamic equilibrium form of the plasma pressure tensor, neglecting stress anisotropy as well as heat flow along the magnetic field. Beginning with exact moments of the kinetic equation, we derive a closed set of Lorentz–covariant fluid equations that allows for anisotropy and heat flow, as would pertain to a collisionless plasma, far from thermodynamic equilibrium. The heart of the derivation is the construction of the plasma stress tensor as the fully general solution to the energy–momentum conservation law in the case of dominant electromagnetic force.

Subject headings: stars: atmospheres, galaxies: jets, methods: analytical, physical data and processes, relativistic plasmas

1. Introduction

Relativistic, magnetized plasma

A relativistic plasma is one in which either the thermal speed—the *rms* speed of individual particles—measured in the fluid rest frame, or the local bulk flow measured in some convenient frame, can approach the speed of light. Various astrophysical and cosmic plasmas (galactic and extra–galactic plasma jets (1), electron–positron streams in pulsar atmospheres and in the accretion disks of active galactic nuclei (2), electron–positron plasma and electron–positron–ion plasmas in the Mev era of the early universe (3)), as well as hot electrons in some laboratory experiments (especially fusion experiments), are relativistic in this sense.

Many relativistic plasmas of interest are magnetized—that is, their dynamics is dominated by the magnetic field. (An appropriate definition of "magnetized" is given in Subsection 2.) Relativistic, magnetized plasmas occur frequently in astrophysical systems; see, for example (4; 5; 6). The conventional description of magnetized plasma dynamics, magnetohydrodynamics (MHD), captures key features of a magnetized plasma, including the electromagnetic nature of its flow ($E \times B$ drift), and its peculiar closure of Maxwell's equations (computing the perpendicular plasma current density from the fluid equation of motion). Relativistic versions of MHD are known (7). However, MHD is based on the use of a stress tensor (or energy–momentum tensor) that does not reflect the dominant electromagnetic force. Indeed the stress tensor of MHD assumes thermal equilibrium; it is set by thermodynamics rather than electrodynamics.

A more consistent treatment determines the stress tensor by electrodynamics, in precise analogy to the determination of the plasma flow velocity. A special case of this tensor is the "gyrotropic" tensor introduced by Chew, Goldberger and Low (8), which we denote by CGL. The CGL tensor, being determined by electrodynamics, displays the characteristic anisotropy between directions parallel and perpendicular to the magnetic field. But we will find that this tensor (and its relativistic generalization) is not the most general one consistent with a dominant electromagnetic force. Furthermore the ("double-adiabatic") laws used to advance the CGL tensor are not obviously physical, especially since heat flow along the field lines of a low-collision-frequency plasma can be rapid.

The object of this work is to derive a closed fluid description of a relativistic, magnetized plasma. The stress tensor that we derive is the *general* solution to the relevant moment equation in the limit of vanishing gyroradius. This fully general tensor differs from the two special cases of MHD and CGL, in particular by allowing for both anisotropy and heat flow along the magnetic field.

Our general framework determines the form of the plasma flow and the plasma stress tensor using only the exact fluid equations, together with the orderings characterizing a magnetized plasma. Unfortunately (and predictably) this framework does not by itself yield a closed description: the scalar functions that appear—such as enthalpy density, and the perpendicular and parallel pressures—outnumber the field equations. To achieve closure we use a representative distribution function for each plasma species, chosen consistently with relativity, magnetization, anisotropy and heat flow.

It is worth noting that the covariant analysis of fluid equations for magnetized plasmas is simpler and more transparent than the nonrelativistic version found in many textbooks. In particular the relativistic derivation suggests straightforward means for the inclusion of finite gyroradius physics. Such generalization will be the subject of future work.

2. Electromagnetic field

Faraday tensor

We use the summation convention, with greek indices varying from 0 to 3, and roman indices from 1 to 3. We occasionally use boldface for the $i = 1, 2, 3$ components ("vector" components), writing an arbitrary four–vector C^μ as

$$C^\mu = (C^0, \boldsymbol{C})$$

We also measure all speeds in terms of the light speed, so that $c = 1$.

The Faraday (or field-strength) tensor is defined by

$$F^{\mu\nu} \equiv \nabla^\mu A^\nu - \nabla^\nu A^\mu$$

where $A^\mu = (\phi, \boldsymbol{A})$ is the four-vector potential and

$$\nabla^\mu = \eta^{\mu\nu} \frac{\partial}{\partial x^\nu}.$$

Here $\eta^{\mu\nu} = \mathrm{diag}\{-1, 1, 1, 1\}$ is the Minkowski tensor.

The tensor dual to $F_{\mu\nu}$ is

$$\mathcal{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda}$$

where $\epsilon^{\mu\nu\kappa\lambda}$ is the antisymmetric tensor. We recall that $\mathcal{F}$ is dual to F in another sense:

$$\mathcal{F}(\boldsymbol{E}, \boldsymbol{B}) = F(\boldsymbol{E} \to \boldsymbol{B}, \boldsymbol{B} \to -\boldsymbol{E}) \tag{1}$$

Here of course $\boldsymbol{E}$ and $\boldsymbol{B}$ are respectively the electric and magnetic fields.

It is also useful to recall that the action of the Faraday tensor on an arbitrary four–vector C^μ is given by

$$F_{\mu\nu} C^\nu = (-\boldsymbol{E} \cdot \boldsymbol{C}, \boldsymbol{E} C^0 + \boldsymbol{C} \times \boldsymbol{B}) \tag{2}$$

The Lorentz force is a special case of this formula.

The well–known Lorentz scalars (relativistic invariants) formed from F and its dual will be denoted by W and λ:

$$\frac{1}{2} F_{\kappa\lambda} F^{\kappa\lambda} = B^2 - E^2 \equiv W$$

and

$$\frac{1}{2} \mathcal{F}^{\mu\kappa} F_{\kappa\mu} = \boldsymbol{E} \cdot \boldsymbol{B} \equiv \lambda W \tag{3}$$

This last relation is especially important because λ, or $E_{||}$, is a small parameter of our theory.

Magnetized plasma

We will consider a plasma system to be magnetized if two criteria are satisfied:

1. The two electromagnetic field invariants, W and λ, must satisfy

$$W > 0 \tag{4}$$
$$\lambda \ll 1. \tag{5}$$

2. The thermal gyroradius must be small compared to any gradient scale length:

$$\delta \ll 1 \tag{6}$$

   where δ is the ratio of the thermal gyroradius of any plasma species to any gradient scale length.

A convenient ordering turns out to be $\lambda \sim \delta$.

This definition of a magnetized plasma will be used implicitly throughout the following analysis. One example of its significance can be seen by considering the well-known relation

$$\mathcal{F}_{\mu\kappa}F^{\kappa\nu} = \eta_\mu{}^\nu \lambda W \tag{7}$$

Thus the Faraday tensor and its dual are inverse tensors, up to a multiplicative constant. However in the magnetized case ($\lambda \to 0$), the relation (7) plays a very different role. It no longer provides a useful inverse, because of the small denominator that would occur. Indeed, in the magnetized case $\mathcal{F}$ becomes an *annihilator* for F rather than its inverse.

That the Faraday tensor and its dual have (two–dimensional) null spaces in a magnetized plasma plays an important role in our argument.

Quasi-projectors

A covariant meaning is given to “perpendicular” and “parallel” by the operators

$$e_\mu{}^\nu \equiv -F_{\mu\kappa}F^{\kappa\nu}/W \tag{8}$$
$$b_\mu{}^\nu \equiv \eta_\mu{}^\nu - e_\mu{}^\nu \tag{9}$$

These tensors become approximate projection operators in the magnetized limit, $\lambda \sim \delta \to 0$. Indeed, from the easily verified identity

$$F^{\mu\kappa}e_\kappa{}^\nu = F^{\mu\nu} - \lambda\mathcal{F}^{\mu\nu} \tag{10}$$

one can show that

$$e^{\mu\kappa}e_{\kappa\nu} = e^{\mu}_{\ \nu} + \lambda^2\delta^{\mu}_{\ \nu}$$

and similarly for $b_{\mu}^{\ \nu}$. Furthermore one can show that, in the rest–frame (R), the action of e and b on an arbitrary four–vector $C = (C_0, \boldsymbol{C})$ is given by

$$b^{\mu\kappa}C_\kappa|_R = (C^0, \boldsymbol{C}_{\|}) \tag{11}$$

$$e^{\mu\kappa}C_\kappa|_R = (0, \boldsymbol{C}_\perp) \tag{12}$$

Here the $\perp$ and $\|$ subscripts have the usual three-dimensional meaning: $\boldsymbol{C}_{\|} = \boldsymbol{B}\boldsymbol{B}\cdot\boldsymbol{C}/B^2 = \boldsymbol{b}\boldsymbol{b}\cdot\boldsymbol{C}$, $\boldsymbol{C}_\perp = \boldsymbol{C} - \boldsymbol{C}_{\|}$. Notice that we use the abbreviation

$$\boldsymbol{b} \equiv \boldsymbol{B}/B.$$

Exact projection operators have been introduced previously by Fradkin (12). Because the exact operators are extremely complicated, it is fortunate that the quasi-projectors defined above are sufficient for our purposes.

3. Fluid closure

Closing Maxwell

A plasma is distinguished from other physical systems by its strong coupling to the electromagnetic field. This coupling enters a fluid description through the second moment equation, the conservation law for energy–momentum (10). If the total (summed over all plasma species) energy–momentum tensor for the plasma is denoted by $\mathcal{T}$, then we have

$$\frac{\partial \mathcal{T}^{\mu\nu}}{\partial x^\nu} - F^{\mu\nu}J_\nu = 0 \tag{13}$$

where J_ν is the four-vector current density.

All fluid descriptions of *magnetized* plasma evolution use this second moment as a constitutive relation, determining the four-vector current density in terms of the fields, and thus providing closure relations for Maxwell's equations:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = J^\mu \tag{14}$$

$$\frac{\partial F_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial F_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x^\beta} = 0. \tag{15}$$

We next review the procedure for computing the current density in a magnetized plasma, in part to display its relative simplicity in the relativistic case. (Note that this procedure is applies exclusively to a magnetized plasma; we do not present a general closure scheme.) By multiplying (13) with $F^{\mu}_{\ \kappa}$ and using the definition of the perpendicular projector, we obtain

$$e^{\mu\nu} J_\nu = -\frac{F^{\mu}_{\ \kappa}}{W} \frac{\partial \mathcal{T}^{\kappa\nu}}{\partial x^\nu}, \tag{16}$$

expressing the two perpendicular components of the current density in terms of the stress. (Because the perpendicular quasi-projector has a 2-dimensional null space, (16) constitutes only two independent equations.) The remaining components of J_μ are found from charge conservation

$$\frac{\partial J^\nu}{\partial x^\nu} = 0, \tag{17}$$

and quasi-neutrality, which has the covariant expression

$$J^\nu U_\nu = 0. \tag{18}$$

Here $U^\mu = (\gamma, \gamma \boldsymbol{V})$ is the local four–velocity of the fluid, with

$$\gamma^2 = (1 - V^2)^{-1} \tag{19}$$

the relativistic dilation factor.

In a relativistic plasma the charge density can be presumed to vanish only in the instantaneous rest–frame, as (18) evidently requires. Note in this regard that the (local) instantaneous rest–frame is an *inertial* frame whose velocity, measured in the laboratory frame, equals the fluid velocity at some arbitrarily chosen space–time point x.)

We conclude that the four–current density is determined, closing Maxwell's equations, once the plasma stress tensor is known. The remainder of this paper is devoted to computing that tensor. Recall that conventional MHD obviates most of the following analysis by assuming the stress tensor to have the thermodynamic form,

$$\mathcal{T}^{\mu\nu} = p\eta^{\mu\nu} + hU^\mu U^\nu \tag{20}$$

in terms of the pressure, p, the enthalpy density, h, and the fluid velocity four–vector U^μ. This form would pertain if thermal relaxation due to collisions occurred more rapidly than any other process of interest. Hence, the present work can be described as an extension of MHD into regimes of much lower collisionality. In fact we ignore collisions altogether, and find the form of the stress tensor for a plasma subject to the electromagnetic force alone.

Small gyroradius

We compute the plasma stress tensor $\mathcal{T}^{\mu\nu}$ in terms of a sum over the stress tensors $T^{\mu\nu}$ of individual species,

$$\mathcal{T}^{\mu\nu} = \sum_{species} T^{\mu\nu}$$

suppressing a species subscript on $T^{\mu\nu}$. Our analysis is based on the three exact (collisionless) conservation laws, for each species,

$$\frac{\partial \Gamma^\nu}{\partial x^\nu} = 0 \tag{21}$$

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} - eF^{\mu\nu}\Gamma_\nu = 0 \tag{22}$$

$$\frac{\partial M^{\mu\alpha\beta}}{\partial x^\mu} - e(F^{\alpha\nu}T_\nu{}^\beta + F^{\beta\nu}T_\nu{}^\alpha) = 0 \tag{23}$$

Here Γ^μ is the four–vector measure of the fluid particle–flux density and $M^{\mu\alpha\beta}$, which we will call the "stress–flow" tensor, is the third–rank moment. For explicitness we express each moment in terms of the (Lorentz–scalar) distribution function $f(x,p)$, where p represents the four–momentum p^μ:

$$\Gamma^\alpha \equiv \int \frac{d^3p}{p^0} f p^\alpha \tag{24}$$

$$T^{\alpha\beta} \equiv \int \frac{d^3p}{p^0} f p^\alpha p^\beta \tag{25}$$

$$M^{\alpha\beta\gamma} \equiv \int \frac{d^3p}{p^0} f p^\alpha p^\beta p^\gamma \tag{26}$$

These formulae use the invariant momentum–space volume element d^3p/p^0. Recall that four–momentum p^μ satisfies the mass–shell condition

$$p^0 = \sqrt{m^2 + \boldsymbol{p}^2}. \tag{27}$$

where m is the particle mass.

In (22) and (23) the first term involves the macroscopic scale, while the remaining terms—those proportional to the charge e—contain the short gyro–scale (gyroradius or gyroperiod). Thus the small–gyroradius limit is obtained formally by allowing the charge to become arbitrarily large, $e \to \infty$. We next consider the forms of the Γ^μ and $T^{\mu\nu}$ in this limit.

Magnetized flow

We express the flux density as

$$\Gamma^\mu = \Gamma^\mu_{(0)} + \Gamma^\mu_{(1)}$$

where $\Gamma^\mu_{(0)}$ denotes the lowest order ($\delta \to 0$) flux density and $\Gamma^\mu_{(1)} \sim \delta$. Then (22) implies

$$F^{\mu\nu}_{(0)} \Gamma_{(0)\mu} = 0 \tag{28}$$

a relation that fixes the two perpendicular components of the flow. Here we distinguish the lowest–order Faraday tensor, $F^{\mu\nu}_{(0)} \equiv F^{\mu\nu}(E_{||} = 0)$ from its first–order contribution

$$F^{\mu\nu}_{(1)} \equiv F^{\mu\nu} - F^{\mu\nu}_{(0)} \propto E_{||}.$$

For the two remaining flow–components, we have the particle conservation law, (21), and an additional equation to be derived presently. Recalling (2), we see that (28) implies

$$\Gamma^0_{(0)} \boldsymbol{E} + \boldsymbol{\Gamma}_{(0)} \times \boldsymbol{B} = 0$$

which reproduces the MHD law $\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} = 0$.

We avoid trivial complications by now restricting our attention to a plasma with a single ion species, and by assuming that the ions and electrons share, approximately, a common rest–frame (see below). Then (18) requires the ions and electrons to have the same rest–frame density, which we denote by n_R, and the flux density is related to the flow velocity by

$$\Gamma^\mu = n_R U^\mu \tag{29}$$

Also at this point we simplify notation by using the symbols Γ^μ and U^μ to refer to the zeroth–order vector fields, suppressing the (0) subscript. In other words, Γ^μ and U^μ represent fields that satisfy (28), and (2) implies

$$\Gamma^\mu = \gamma n_R (1, \boldsymbol{V}_{||} + \boldsymbol{V}_E) \tag{30}$$

where $\boldsymbol{V}_E \equiv \boldsymbol{E} \times \boldsymbol{b}/B$ and $\boldsymbol{V}_{||} = \boldsymbol{b}\boldsymbol{b} \cdot \boldsymbol{V}$. Note in particular that all factors of γ are evaluated at the lowest–order flow $\boldsymbol{V} = \boldsymbol{V}_{||} + \boldsymbol{V}_E$. We similarly suppress ordering subscripts on the Faraday tensor whenever they are not essential to the argument: F will denote $F_{(0)}$ unless the context indicates otherwise.

Equation (30) shows that the perpendicular flows of both plasma species are the same. Note that otherwise the perpendicular current would be larger, $\boldsymbol{J}_\perp \sim \delta^{-1}$, than allowed by (16). We do not force the (lowest-order) parallel flows of the two species to coincide. However, we require that the *relative* velocity of the rest–frames of any two species s and s' be nonrelativistic:

$$|\boldsymbol{V}_{\|s} - \boldsymbol{V}_{\|s'}| \ll 1.$$

Thus we allow arbitrarily large plasma flow, but not arbitrarily large current density. Only in that case does it make sense to refer to a fluid rest–frame, in which the flows of both species can be assumed small.

In this regard, notice that we can choose an approximate rest–frame, in which the lowest–order flow, rather than the total (exact) flow, vanishes. Such a choice makes it clear that the electric field is negligibly small ($E \ll B$) in the rest–frame, since $E_{\|}$ has been neglected and the lowest–order perpendicular flow is $\boldsymbol{E} \times \boldsymbol{B}$. It should also be noticed that a true rest–frame in which the electric field exactly vanishes cannot occur in a plasma that is dominated by the electromagnetic field: if there were such a frame the Lorentz force would vanish in every frame.

For later application we point out that (22) has become, in view of (28),

$$\frac{\partial T^{\mu\nu}_{(0)}}{\partial x^{\nu}} = eF^{\mu\nu}_{(0)}\Gamma_{(1)\nu} + eF^{\mu\nu}_{(1)}\Gamma_{(0)\nu} \tag{31}$$

an equation in which all terms are formally of the same order in δ.

Magnetized stress

The stress tensor is computed in close analogy to the flow. Again allowing $e \to \infty$ we find that (23) reduces to

$$F^{\alpha\nu}T_{\nu}{}^{\beta} + F^{\beta\nu}T_{\nu}{}^{\alpha} = 0 \tag{32}$$

To find the general solution to (32), we use the indicial symmetry of the stress tensor and antisymmetry of the Faraday tensor. Then properties of the projection operators can be combined to show that $T^{\mu\nu}$ must have the form

$$T^{\mu\nu} = b^{\mu\nu}p_{\|} + e^{\mu\nu}p_{\perp} + hU^{\mu}U^{\nu} + q^{\mu}U^{\nu} + U^{\mu}q^{\nu} \tag{33}$$

where $p_{\|}$, $p_{\perp}$ and h are Lorentz scalars corresponding respectively to parallel pressure, perpendicular pressure and enthalpy density, and where the four–vector q^{μ} must satisfy

$$e_{\alpha\beta}q^{\alpha} = 0 \tag{34}$$

in order to satisfy force-balance, and

$$U_\alpha q^\alpha = 0 \tag{35}$$

in order to preserve the significance of $p_{||}$ and $p_\perp$. Thus there is only one independent component in q^μ; this represents parallel heat flow in the rest–frame and is denoted by $q_{||}$.

It is convenient to introduce the dimensionless four–vector k^α such that

$$q^\alpha \equiv q_{||} k^\alpha$$

Notice that (34) and (35) require k^α to reduce to the unit vector $\boldsymbol{b} = \boldsymbol{B}/B$ in the local rest–frame. Hence, by applying a Lorentz boost to $\boldsymbol{b}$ we find that

$$k^\alpha = (k^0, \boldsymbol{k}) = \gamma\sqrt{\frac{W}{B^2}}\left(\frac{B^2}{W}V_{||}, \boldsymbol{b} + \frac{B^2}{W}V_{||}\boldsymbol{V}_E\right) \tag{36}$$

It is easily verified that the resulting q^α indeed satisfies (34) and (35).

That q^μ represents heat flow will be clear from the fluid equations to be derived presently. It is noteworthy, however, that it can also be demonstrated from general physical considerations, based on (33) alone. Thus de Groot (9) shows that the heat flow is generally related to the stress tensor by

$$q^\mu = (U^\nu T_{\nu\sigma} - hU_\sigma)(U^\mu U^\sigma + \eta^{\mu\sigma})$$

which is consistent with (33).

We emphasize that (33) represents the unique, general form of the stress tensor in a plasma dominated by the electromagnetic force. It is instructively compared to the special case (20) used in MHD; evidently collisional dissipation has been allowed to remove stress anisotropy in the latter. Compared to the CGL stress tensor, (33) differs in allowing heat flow. The fact that the electromagnetic field appears in the stress tensor only through quasi-projectors $b^{\mu\nu}$ and $e^{\mu\nu}$ is a reflection of gauge–invariance and the indicial symmetry of $T^{\mu\nu}$; recall (25).

The stress tensor contains eight unknown scalar functions: $n_R(x,t)$, $p_{||}(x,t)$, $p_\perp(x,t)$, $h(x,t)$, the three independent components of $\Gamma^\mu(x,t)$, and the single independent component of $q^\mu(x,t)$. Since (21) provides the evolution of the density, and (28) determines the two perpendicular components of the flow, closure of our system requires five additional equations (for each plasma species); deriving those equations is our remaining task.

Two of the needed equations are derived from (31). We multiply this relation by $\mathcal{F}$, use (7) and consider the $\lambda \sim \delta \to 0$ limit to find

$$\mathcal{F}_{\mu\kappa}\frac{\partial T^{\kappa\nu}}{\partial x^\nu} = eE_{||}B\Gamma_\mu \tag{37}$$

which constitutes two independent equations. These can be taken to advance the parallel flows and parallel pressures of each species.

The right–hand side of (37) comes from the $F_{(1)}$ term in (31). We temporarily restore ordering–subscripts to show the essential point: multiplication by $\mathcal{F}_{(0)}$ annihilates the $F_{(0)}\Gamma_{(1)}$ term in (31), leaving only

$$e\mathcal{F}_{(0)\alpha\mu}F^{\mu\nu}_{(1)}\Gamma_{(0)\nu} = eE_{\|}B\Gamma_{(0)\alpha}. \tag{38}$$

Advancing the remaining variables in $T^{\mu\nu}$ is less straightforward. Indeed, finding a closed fluid description of a magnetized plasma is more complicated than the neutral gas case. For the latter, external forces are presumed known, and only the stress components themselves need to be advanced in time. But the key "external" force acting on a plasma comes from $\boldsymbol{J} \times \boldsymbol{B}$, and this force must be computed, as we have shown, from the stress tensor, rather than being used to advance that tensor in time. Thus only two components of the energy-momentum law—the two components of (37)—are available for calculating the stress tensor.

An equivalent statement is the observation that (31) is useless as an equation for $T^{\mu\nu}$—because we have no independent equation for $\Gamma_{(1)}$—unless the term involving $\Gamma_{(1)}$ is annihilated. But such annihilation leaves only two independent equations. It is this circumstance that forces us to consider the higher–rank tensor $M^{\alpha\beta\gamma}$.

Magnetized stress-flow

We study the tensor $M^{\alpha\beta\gamma}$ in a magnetized plasma by following the procedure used for Γ^α and $T^{\alpha\beta}$. We use curly brackets to indicate indicial symmetrization; for example

$$\eta^{\{\alpha\beta}U^{\gamma\}} \equiv \eta^{\alpha\beta}U^\gamma + \eta^{\alpha\gamma}U^\beta + \eta^{\beta\gamma}U^\alpha$$

It is straightforward to see that the $\delta \to 0$ limit of the fourth-rank conservation law is

$$F^{\{\alpha\kappa}M_\kappa{}^{\beta\gamma\}} = 0,$$

an equation that determines the form of the stress–flow in the magnetized case. Assuming that the only four–vectors in $M^{\alpha\beta\gamma}$ are Γ^α and q^α (or, equivalently, U^α and k^α), and using identities implied by (27), we find that it must have the form

$$M^{\alpha\beta\gamma} = m^2 n_R U^\alpha U^\beta U^\gamma + \sum_k m_k M_k^{\alpha\beta\gamma} \tag{39}$$

where

$$
\begin{aligned}
M_1^{\alpha\beta\gamma} &= \eta^{\{\alpha\beta}U^{\gamma\}} + 6U^\alpha U^\beta U^\gamma && (40)\\
M_2^{\alpha\beta\gamma} &= b^{\{\alpha\beta}U^{\gamma\}} + 4U^\alpha U^\beta U^\gamma && (41)\\
M_3^{\alpha\beta\gamma} &= \eta^{\{\alpha\beta}k^{\gamma\}} + 6U^{\{\alpha}U^\beta k^{\gamma\}} && (42)
\end{aligned}
$$

and the m_k are arbitrary scalars. Here it should be kept in mind that the quantity U^μ refers to the lowest–order flow velocity, given by (30).

It can be seen that the M_k satisfy

$$M_{k\ \alpha}^{\alpha\ \gamma} = 0$$

so that

$$M_{\ \alpha}^{\alpha\ \gamma} = -m^2 n_R U^\gamma$$

as follows from its definition: compare (24) and (26).

The evolution of the magnetized stress–flow is governed by two equations, analogous to (37), that are obtained by annihilating the right-hand side of (23). Appealing as usual to indicial symmetries and properties of the quasi-projectors, we find two constraints:

$$
\begin{aligned}
e_{\alpha\beta}\frac{\partial M^{\kappa\beta\alpha}}{\partial x^\kappa} &= 0 && (43)\\
(U_\alpha k_\beta + U_\beta k_\alpha)\frac{\partial M^{\kappa\alpha\beta}}{\partial x^\kappa} &= -2eE_\parallel h && (44)
\end{aligned}
$$

Recall here that $U^\mu = \Gamma^\mu / n_R$ refers to the lowest–order flow, satisfying (28). An explicit derivation of these constraints is the subject of Appendix A.

We will find in Section 5 that (43) can be expressed as a conservation law:

$$\frac{\partial}{\partial x^\nu}\left(\frac{m_1 U^\nu + m_3 k^\nu}{\sqrt{W}}\right) = 0 \qquad (45)$$

In other words the four–vector

$$G^\mu \equiv W^{-1/2}(m_1 U^\nu + m_3 k^\nu) \qquad (46)$$

is conserved in the same manner as, for example, Γ^μ. The new conservation law

$$\frac{\partial G^\nu}{\partial x^\nu} = 0$$

replaces, in a sense, the double-adiabatic assumption of previous literature (8). What is especially interesting is that the new conserved four–vector is not simply a fluid quantity, like Γ^μ or q^μ. Because of the factor $W^{-1/2}$ in (46) as well as such electromagnetic constraints as (34), G^μ depends crucially upon the electromagnetic field.

Of course the conservation law implies that G^0 is a conserved density: if there is some closed, space–like surface S on which $\boldsymbol{G}$ vanishes, then

$$\frac{d}{dt}\int d^3x G^0 = 0$$

where the integration volume is the region enclosed by S.

If the m_k were known, the relations (43) and (44) would determine the evolution of the perpendicular pressure and the parallel heat flow. Thus we have found dynamical equations for the variables

$$\Gamma^\mu,\ p_{||}, p_\perp \text{ and } q_{||},$$

which we therefore take as the basic dynamical variables of our system. It is convenient to consider the remaining four scalars—the enthalpy h and the three m_k—as 'auxiliary parameters.'

We have reduced the closure problem to that of relating the auxiliary parameters to the dynamical variables. The desired relations cannot be found from moment equations but, as noted in Section 1, require separate information about the distribution function.

It is convenient to take note here of the form of the tensor $M^{\alpha\beta\gamma}$ in the (approximate) instantaneous rest–frame of the fluid. For convenience, we orient this frame so that $\boldsymbol{B} = (0, 0, B)$; also recall that $\boldsymbol{E}$ is negligible in the rest–frame. It is then straightforward to show that the only non-vanishing components of the rest–frame tensor $M_R^{\alpha\beta\gamma}$ are

$$\begin{aligned}
M_R^{000} &= m^2 n_R + 3m_1 && (47)\\
M_R^{003} &= 5m_3 && (48)\\
M_R^{011} &= m_1 = M_R^{022} && (49)\\
M_R^{033} &= m_1 + m_2 && (50)\\
M_R^{113} &= m_3 = M_R^{223} && (51)\\
M_R^{333} &= 3m_3 && (52)
\end{aligned}$$

4. Distribution function

Relation to kinetic theory

Our analysis has begun with exact (collisionless) moments of the kinetic equation. After defining what is meant by a magnetized plasma, we have used the limit $\delta \sim \lambda \to 0$ to find magnetized fluid equations for advancing the dynamical variables of the system. However this description requires additional auxiliary parameters, not fixed by moment equations. We next compute these parameters from a distribution function.

The key properties of the lowest-order distribution function in a magnetized plasma are well known: it is gyrotropic (independent of the velocity–space angle corresponding to rotation about the direction of $\boldsymbol{B}$) and it solves a 'drift–kinetic' equation. The first requirement is easily implemented exactly, but a fully general *fluid* implementation of the second is difficult, especially in nonlinear regimes. For this reason, the only fully rigorous magnetized system, so–called "kinetic MHD," abandons the fluid point of view at this point, making the drift–kinetic equation an essential part of the closed system (13).

Note however that the details of the distribution are not relevant to closure: any one of the countless distributions that reproduce the general stress tensor (33) would yield the same closure of Maxwell's equations and therefore the same dynamics. In other words, what is needed is a *representative* of the equivalence class of distributions that are consistent with (33).

With this in mind, we replace the drift–kinetic equation by a parametrized distribution that reproduces the form of the magnetized stress tensor and is sufficiently flexible, through its space–time varying parameters, to consistently represent the fluid equations of motion. Indeed, the parameters in the distribution are proportional to the dynamical variables of the fluid system, and therefore evolve according to the fluid equations. Because the distribution function allows evaluating the auxiliary parameters in terms of the dynamical variables, it closes the system.

The assumed form of our representative distribution will describe a wide variety of physical conditions. On the other hand, as a function with moderate velocity–space gradients, it will poorly represent situations, involving for example beams or velocity–space boundary layers, where the actual distribution is not smooth.

It is clear that this approach agrees in spirit with the "thirteen–moment" closure due to Grad (14; 15). However, we point out that the thirteen–moment approximation is significantly modified by relativity, which rules out the polynomial expansion used in non-relativistic theory, and by magnetization, which imposes new symmetries on the distribution and makes half of the conventional closure equations unavailable; recall the discussion following (37).

Explicit form

Our assumed distribution is the simplest Lorentz-scalar distribution that is gyrotropic (and therefore consistent with lowest order kinetic theory), and that allows for both stress anisotropy and heat flow. It has the form

$$f(x,p) = f_M[1 + \hat{\Delta} + \Delta p_\alpha e^{\alpha\beta} p_\beta + Q_\alpha b^{\alpha\beta} p_\beta (1 + \hat{Q} U^\alpha p_\alpha)] \tag{53}$$

where f_M is a relativistic Maxwellian, discussed presently. It can be seen that the scalar Δ measures the anisotropy, while $Q_\alpha \propto q_\alpha$ measures heat flow. Note in particular that Q_α satisfies the constraints (34) and (35) and therefore has only one independent component.

The remaining scalar parameters will be chosen presently for convenience. Thus our distribution can be parametrized by the density n_R, the two pressures $p_{||}$ and $p_\perp$, and the parallel heat flow $q_{||}$.

At this point we recall the relativistic definition of a Maxwellian distribution. The canonical momentum $P^\mu = p^\mu + eA^\mu$ allows us to define the invariant energy $U_\mu P^\mu$; then we have

$$f_M(x,p) = N_M e^{U_\mu P^\mu / T}$$

where x and p represent the corresponding four–vectors and the normalization $N_M(x)$ and temperature $T(x)$ are Lorentz–scalars. In the rest–frame this becomes

$$f_{MR} = N_M e^{-P^0/T}$$

Because of the mass–shell condition (27), moments of the rest–frame Maxwellian have the form

$$\int_0^\infty \frac{ds}{\sqrt{1+s^2}} s^{2n} e^{-\zeta\sqrt{1+s^2}} = \frac{1 \cdot 2 \cdots (2n-1) K_n(\zeta)}{z^n}$$

where K_2 is a MacDonald function, $s = |\boldsymbol{p}|/m$, and $\zeta = m/T$. We use this formula to find the normalization

$$N_M = \frac{n_R e^{\Phi/T}}{4\pi m^2 T K_2(\zeta)}$$

where $\Phi = A^0$ is the electrostatic potential.

Thus the rest–frame Maxwellian is

$$f_{MR} = \frac{n_R e^{-p^0/T}}{4\pi m^2 T K_2(\zeta)} \tag{54}$$

Returning to (53), we now choose the parameters $\hat{\Delta}$ and $\hat{Q}$ to insure that the rest-frame density coincides with that of the Maxwellian alone, and that the rest-frame flow velocity vanishes. The result is

$$f(x,p) = f_M \left[1 + \Delta \left(\frac{e^{\alpha\beta} p_\alpha p_\beta}{m^2} - \frac{2K_3}{\zeta K_2} \right) + \frac{Q_\alpha b^{\alpha\beta} p_\beta}{m} \left(1 + \frac{K_2 U^\kappa p_\kappa}{m K_3} \right) \right]$$

Here and below all MacDonald functions are evaluated at ζ. Choosing the same coordinate orientation as in (47), we find the form of our distribution in the instantaneous rest–frame:

$$f_R = f_{MR} \left[1 + \Delta \left(\frac{\boldsymbol{p}^2 - p_3^2}{m^2} - \frac{2K_3}{\zeta K_2} \right) + \frac{Q_3 p_3}{m} \left(1 - \frac{K_2 p^0}{K_3 m} \right) \right] \tag{55}$$

Here Q_3 is the only non-vanishing component, in our special reference frame, of Q_α.

Scalar moments

Since any scalar can be computed in the rest–frame, we use (55) to compute the various scalar parameters of interest. In particular we find that the dynamical variable

$$p_{||} = \int \frac{d^3 p}{p^0} f_R p_3^2$$

turns out to be

$$p_{||} = n_R T$$

It follows that the parameter ζ has a simple expression,

$$\zeta = m n_R / p_{||} \tag{56}$$

in terms of the dynamical variables n_R and $p_{||}$.

Next we relate the two parameters Δ and Q^3 to dynamical variables. First we compute

$$p_{||} - p_\perp = -2 n_R T \Delta \frac{K_3}{\zeta K_2}, \tag{57}$$

allowing Δ to be expressed in terms of the density and pressures. Similarly, by computing the $(0,3)$–component of the stress tensor in the rest–frame, we find that

$$q_{||} = \frac{m n_r}{\zeta^2} Q_3 (1 + \zeta \mathcal{K}) \tag{58}$$

where we use the abbreviation

$$\mathcal{K} \equiv \frac{K_3}{K_2} - \frac{K_4}{K_3} \tag{59}$$

Finally we turn to the auxiliary parameters h and the m_k, which are also scalars and therefore computable in the rest–frame. For the enthalpy density we find

$$h = \frac{mn_R K_3}{K_2}\left(1 - \frac{2\Delta}{\zeta}\mathcal{K}\right) \tag{60}$$

This expression generalizes a well-known isotropic ($\Delta \to 0$) result (3). In view of (56) and (57), it expresses h in terms of the dynamical variables n_R, $p_{||}$ and $p_\perp$. Computing the m_k is more complicated but still straightforward; one first solves (47)–(52) for the m_k in terms of components of the rest–frame stress–flow. Then, after computing those components from their definitions and (55), one finds that

$$m_1 = \frac{m}{K_2}\left[K_3 p_{||} + (p_{||} - p_\perp)\left(K_3 - 2\frac{K_4}{K_3}K_2\right)\right] \tag{61}$$

$$m_2 = m(p_{||} - p_\perp)\frac{K_4}{K_3} \tag{62}$$

$$m_3 = q_{||}\frac{m\mathcal{K}}{1 + \zeta\mathcal{K}} \tag{63}$$

It is helpful to notice that

$$m_1 + m_2 = Th \tag{64}$$

5. Alternative forms

Gradients of projectors

For many applications it is convenient to express the fluid equations in terms of three–vectors, sacrificing manifest Lorentz covariance. We display the three–vector versions here. Since the three–vector form of the plasma flow coincides with the conventional MHD result and is given by (30), we need consider only the relations (37) for the parallel pressure and parallel flow, and the two constraints on stress–flow, (43) and (44).

Making our results explicit will require expressions for the gradients of the quasi-projectors $b^{\mu\nu}$ and $e^{\mu\nu}$. To this end we recall the Maxwell stress tensor

$$\Theta^{\alpha\beta} = F^{\alpha}_{\ \kappa}F^{\beta\kappa} - \frac{1}{4}\eta^{\alpha\beta}F_{\kappa\lambda}F^{\kappa\lambda}$$

and observe that

$$e^{\alpha\beta} = \frac{\eta^{\alpha\beta}}{2} + \frac{\Theta^{\alpha\beta}}{W} \tag{65}$$

$$b^{\alpha\beta} = \frac{\eta^{\alpha\beta}}{2} - \frac{\Theta^{\alpha\beta}}{W} \tag{66}$$

Then, since Maxwell's equations (15) and (14) imply

$$\frac{\partial \Theta^{\mu\nu}}{\partial x^\nu} = -F^{\mu\kappa} J_\kappa,$$

it is not hard to show that

$$\frac{\partial b_\mu{}^\nu}{\partial x^\nu} = \frac{F_{\mu\nu}}{W} J^\nu + \left(\frac{1}{2}\eta_\mu{}^\nu - b_\mu{}^\nu\right) \frac{\partial \log W}{\partial x^\nu}, \tag{67}$$

$$\frac{\partial e_\mu{}^\nu}{\partial x^\nu} = -\frac{F_{\mu\nu}}{W} J^\nu + \left(\frac{1}{2}\eta_\mu{}^\nu - e_\mu{}^\nu\right) \frac{\partial \log W}{\partial x^\nu}. \tag{68}$$

New conservation law

It is now straightforward to show that the conservation law, (45), follows from (43). We need to compute

$$e_{\alpha\beta} \frac{\partial M^{\alpha\beta\gamma}}{\partial x^\gamma} = e_{\alpha\beta} \frac{\partial}{\partial x^\gamma} \left(m_1 M_1^{\alpha\beta\gamma} + m_2 M_2^{\alpha\beta\gamma} + m_3 M_3^{\alpha\beta\gamma} \right) \tag{69}$$

where the $M_i^{\alpha\beta\gamma}$ are given by (40)–(42). Observing first that neither k^α nor U^α survive perpendicular projection, we have

$$e_{\alpha\beta} \frac{\partial}{\partial x^\gamma} \left(m_k U^\alpha U^\beta U^\gamma \right) = 0,$$

$$e_{\alpha\beta} \frac{\partial}{\partial x^\gamma} \left(m_k U^{\{\alpha} U^\beta k^{\gamma\}} \right) = 0$$

It is similarly straightforward to show that

$$e_{\alpha\beta} U^\gamma \frac{\partial b^{\alpha\beta}}{\partial x^\gamma} = 0.$$

On the other hand (68) and (21) can be used to show that

$$e_{\alpha\beta} \frac{\partial}{\partial x^\gamma} \eta^{\{\alpha\beta} U^{\gamma\}} = -2U^\gamma \frac{\partial}{\partial x^\gamma} (\log n_R + \log W^{1/2})$$

A similar calculation gives

$$e_{\alpha\beta}\frac{\partial}{\partial x^\gamma}\eta^{\{\alpha\beta}k^{\gamma\}} = 2\frac{\partial k^\gamma}{\partial x^\gamma} - k^\gamma\frac{\partial \log W}{\partial x^\gamma}$$

Substituting these results into (69) we obtain

$$2\sqrt{W}\frac{\partial}{\partial x^\gamma}\left(\frac{m_1 U^\gamma + m_3 k^\gamma}{\sqrt{W}}\right) = 0$$

which is equivalent to (45).

To express the conservation law in terms of three–vectors, we recall (36) and find

$$W^{1/2}G^0 = \gamma\left(m_1 + m_3\sqrt{\frac{B^2}{W}}V_\parallel\right) \tag{70}$$

$$W^{1/2}\boldsymbol{G} = \gamma\left[\boldsymbol{b}\left(m_1 V_\parallel + m_3\sqrt{\frac{W}{B^2}}\right) + \boldsymbol{V}_E\left(m_1 + m_3\sqrt{\frac{B^2}{W}}V_\parallel\right)\right] \tag{71}$$

where the m_k are given by (61) and (63).

Other equations

The remaining three equations can be treated similarly to (43); we omit the details and show only the results. It is convenient to use the identity

$$U^\nu\frac{\partial}{\partial x^\nu} = \gamma\frac{d}{dt}$$

where $d/dt = \partial/\partial t + \boldsymbol{V}\cdot\nabla$ is the conventional convective derivative. We also use the analogous abbreviation

$$k^\mu\frac{\partial}{\partial x^\mu} \equiv \frac{d}{ds}$$

Now the explicit form of (44) can be expressed as

$$\begin{aligned} 5m_3\gamma\frac{d\log(m_3 n_R^{-6/5})}{dt} + (m^2 n_R \quad &+ \quad 5Th - 2m_3)\gamma^2\boldsymbol{k}\cdot\frac{d\boldsymbol{V}}{dt} \\ +\frac{dTh}{ds} - m_2\frac{d\log\sqrt{W}}{ds} \quad &+ \quad 7m_3\gamma\boldsymbol{k}\cdot\frac{d\boldsymbol{V}}{ds} = ehE_\parallel \end{aligned} \tag{72}$$

Here we have recalled (64).

Turning to (37), we express its temporal component as

$$\sqrt{W}\nabla_{\|}\left(\frac{p_{\|}}{\sqrt{W}}\right)+\frac{p_{\perp}}{2}\nabla_{\|}\log W+\gamma n_R \boldsymbol{b}\cdot\frac{d}{dt}\left(\frac{h\gamma\boldsymbol{V}+\boldsymbol{q}}{n_R}\right)$$
$$+q_{\|}\boldsymbol{b}\cdot\frac{d\gamma\boldsymbol{V}}{ds}+\gamma V_{\|}\left(\frac{\partial q^0}{\partial t}+\nabla\cdot\boldsymbol{q}\right)=\gamma e n_R E_{\|} \tag{73}$$

The scalar product of the spatial components of (37) with $\boldsymbol{b}$ conveys all the information in those components; its rather lengthy form simplifies when it is added to the product of $V_{\|}$ with (73). We thus find that

$$\sqrt{W}\frac{d}{dt}\frac{p_{\|}}{\sqrt{W}}+\frac{p_{\perp}}{2}\frac{d\log W}{dt}-n_R\frac{d}{dt}\frac{h}{n_R}-\gamma\boldsymbol{q}\cdot\frac{d\boldsymbol{V}}{dt}-\frac{1}{\gamma}\left(\frac{\partial q^0}{\partial t}+\nabla\cdot\boldsymbol{q}\right)=0 \tag{74}$$

An alternative derivation of (74) is of interest. Instead of combining the two independent components of (37) as we have done, one can begin with the scalar relation

$$\Gamma_\mu\frac{\partial T^{\mu\nu}}{\partial x^\nu}=0 \tag{75}$$

which follows easily from (22) because of indicial symmetries. It is not hard to show that (75) reproduces (74) directly. A similar comment pertains to (44): indicial symmetry implies the exact constraint

$$T_{\mu\nu}\frac{\partial M^{\mu\nu\kappa}}{\partial x^\kappa}=0 \tag{76}$$

After substituting from (33), one finds that the terms in $T_{\mu\nu}$ involving $b_{\mu\nu}$ and $e_{\mu\nu}$ cannot contribute, so that (76) immediately reduces to

$$(hU_\mu U_\nu+q_\mu U_\nu+U_\mu q_\nu)\frac{\partial M^{\mu\nu\kappa}}{\partial x^\kappa}=0$$

From here, a straightforward but more complicated argument shows that

$$U_\mu U_\nu\frac{\partial M^{\mu\nu\kappa}}{\partial x^\kappa}=-2eq_{\|}E_{\|}+2\frac{h-p_{\|}}{n_R^2}\Gamma_\mu\frac{\partial T^{\mu\nu}}{\partial x^\nu}$$

where the second term on the right–hand side vanishes by (75). Hence (76) is equivalent to (44).

6. Closure summary

Maxwell's equations are closed in a magnetized plasma by expressing the four–vector current density in terms of the stress tensor of the plasma,

$$\mathcal{T}^{\mu\nu}=\sum_{species}T^{\mu\nu}.$$

where $T^{\mu\nu}$ is the stress of a single plasma species. This closure procedure (13) is implicit, at least, in most textbook descriptions of a magnetized plasma; its relativistic expression is given by (16) *et seq.*

A closed fluid description of plasma dynamics is therefore contained in a set of equations that fix the evolution of the stress tensor $T^{\mu\nu}$ of each plasma species. The crucial step in our closure is the observation that the electromagnetic force imposes a certain form on the stress, given by (33):

$$T^{\mu\nu} = b^{\mu\nu}p_{||} + e^{\mu\nu}p_{\perp} + hU^{\mu}U^{\nu} + q_{||}\left(k^{\mu}U^{\nu} + U^{\mu}k^{\nu}\right) \tag{77}$$

Here $b^{\mu\nu}$ and $e^{\mu\nu}$ are quasi-projection operators defined in Section 2, while the fluid four–velocity U^{μ} and heat flux density $q_{||}k^{\mu}$ are constrained by

$$F^{\mu}_{\ \nu}U^{\nu} = 0 \tag{78}$$
$$e^{\mu}_{\ \nu}k^{\nu} = 0 \tag{79}$$
$$U_{\nu}k^{\nu} = 0 \tag{80}$$

We have noted that (78) reproduces the familiar $\boldsymbol{E}\times\boldsymbol{B}$ drift for the motion perpendicular to the magnetic field. Thus Γ^{μ} has two independent components, corresponding to the rest–frame density n_R and the parallel flow $V_{||}$, while q^{μ} has a single independent component, corresponding to the parallel flow of heat in the rest–frame, $q_{||}$. The remaining three parameters in the stress tensor—$p_{||}$, $p_{\perp}$ and h—are Lorentz scalars.

The relativistic expression of quasineutrality, (18), forces n_R to be the same for both species; the other quantities appearing in $T^{\mu\nu}$ will generally differ between species. Thus the stress tensor for each species is determined by six parameters. Five of these,

n_R, $p_{||}$, $p_{\perp}$, $V_{||}$ and $q_{||}$

constitute the basic dynamical variables of our system.

The evolution of the five dynamical variables is set by the following five evolution equations:

$$\frac{\partial\Gamma^{\nu}}{\partial x^{\nu}} = 0 \tag{81}$$
$$\mathcal{F}_{\mu\kappa}\frac{\partial T^{\kappa\nu}}{\partial x^{\nu}} = eE_{||}B\Gamma_{\mu} \tag{82}$$
$$e_{\alpha\beta}\frac{\partial M^{\kappa\beta\alpha}}{\partial x^{\kappa}} = 0 \tag{83}$$
$$(U_{\alpha}k_{\beta} + U_{\beta}k_{\alpha})\frac{\partial M^{\kappa\alpha\beta}}{\partial x^{\kappa}} = 2eE_{||}h \tag{84}$$

where (82) constitutes two independent equations, and where $M^{\alpha\beta\gamma}$ is the stress–flow tensor of (26). These relations hold to lowest order in the small–gyroradius ordering; they follow directly from the definition of a magnetized plasma (Section 2) and the neglect of collisions.

The magnetized form of the stress–flow tensor depends on three additional scalar functions, m_k, which are grouped with h in a set of four auxiliary parameters. These quantities are computed (that is, expressed in terms of the dynamical variables) from a representative distribution function, which is chosen to have the simplest form consistent with magnetization, anisotropy and heat flow. The distribution is parametrized by the dynamical variables, and therefore evolves according (81)–(84).

The parametrization of the distribution is displayed in (53) *et seq.*; the enthalpy is expressed in terms of the dynamical variables by (60),

$$h = \frac{mn_R K_3}{K_2}\left(1 - \frac{2\Delta}{\zeta}\mathcal{K}\right); \tag{85}$$

and the m_k by (61)–(63).

Our fluid description of a magnetized plasma is relatively simple; it involves only two additional variables, the anisotropy and heat flow, beyond ordinary MHD. The entire closed set, in manifestly covariant form, fits on a single page. Even the more complicated three–vector form, given by (70)–(74), is hardly forbidding.

One of the equations in our closure, (83), is found to be expressible as a conservation law,

$$\frac{\partial G^\nu}{\partial x^\nu} = 0.$$

The conserved four–vector,

$$G^\mu \equiv W^{-1/2}(m_1 U^\mu + m_3 k^\mu) \tag{86}$$

involves a novel combination of fluid and electromagnetic variables.

The system derived here is intended to allow detailed studies of astrophysical and cosmic plasmas at a level more realistic than MHD. Its nonrelativistic limit, now under investigation, should similarly assist 'post–MHD' investigations of magnetized laboratory plasmas.

Appendix A

Here we verify the constraints (43) and (44) by manipulation of (23).

Consider first

$$e_{\alpha\beta}(F^{\alpha\nu}T_\nu{}^\beta + F^{\beta\nu}T_\nu{}^\alpha) = (F_\beta{}^\nu - \lambda\mathcal{F}_\beta{}^\nu)T_\nu{}^\beta + (F_\alpha{}^\nu - \lambda\mathcal{F}_\alpha{}^\nu)T_\nu{}^\alpha$$

where we have used the (exact) identity (10). However the first term on the right–hand side is

$$F_\beta{}^\nu T_\nu{}^\beta = F^{\beta\nu} T_{\nu\beta} = 0$$

because the first factor is antisymmetric in its indices and the second symmetric. Applying the same argument to each term, we conclude

$$e_{\alpha\beta}(F^{\alpha\nu} T_\nu{}^\beta + F^{\beta\nu} T_\nu{}^\alpha) = 0.$$

Hence (23) implies (43). Significantly, the smallness of δ or λ plays no role in the present argument, which uses exact relations exclusively.

The remaining constraint is analogous to (37) and derived in the same way. Recalling (28) one quickly finds that

$$(U_\alpha k_\beta + U_\beta k_\alpha) \frac{\partial M^{\kappa\alpha\beta}}{\partial x^\kappa} = 2e(U_\alpha k_\beta + U_\beta k_\alpha) F_{(1)}^{\alpha\nu} T_\nu{}^\beta$$

where $F_{(1)}$ is as usual the part of the Faraday tensor that contains the parallel electric field, and all other quantities are evaluated in lowest order ($\delta = 0$). The scalar on the right–hand side of this equation can be evaluated in the instantaneous rest–frame; one finds that

$$(U_\alpha k_\beta + U_\beta k_\alpha) F_{(1)}^{\alpha\nu} T_\nu{}^\beta = -hE_{\|}$$

and the constraint (44) follows.

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