The Twisted Top

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Abstract

We describe a new type of top, the twisted top, obtained by appending a cocycle to the Lie–Poisson bracket for the charged heavy top, thus breaking its semidirect product structure. The twisted top has an integrable case that corresponds to the Lagrange (symmetric) top. We give a canonical description of the twisted top in terms of Euler angles. We also show by a numerical calculation of the largest Lyapunov exponent that the Kovalevskaya case of the twisted top is chaotic.

1 Introduction

We present a new top, called the twisted top, obtained by modifying the Lie–Poisson bracket for the charged heavy top. The charged heavy top, also introduced in this paper, is a heavy top [1–3] immersed in an electric field. The bracket for the charged heavy top

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arises from a semidirect product of SO(3) and $\mathbb{R}^3 \times \mathbb{R}^3$. The twisted top is not a top in the classical sense of a rigid body in a gravitational field. Rather, it is a mathematical construction obtained by using a different Lie group to build the Lie–Poisson bracket for the system. This abstract procedure is analogous to the manner in which tops are derived for SO(N) [4], for SU(N) (obtained in Hamiltonian truncations of the Euler equation [5,6]), and for other groups [7,8]. The construction method of the twisted top is also related to tops obtained by deformations of algebras [9].

The bracket for the twisted top results from adding a cocycle to the charged heavy top bracket, so that the structure is no longer semidirect. Such brackets are classified in Thiffeault [10] and Thiffeault and Morrison [11], and the case we are considering is the simplest example of a Leibniz extension [10–12]. Because we are interested in how the nontrivial cocycle affects the dynamics of the system, we use the same Hamiltonian for the twisted top as for the heavy top. The bracket for the twisted top possesses three Casimir invariants, one of which differs from that possessed by the charged heavy top.

A most interesting feature of the twisted top is that it retains the integrability property of the Lagrange top: it is integrable when it has an axis of symmetry (two moments of inertia are equal), its centre of rotation lies on the symmetry axis, and the electric field vanishes (or, equivalently, the top is uncharged). The conserved quantities are the energy, the angular momentum along the symmetry axis, and a third invariant which is a modification of the conserved component of the canonical momentum in the Lagrange case.

The outline of this paper is as follows. In Section 2 we discuss the charged heavy top and describe its invariants and some of its integrable cases. In Section 3 we introduce the twisted top and its invariants. We show that it has an integrable case analogous to the Lagrange case of the heavy top. We give a canonical description of the twisted top in terms of Euler

angles in Section 4. In canonical coordinates the difference between the twisted top and the charged heavy top is transferred from the bracket to the Hamiltonian, and appears as a term that can be interpreted as a momentum-dependent potential. Finally, in Section 5 we discuss our results and show by numerical calculation that the Kovalevskaya case of the twisted top is not integrable.

2 The Charged Heavy Top

Consider a heavy, charged top in constant gravitational and electric fields. The angular momentum vector is denoted by $\boldsymbol{\ell}$, the position of the centre of mass is a vector $\mathbf{a}$, and the position of the centre of charge is $\mathbf{b}$. The direction and strength of the fixed gravitational and electric forces are given by the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively. The frame of reference is the body frame, so that $\mathbf{a}$ and $\mathbf{b}$ are constant. The energy of such a top is

$$H(\boldsymbol{\ell}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \tfrac{1}{2}\boldsymbol{\ell} \cdot \boldsymbol{\omega} + \boldsymbol{\alpha} \cdot \mathbf{a} + \boldsymbol{\beta} \cdot \mathbf{b} \tag{1}$$

where $\boldsymbol{\omega} := I^{-1}\boldsymbol{\ell}$ is the angular velocity and I is the moment of inertia tensor, which can be taken to be diagonal by an appropriate choice of frame. We assume that the top's rotation is slow enough that the magnetic fields set up by the motion of charges is negligible, and that the top is a perfect insulator, so that the centre of charge remains fixed within the body. The charge, like the mass, does not have to be distributed uniformly, but only the centres of charge and mass couple to uniform gravitational and electric fields.

The vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, being fixed in space, rotate in the body frame. The dynamics of such a configuration can be generated by a Lie–Poisson bracket with a semidirect product

structure,

$$\{f\,,g\}_{\rm SD} = -\boldsymbol{\ell}\cdot(\nabla_{\boldsymbol{\ell}}\,f\times\nabla_{\boldsymbol{\ell}}\,g) - \boldsymbol{\alpha}\cdot(\nabla_{\boldsymbol{\ell}}\,f\times\nabla_{\boldsymbol{\alpha}}\,g + \nabla_{\boldsymbol{\alpha}}\,f\times\nabla_{\boldsymbol{\ell}}\,g) \\ - \boldsymbol{\beta}\cdot(\nabla_{\boldsymbol{\ell}}\,f\times\nabla_{\boldsymbol{\beta}}\,g + \nabla_{\boldsymbol{\beta}}\,f\times\nabla_{\boldsymbol{\ell}}\,g)\,, \quad (2)$$

where f and g are functions of $(\boldsymbol{\ell},\boldsymbol{\alpha},\boldsymbol{\beta})$, and ∇ is a gradient with respect to its subscript. Equation (2) is a simple extension of the bracket for the heavy top, which also has a semidirect product structure [2, 3, 13, 14] (without $\boldsymbol{\alpha}$). The Casimir invariants of Eq. (2) are

$$C_1 = \|\boldsymbol{\alpha}\|^2\,, \quad C_2 = \boldsymbol{\alpha}\cdot\boldsymbol{\beta}, \quad C_3 = \|\boldsymbol{\beta}\|^2\,.$$

The invariant C_2 says that the angle between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is constant, because by C_1 and C_3 their length is conserved. Therefore, the two vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ fully describe the orientation of the rigid body, and there is a one-to-one mapping between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and Euler angles. The phase space of the motion is thus $\mathrm{SO}(3)\times\mathbb{R}^3 \cong T^*\mathrm{SO}(3)$. This is the same phase space as in the unreduced (canonical) system [15].

For the case with $I_1 = I_2$, $\mathbf{a} = (0,0,a_3)^T$, and $\mathbf{b} = 0$, the charged heavy top reduces to the Lagrange top, also called the heavy symmetric top, and so is integrable (see for example Audin [16]). The invariants are the energy H, ℓ_3, and $\boldsymbol{\ell}\cdot\boldsymbol{\alpha}$. (There is also an integrable case with $\mathbf{b} = (0,0,b_3)^T$ and $(\boldsymbol{\alpha}\times\boldsymbol{\beta})\cdot\mathbf{b} = 0$, i.e., where the forces are in the equatorial plane.)

3 The Twisted Top

In Thiffeault [10] and Thiffeault and Morrison [11], it is shown that the only bracket extension of two field variables (such as $\boldsymbol{\ell}$ and $\boldsymbol{\alpha}$) is of the semidirect product type. To obtain an extension that is not semidirect, one requires at least three variables, which we take to be the same variables as for the charged heavy top. The simplest non-semidirect

extension is then the Leibniz bracket

$$\{f\,,g\}_{\rm Leib} = \{f\,,g\}_{\rm SD} - \varepsilon\,\boldsymbol{\beta}\cdot(\nabla_{\boldsymbol{\alpha}}\,f\times\nabla_{\boldsymbol{\alpha}}\,g)\,, \tag{3}$$

where ε is a parameter measuring the deviation from a semidirect bracket and is not necessarily small. Using the same Hamiltonian (1) as for the charged heavy top in the bracket (3), we obtain the equations

$$\dot{\boldsymbol{\ell}} = \{\boldsymbol{\ell}\,,H\}_{\rm Leib} \;= \boldsymbol{\ell}\times\boldsymbol{\omega} + \boldsymbol{\alpha}\times\mathbf{a} + \boldsymbol{\beta}\times\mathbf{b}, \tag{4}$$

$$\dot{\boldsymbol{\alpha}} = \{\boldsymbol{\alpha}\,,H\}_{\rm Leib} = \boldsymbol{\alpha}\times\boldsymbol{\omega} + \varepsilon\,\boldsymbol{\beta}\times\mathbf{a}, \tag{5}$$

$$\dot{\boldsymbol{\beta}} = \{\boldsymbol{\beta}\,,H\}_{\rm Leib} = \boldsymbol{\beta}\times\boldsymbol{\omega}. \tag{6}$$

These are the equations for the twisted top. The term proportional to ε in the $\dot{\boldsymbol{\alpha}}$ equation adds a "twist" which means that $\boldsymbol{\alpha}$ does not simply rotate rigidly (though $\boldsymbol{\beta}$ still does). This is reflected in the Casimir invariants, which are now

$$C_1 = \|\boldsymbol{\alpha}\|^2 + 2\varepsilon\,\boldsymbol{\ell}\cdot\boldsymbol{\beta}, \quad C_2 = \boldsymbol{\alpha}\cdot\boldsymbol{\beta}, \quad C_3 = \|\boldsymbol{\beta}\|^2\,. \tag{7}$$

Since the length of $\boldsymbol{\alpha}$ is no longer preserved, the invariant C_2 does not imply that the angle between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is constant. However, the length of the *projection* of $\boldsymbol{\alpha}$ onto $\boldsymbol{\beta}$ is preserved.

For a positive-definite moment of inertia tensor, the energy surfaces of the twisted top are bounded, as can be seen from the following argument. First note that the components of $\boldsymbol{\ell}$ cannot diverge without $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ also diverging, since $\boldsymbol{\ell}$ enters the Hamiltonian in a positive-definite quadratic form. From the invariant C_3, the components of $\boldsymbol{\beta}$ are finite. To have unbounded surfaces, and still conserve C_1, both $\|\boldsymbol{\ell}\|$ and $\|\boldsymbol{\alpha}\|$ must go to infinity, with $\|\boldsymbol{\alpha}\|^2 \sim -2\varepsilon\,\boldsymbol{\ell}\cdot\boldsymbol{\beta}$. But with this functional relation it is not possible to have $\|\boldsymbol{\ell}\| \to \infty$ whilst preserving the Hamiltonian H, since its kinetic part is proportional to $\|\boldsymbol{\ell}\|^2$ and its potential part to $\|\boldsymbol{\alpha}\| \sim \|\boldsymbol{\ell}\|^{1/2}$, precluding any balance. We conclude that the energy

surfaces are bounded. This will be important in Section 5 where we try to demonstrate chaotic behaviour by computing the largest Lyapunov exponent.

The twisted top also has an integrable Lagrange case. It is obtained, as for the charged heavy top, by letting $I_1 = I_2$, $\mathbf{a} = (0, 0, a_3)^T$, and $\mathbf{b} = 0$. The energy H and the third component of the angular momentum ℓ_3 are still conserved, whereas the third invariant becomes

$$P = \boldsymbol{\ell} \cdot \boldsymbol{\alpha} + \varepsilon\, I_1 a_3\, \beta_3\,. \tag{8}$$

We call this integrable case the twisted Lagrange top. We can verify that P is conserved directly from the equations of motion (4)–(6),

$$\begin{aligned}
\dot{P} &= \dot{\boldsymbol{\ell}} \cdot \boldsymbol{\alpha} + \boldsymbol{\ell} \cdot \dot{\boldsymbol{\alpha}} + \varepsilon\, I_1 a_3 \dot{\beta}_3 \\
&= (\boldsymbol{\ell} \times \boldsymbol{\omega}) \cdot \boldsymbol{\alpha} + \boldsymbol{\ell} \cdot (\boldsymbol{\alpha} \times \boldsymbol{\omega} + \varepsilon\, \boldsymbol{\beta} \times \mathbf{a}) + \varepsilon\, I_1\, a_3\, (\boldsymbol{\beta} \times \boldsymbol{\omega})_3 \\
&= \varepsilon\, \mathbf{a} \cdot (\boldsymbol{\ell} \times \boldsymbol{\beta}) + \varepsilon\, \mathbf{a} \cdot (\boldsymbol{\beta} \times I_1\, \boldsymbol{\omega}) = 0,
\end{aligned}$$

where we equated $I_1\, \boldsymbol{\omega}$ to $\boldsymbol{\ell}$ in the last triple product because only the first two components of $\boldsymbol{\omega}$ are involved, and $I_1 = I_2$. It is straightforward to verify that the invariants $\{H, \ell_3, P\}$ are in involution, i.e., they commute with respect to the bracket (3)—a necessary condition for integrability. The commutativity of the invariants carries over to the canonical variables of Section 4.

4 Canonical Description

Since the twisted top is a Hamiltonian system, there exists a coordinate transformation on the symplectice leaves (the constraint surfaces described by the Casimirs) that makes the system canonical. We now proceed to find such a coordinate transformation, in a manner analogous to the reduction of the rigid body and the heavy top [14,17]. The transformation we

describe will be from the three Euler angles $\mathbf{q} = (\phi, \psi, \theta)^T$ and their corresponding canonical momenta $\mathbf{p} = (p_\phi, p_\psi, p_\theta)^T$ (6 coordinates) to the vectors $(\boldsymbol{\ell}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ (9 coordinates, 3 Casimirs). We show that the transformation is invertible on the symplectic leaves, so that it can be used to canonize the system.

Following the heavy top reduction [14], since the vector $\boldsymbol{\beta}$ rotates rigidly (length conserved), it is fixed in the space frame, and we write

$$\boldsymbol{\beta} = A(\mathbf{q})\,\mathbf{w}, \tag{9}$$

where the rotation matrix A is

$$\begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}.$$

The matrix A transforms vectors from the space frame to the body frame (we are following the convention of Goldstein [18, p. 147] for the definition of A.) The vector $\mathbf{w}$ is constant and fixed in space. Since rotations preserve lengths, we have $C_3 = \|\boldsymbol{\beta}\|^2 = w^2$, where $w = \|\mathbf{w}\| \geq 0$.

For the angular momentum, we take

$$\boldsymbol{\ell} = L(\mathbf{q})\,\mathbf{p}, \tag{10}$$

where L is more concisely defined via its inverse,

$$L^{-1} := \begin{pmatrix} \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \\ 0 & 0 & 1 \\ \cos\psi & -\sin\psi & 0 \end{pmatrix}.$$

This is the usual transformation one makes when reducing the rigid body [17, p. 499], where L^{-1} maps $T^*\mathrm{SO}(3)$ to $\mathfrak{g}^*$.

Finally, for $\boldsymbol{\alpha}$ we try the form

$$\boldsymbol{\alpha} = A(\mathbf{q})\,\mathbf{v}(\mathbf{q},\mathbf{p}), \tag{11}$$

where we have allowed $\mathbf{v}$ to depend on the Euler angles and canonical momenta in an effort to conserve the Casimirs C_1 and C_2. We then have

$$C_2 = \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = \mathbf{v}\cdot A^T A\,\mathbf{w} = \mathbf{v}\cdot\mathbf{w},$$

where we used the orthogonality of A. Decomposing $\mathbf{v}$ into a part $v_\perp\,\hat{\mathbf{e}}_\perp$ perpendicular to $\mathbf{w}$ and a part $v\,\hat{\mathbf{e}}_\parallel$ parallel to $\mathbf{w}$, we obtain $C_2 = v\,w$. Since w is constant, we require v to also be constant.

The norm of $\boldsymbol{\alpha}$ is

$$\|\boldsymbol{\alpha}\|^2 = \mathbf{v}\cdot A^T A\,\mathbf{v} = v_\perp^2 + v^2 = C_1 - 2\varepsilon\,\boldsymbol{\ell}\cdot\boldsymbol{\beta},$$

where we used the definition (7) of C_1. We solve this for $v_\perp^2$,

$$v_\perp^2 = C_1 - v^2 - 2\varepsilon\,\boldsymbol{\ell}\cdot\boldsymbol{\beta} = \|\boldsymbol{\alpha}\|^2 - \frac{(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})^2}{\boldsymbol{\beta}^2} \geq 0, \tag{12}$$

with $v_\perp^2 = 0$ if and only if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are collinear. For convenience, define the constant

$$\eta \coloneqq C_1 - v^2 = \|\boldsymbol{\alpha}\|^2 - \frac{(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})^2}{\boldsymbol{\beta}^2} + 2\varepsilon\,\boldsymbol{\ell}\cdot\boldsymbol{\beta},$$

which we will use from now on instead of C_1. Then Eq. (11) becomes

$$\boldsymbol{\alpha} = A\left[v\,\hat{\mathbf{e}}_\parallel + \sqrt{\eta - 2\varepsilon\,\boldsymbol{\ell}\cdot\boldsymbol{\beta}}\,\hat{\mathbf{e}}_\perp\right] \tag{13}$$

Note that the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are collinear ($\boldsymbol{\alpha}\times\boldsymbol{\beta} = 0$) if and only if $\eta = 2\varepsilon\,\boldsymbol{\ell}\cdot\boldsymbol{\beta}$. Assume that they are initially not collinear ($\eta \neq 2\varepsilon\,\boldsymbol{\ell}\cdot\boldsymbol{\beta}$). The time evolution of $2\varepsilon\,\boldsymbol{\ell}\cdot\boldsymbol{\beta}$ is obtained

from (5) and the conservation of C_1, yielding

$$2\varepsilon \frac{d}{dt}(\boldsymbol{\ell} \cdot \boldsymbol{\beta}) = -\frac{d}{dt} \|\boldsymbol{\alpha}\|^2 = -2\varepsilon\, \mathbf{a} \cdot (\boldsymbol{\alpha} \times \boldsymbol{\beta}). \tag{14}$$

If $\eta = 2\varepsilon\, \boldsymbol{\ell} \cdot \boldsymbol{\beta}$ initially, then it remains so for all times, because then the right-hand side of (14) vanishes. Conversely, if $\eta \neq 2\varepsilon\, \boldsymbol{\ell} \cdot \boldsymbol{\beta}$ initially, then the two vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are never collinear.

This is crucial because it tells us that we can always invert Eqs. (9) and (13) for (ϕ, ψ, θ), as long as $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are not initially collinear. The inversion is done as follows: take $\mathbf{w}$ as the spatial z-axis. Then $\boldsymbol{\beta}$ is z', the transformed z-axis, which allows us to determine ψ and θ, but not ϕ since it represents a rotation about the z-axis. We then use $\mathbf{v}_\perp$ to define the x-axis, which allows us to find ϕ from $\boldsymbol{\alpha}$ (as long as $\mathbf{v}_\perp \neq 0$, but we showed that it is sufficient to require this initially). But since the $2\varepsilon\, \boldsymbol{\ell} \cdot \boldsymbol{\beta}$ term only affects the magnitude of $\mathbf{v}_\perp$, not its orientation, we conclude that the Euler angles are only a function of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, not of $\boldsymbol{\ell}$. We can then go back and solve (10) for the canonical momenta. (Provided $\det L^{-1} = \sin\theta \neq 0$, the coordinate singularity inherent to Euler angles. This singularity can be avoided by "inflating" the phase space [19].)

It remains to be shown that the canonical coordinates do indeed transform the bracket (3) to canonical form.

In canonical coordinates, the 'potential' part of the Hamiltonian (1) becomes

$$\begin{aligned} V(\boldsymbol{\alpha}, \boldsymbol{\beta}) &= \boldsymbol{\alpha} \cdot \mathbf{a} + \boldsymbol{\beta} \cdot \mathbf{b} \\ &= (v\, \mathbf{a} + w\, \mathbf{b}) \cdot A\, \hat{\mathbf{e}}_\parallel + \sqrt{\eta - 2\varepsilon\, \boldsymbol{\ell} \cdot \boldsymbol{\beta}}\, \mathbf{a} \cdot A\, \hat{\mathbf{e}}_\perp \end{aligned} \tag{15}$$

with

$$\boldsymbol{\ell} \cdot \boldsymbol{\beta} = \mathbf{p} \cdot L^T(\mathbf{q})\, A(\mathbf{q})\, \mathbf{w}.$$

The matrix $L^T A$ is

$$L^T A = \begin{pmatrix} -\cot\theta\sin\phi & \cot\theta\cos\phi & 1 \\ \csc\theta\sin\phi & -\csc\theta\cos\phi & 0 \\ \cos\phi & \sin\phi & 0 \end{pmatrix}.$$

The integrable case of the twisted top has $I_1 = I_2$, $\mathbf{a} = (0, 0, a_3)^T$, $\mathbf{b} = 0$, for which the kinetic energy is independent of ϕ and ψ, and the potential becomes

$$V(\phi, \theta, \mathbf{p}) = a_3(v\,\hat{\mathbf{e}}_3 \cdot A\,\hat{\mathbf{e}}_\parallel + \sqrt{\eta - 2\varepsilon\,\boldsymbol{\ell}\cdot\boldsymbol{\beta}}\;\hat{\mathbf{e}}_3 \cdot A\,\hat{\mathbf{e}}_\perp).$$

Note that both $L^T A$ and $\hat{\mathbf{e}}_3 \cdot A = (\sin\theta\sin\phi, -\sin\theta\cos\phi, \cos\theta)$ are independent of ψ, so that in the twisted Lagrange top case ψ is cyclic (p_ψ conserved).

A particularly simple choice is $\hat{\mathbf{e}}_\parallel = \hat{\mathbf{e}}_3$, $\hat{\mathbf{e}}_\perp = \hat{\mathbf{e}}_1$, $w = 1$, for which the potential is

$$V(\phi, \theta, \mathbf{p}) = a_3(v\cos\theta + \sqrt{\eta - 2\varepsilon\,p_\phi}\,\sin\theta\sin\phi).$$

Though simple, this description does not reduce nicely to the Lagrange top when $\varepsilon \to 0$. The choice $\hat{\mathbf{e}}_\perp = \hat{\mathbf{e}}_3$, $\hat{\mathbf{e}}_\parallel = \hat{\mathbf{e}}_1$, $w = 1$, which gives

$$V(\phi, \theta, \mathbf{p}) = a_3(v\,\sin\theta\sin\phi + \sqrt{\eta - 2\varepsilon\,\boldsymbol{\ell}\cdot\boldsymbol{\beta}}\,\cos\theta),$$

with

$$\boldsymbol{\ell}\cdot\boldsymbol{\beta} = -p_\phi\cot\theta\sin\phi + p_\psi\csc\theta\sin\phi + p_\theta\cos\phi.$$

has the Lagrange top form when $v = \varepsilon = 0$, at the cost of a more complicated expression for $\boldsymbol{\ell}\cdot\boldsymbol{\beta}$.

5 Discussion

We have introduced a simple generalisation of the heavy top by giving it charge and placing it in a constant electric field. By deforming the bracket of this charged heavy top, we have obtained a new top that we call twisted. We have found that twisted top possesses an integrable case analogous to the Lagrange top. It is then natural to ask if such a deformation always preserves integrability. An affirmative answer would be very surprising, considering the delicate nature of integrable systems, and indeed it does not seem to be so for the system at hand.

We investigate this by looking at the Kovalevskaya case of the twisted top. For the uncharged heavy top ($\mathbf{b} = 0$, $\varepsilon = 0$), the Kovalevskaya case involves setting $I_1 = I_2$, $I_3 = 2I_1$, and $a_3 = 0$. This top is integrable [16, 20]. The analogous case for the twisted top, as for the Lagrange top, simply involves a change of bracket by setting $\varepsilon \neq 0$. Specifically, we choose $\varepsilon = 1$ and $\mathbf{a} = (-1, 0, 0)^T$.

Figure 1 shows a plot of the instantaneous largest Lyapunov exponent of the twisted Kovalevskaya top, after averaging over $20,000$ random initial conditions integrated numerically (dotted line). The solid line is a least-squares fit to help determine the Lyapunov exponent to greater accuracy, using an asymptotic form of the averaged exponent [21]. The Lyapunov exponent is $\lambda \simeq 0.122$, suggesting that the twisted Kovalevskaya top is chaotic. For comparison, the dashed line shows the same calculation for the untwisted (ordinary) Kovalevskaya top, which shows the Lyapunov exponent going to zero. Thus, the twisted case appears to be chaotic, whilst the untwisted case is not. We conclude that integrability does not always survive deformation, contrary to the Lagrange case. Note that we can infer chaos from a positive Lyapunov exponent because we showed in Section 3 that the motion takes place in a bounded region of phase space [22]. The presence of chaos does not rule out the existence

of an integrable case with parameters "close" to the Kovalevskaya values (in the sense of differing only by terms involving ε). We have not found such a case.

A rigorous demonstration that the twisted Kovalevskaya top is chaotic could in principle be achieved using a Melnikov analysis, as was done by Holmes and Marsden [14] for the heavy top. The present case is less straightforward because of the complicated form of the homoclinic orbits.

It would of course be of great value to find a physical realisation of the twisted top. One could use either the noncanonical picture, Eqs (4)–(6), or the canonical picture, given by the standard free rigid body Hamiltonian [14] with Eq. (15) for a potential. Regardless of the physical interpretation, the twisted top remains an object worthy of study in its own right, because of its interesting integrable case and peculiar geometry. It would be worthwhile to carry out a topological classification of the bifurcations of the phases space of twisted Lagrange top, as was done by Dullin et al. [23] for the Kovalevskaya top. This is complicated by the need to find a good surface to make Poincaré sections in the canonical coordinate space.

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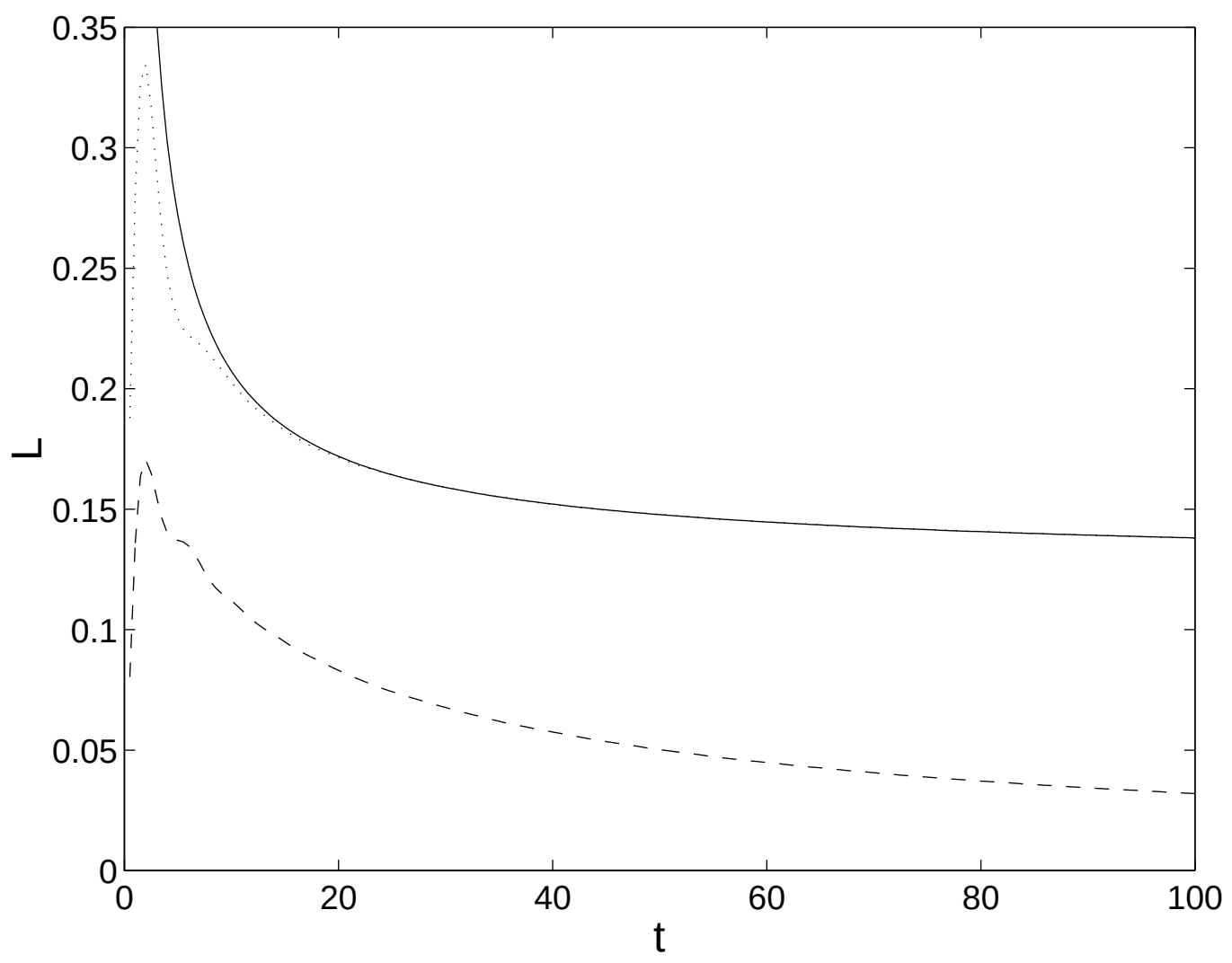
FIGURE CAPTIONS

FIG. 1. Lyapunov exponent for the twisted top, averaged over initial conditions. The dotted line is for the Kovalevskaya top and the solid line is the function $0.497/t+0.113/\sqrt{t}+0.122$, obtained by a least-squares fit and yielding the value $\lambda \simeq 0.122$ (See Ref. [21]). For comparison, the dashed line is the averaged Lyapunov exponent for the ordinary Kovalevskaya top.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