Low frequency stability of geotail plasma

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Abstract

The release of stored energy in the magnetosphere during magnetic storms may be triggered by plasma instabilities. We investigate the local stability of a simple but representative model of the flux surfaces of the Earth's magnetosphere in the MHD and drift frequency regimes. Magnetospheric flux surfaces at 6-10 Earth radii , with plasma beta ~ 5, are stable to MHD ballooning modes unless $\kappa_v x_p \leq 2/5$ where x_p is the plasma gradient scale length and κ_v the vacuum field line curvature at the equatorial plane. Drift modes may also be unstable unless $\eta \sim 2/3$, where η is the density gradient scale length divided by the temperature gradient scale length.

I. INTRODUCTION

During magnetic substorms in the Earth's magnetosphere, auroral brightening due to particle precipitation has been observed on field lines which intersect the night-side equatorial plane at about 6 to 10 Earth radii [1]. It has been suggested that plasma instabilities are responsible for the perturbations of these field lines, driven by large plasma pressure gradients in the neighbourhood of the equatorial plane where the field line curvature is large [2,3].

At 6 to 10 Earth radii, the magnetospheric plasma is essentially collisionless and the plasma beta at the equatorial plane is typically larger than unity ($\beta = 8\pi p/B^2 > 1$ where p is the plasma pressure and B the magnetic field magnitude). Thus, to test the relevance of the above suggestion, it is necessary to establish the conditions required for the onset of plasma instabilities in collisionless high beta magnetic dipole configurations and to determine whether such conditions are met in the Earth's magnetosphere [4-11].

In this paper, we investigate the stability of very high-beta magnetic dipole configurations using a quadratic variational form of the low frequency eigenmode equations for electromagnetic perturbations (derivable from the drift kinetic equation), and we discuss the local stability of the Earth's magnetosphere in the MHD and drift frequency regimes.

II. QUADRATIC VARIATIONAL FORM

We choose the gauge in which the parallel component of the magnetic vector potential is zero ($A_\parallel = 0$), and we express the perturbed fields in terms of the electrostatic potential $\Phi = \phi\, \exp(iky - i\omega t)$ and the perpendicular component of the vector potential $\boldsymbol{A}_\perp = \boldsymbol{\xi} \times \boldsymbol{B}$, with vector field $\boldsymbol{\xi} = \nabla y\, \{(i/kB)(Q_L + \kappa\xi^\psi)\exp(iky - i\omega t)\} - \boldsymbol{b} \times \nabla\{(i/kB)\xi^\psi \exp(iky - i\omega t)\}$. The field amplitude Q_L is related to the perturbed parallel magnetic field, and ξ^ψ to the field line displacement. Also ω is the mode frequency, and k is the perpendicular wave number in the y coordinate. The equilibrium magnetic field is $\boldsymbol{B} = \nabla y \times \nabla\psi(x, z)$, with $\boldsymbol{b}$ the unit magnetic field vector and ψ the equilibrium magnetic flux function. Instead of cartesian coordinates x, y, z, it will be convenient to use curvilinear coordinates ψ, y, s, where s is the coordinate measuring distance along the field line. We assume that the perpendicular

wavelength $2\pi/k$ is long compared to the particle Larmor radius r_L ($kr_L < 1$) but short compared to the field line curvature κ and spatial scale lengths of the field amplitudes ($k > \{\kappa, \partial/\partial s, B\partial/\partial\psi\}$).

In the limit of $k \gg \kappa$, the quadratic form, symmetric in the field amplitudes, can be approximated as [12,13]:

$$\mathcal{L}(\xi^\psi, Q_L, \phi) = \int_{-L}^{L} \frac{ds}{B} \left[-\frac{mN\omega^2}{B^2}\, \xi^\psi \xi^\psi + \frac{1}{4\pi} \frac{\partial \xi^\psi}{\partial s} \frac{\partial \xi^\psi}{\partial s} + \frac{1}{4\pi} Q_L Q_L - 2\frac{\kappa}{B}\, \frac{\partial p}{\partial \psi} \xi^\psi \xi^\psi \right]$$
$$- \sum_j \int_{-L}^{L} \frac{ds}{B} \int d^3v \frac{F_{0j}}{T_j} \left\{ q_j^2 \phi\phi - \frac{\omega - \omega_j^*}{\omega - \overline{\omega}_{Dj}} \overline{K}_j \overline{K}_j \right\}. \tag{1}$$

Here the sum is over particle species with label j, and m_j is the particle mass, N_j the equilibrium density, q_j the particle charge, and $p(\psi)$ the plasma pressure. The quantity $K_j = q_j\phi + \mu Q_L + (2\mathcal{E} - \mu B)\,\kappa\xi^\psi/B$ is the particle interaction Hamiltonian for guiding center motion, with $\mathcal{E}$ the particle energy and μ the particle magnetic moment; $F_{0j} = N_j(m_j/2\pi T_j)^{3/2} \exp(-\mathcal{E}/T_j)$ is the equilibrium particle distribution function; $\omega_j^* = (kc/q_jB)(\nabla y \cdot \boldsymbol{b} \times \nabla F_{0j})/(\partial F_{0j}/\partial \mathcal{E})$ is the particle diamagnetic frequency; and $\omega_{Dj} = -(kc/q_jB)(\mu \nabla y \cdot \boldsymbol{b} \times \nabla B + 2(\mathcal{E} - \mu B)\nabla y \cdot \boldsymbol{b} \times \boldsymbol{\kappa})$ is the particle magnetic drift frequency.

The overline on K in Eq. (1) denotes the time average by integration along a field line following one period of the particle trajectory: $\overline{K} = \int\, ds/v_\parallel K / \int\, ds/v_\parallel$, where $v_\parallel = \{2/m(\mathcal{E} - \mu B)\}^{1/2}$ is the parallel component of the particle velocity. It has been assumed that the mode frequencies of interest are smaller than the particle bounce frequencies; otherwise the overline should be removed. The coupling to harmonics of the particle bounce frequency has been ignored.

The spatial integral is taken over the field line variable s on a particular flux surface ψ. For closed field lines, the end points $s = -L$ and $s = L$ coincide. The origin of the field line variable is taken to be the magnetic field minimum $B(\psi, s = 0) = B_0$, and the plasma equilibrium is considered to be symmetric $B(\psi, s) = B(\psi, -s)$ in s about the origin $s = 0$.

The perturbed magnetic field is $\boldsymbol{Q} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B})$, with perpendicular component $\boldsymbol{Q}_\perp = \boldsymbol{b} \times (\boldsymbol{Q} \times \boldsymbol{b}) = \boldsymbol{b} \times \nabla y\; \boldsymbol{b} \cdot \nabla \xi^\psi + \cdots$ and parallel component (in a translated frame of reference) given by $Q_L = \boldsymbol{b} \cdot \boldsymbol{Q} + \boldsymbol{\xi} \cdot \nabla B - \kappa \xi^\psi$.

The kinetic term in Eq. (1) can be rewritten as follows:

$$\sum_j \int_{-L}^{L} \frac{ds}{B} \int d^3v \frac{F_{0j}}{T_j} \frac{\omega - \omega_j^*}{\omega - \overline{\omega}_{Dj}} \overline{K}_j \overline{K}_j$$
$$= \frac{N_0 T_0}{B_0} \left\langle \epsilon_0(\lambda,\omega) \frac{e^2 \overline{\phi}^2}{T_0^2} + 2\epsilon_1(\lambda,\omega) \left(\lambda \overline{Q}_L + \overline{w_c(\lambda,s)\xi^\psi} \right) \frac{e\overline{\phi}}{B_0 T_0} + \epsilon_2(\lambda,\omega) \frac{1}{B_0^2} \left(\lambda \overline{Q}_L + \overline{w_c(\lambda,s)\xi^\psi} \right)^2 \right\rangle$$

where the volume element in velocity space is $d^3v = (\frac{2}{m})^{3/2} (\frac{\pi B}{B_0}) \int_0^\infty d\mathcal{E}\, \mathcal{E}^{1/2} \int_0^{B_0/B} d\lambda\, g(\lambda, B)$. Here $\lambda = \mu B_0/\mathcal{E}$ is the pitch angle variable, $g(\lambda, B) \equiv (1 - \lambda B/B_0)^{-1/2}$ is inversely proportional to $v_\parallel$, and $w_c(\lambda, s) \equiv (2B_0/B - \lambda)\kappa$ is proportional to the particle curvature drift. Also $N_j = N_0$ is the equilibrium density, and T_0 is a nominal temperature which we can conveniently take to be T_i or T_e. We consider a plasma of protons and electrons with electric charge $q_i = e$ and $q_e = -e$, respectively, and equal densities $N_i = N_e = N_0$.

The angular brackets denote the following integration over the field line coordinate s and the pitch angle λ: $\langle \{\cdots\} \rangle = \int ds \int_0^{B_0/B} d\lambda\, g(\lambda, B) \{\cdots\}$. Note that $\langle \{1\} \rangle = 2 \int ds\ B_0/B$. The field line integration in s is either over a closed field line or over a periodic particle trajectory. There should be no confusion as to the appropriate limits, and for convenience in notation we will not specify explicitly the end point limits.

The functions $\epsilon_n(\lambda, \omega)$ for $n = 0, 1, 2$ are defined by the following integrals in the energy variable $\mathcal{E}$:

$$\epsilon_n(\lambda,\omega) \equiv \sum_j \frac{N_j T_0^{1-n}}{N_0 (\pi)^{1/2}} \left(\frac{q_j}{e} \right)^{2-n} \int_0^\infty d\mathcal{E}\ \exp(-\mathcal{E}/T_j) \frac{\mathcal{E}^{1/2+n}}{T_j^{5/2}} \frac{(\omega - \omega_j^*)}{(\omega - \overline{\omega}_{Dj})}. \tag{2}$$

Trapped particles have pitch angle values in the range $1 \geq \lambda \geq B_0/B_{\max}$ and passing particles in the range $B_0/B_{\max} > \lambda \geq 0$, where $B_{\max}$ is the maximum in the magnetic field.

The eigenmode equations for ϕ, Q_L, and ξ^ψ are the Euler-Lagrange equations, derivable from the first variation of the quadratic form with respect to ϕ, Q_L, and ξ^ψ. Thus for Q_L we have:

$$\frac{Q_L}{4\pi} = -\frac{N_0 T_0 B}{B_0^2} \int_0^{B_0/B} d\lambda\, g(\lambda, B)\, \lambda \left\{ \epsilon_1 \frac{e\overline{\phi}}{T_0} + \frac{\epsilon_2}{B_0} \left(\lambda \overline{Q}_L + \overline{w_c(\lambda,s)\xi^\psi} \right) \right\}. \tag{3}$$

We separate the discussion of plasma stability into two frequency regimes: (1) kinetic MHD frequencies $|\omega| \sim (T_i/m_i)^{1/2}\kappa \gg |\omega_j^*|, |\omega_{Dj}|$; and (2) drift frequencies $|\omega| \sim |\omega_j^*|, |\omega_{Dj}|$.

III. KINETIC MHD FREQUENCY REGIME

In the MHD frequency range $|\omega| \sim (T_j/m_j)^{1/2}\kappa \gg |\omega_j^*|, |\omega_{Dj}|$, it is appropriate to have $\overline{\phi} = \phi$ (that is, zero perturbed parallel electric field, $\boldsymbol{b}\cdot\nabla\phi = 0$). The quadratic form can then be approximated by the following variational functional for ω^2:

$$\omega^2 \int \frac{ds}{B}\frac{mN}{B^2}\xi^\psi\xi^\psi = \delta W^{KO}(Q_L, \xi^\psi), \tag{4}$$

where the functional $\delta W^{KO}(Q_L, \xi^\psi)$ is the Kruskal-Oberman energy functional [14]:

$$\delta W^{\rm KO} = \int \frac{ds}{4\pi B}\left\{\left(\frac{\partial\xi^\psi}{\partial s}\right)^2 + (Q_L)^2 - \frac{8\pi\kappa}{B}\frac{\partial p}{\partial\psi}\left(\xi^\psi\right)^2\right\} + \frac{15p}{8B_0^3}\left\langle\left\{\lambda\overline{Q}_L + \overline{w_c\xi^\psi}\right\}^2\right\rangle. \tag{5}$$

The first term in Eq. (5) is the "line bending" energy, the second term is the magnetic compressional energy, and the fourth term (kinetic) is the plasma compressional energy. These terms are all positive definite whereas the third term, the interchange free energy, is negative on the outer flux surfaces of a magnetic dipole equilibrium where the plasma density gradient is negative. A necessary and sufficient condition for stability is $\delta W^{\rm KO} > 0$.

The energy functional can be reduced to a functional of the field variable ξ^ψ by solving the eigenmode equation which determines Q_L in terms of ξ^ψ:

$$\frac{Q_L}{4\pi} = -\frac{15pB}{8B_0^3}\int_0^{B_0/B} d\lambda\, g(\lambda, B)\,\lambda\left\{\lambda\overline{Q}_L + \overline{w_c(\lambda,s)\xi^\psi}\right\}. \tag{6}$$

The right-hand side of Eq. (6) is proportional to the plasma beta $\beta_0 = 8\pi p/B_0^2$. We can therefore obtain (by iteration) approximate solutions for Q_L in terms of ξ^ψ in the limit of low beta $\beta_0 < 1$ or very high beta $\beta_0 > 1$.

In the case of low plasma beta $\beta_0 < 1$, we express the solution $Q_L = Q_L^{(0)} + Q_L^{(1)} + \cdots$ as a power series expansion in β_0. To lowest order in β_0, we have $Q_L^{(0)} = -\frac{15\beta_0 B}{16B_0}\int_0^{B_0/B} d\lambda\, g(\lambda, B)\,\lambda\,\overline{w_c\xi^\psi} + \cdots$. Then Eq. (5) for $\delta W^{\rm KO}(\xi^\psi)$ to order β_0^2 is given by

$$\delta W_0(\xi^\psi) = \int \frac{ds}{4\pi B}\left(\frac{\partial\xi^\psi}{\partial s}\right)^2 - \frac{3}{4B_0^2}\left\langle w_c\xi^\psi\xi^\psi\frac{\partial p}{\partial\psi}\right\rangle + \frac{15p}{8B_0^3}\left\langle\left(\overline{w_c\xi^\psi} + \frac{\lambda}{2}\overline{Q}_L^{(0)}\right)^2\right\rangle + O(\beta_0^3). \tag{7}$$

Using the Schwartz inequality, we have

$$\left\langle\left(\overline{w_c\xi^\psi} + \frac{\lambda}{2}\overline{Q}_L^{(0)}\right)^2\right\rangle \geq \frac{\left\langle\left(\overline{w_c\xi^\psi} + \frac{\lambda}{2}\overline{Q}_L^{(0)}\right)\right\rangle^2}{\langle(1)\rangle} = \frac{\left\langle\overline{w_c\xi^\psi}\right\rangle}{\langle(1)\rangle}\left\{\left\langle\overline{w_c\xi^\psi}\right\rangle - \frac{5\beta_0}{4}\left\langle\frac{\lambda B_0}{B}\right\rangle\right\} + \ O(\beta_0^2).$$

Thus for flute perturbations $\partial\xi^\psi/\partial s = 0$, $\delta W(\xi^\psi)$ is bounded from below by

$$\delta W(\xi^\psi) \geq -\frac{3N_0 e}{2kcB_0^2}\left(\xi^\psi\right)^2 \frac{\langle w_c(\lambda,s)\rangle}{\langle(1)\rangle}\left\langle\left\{1-\frac{5\beta_0}{4}\left(\frac{\overline{\lambda B_0}}{B}\right)\right\}\left(\omega_p^* - \frac{5}{2}\overline{\omega}_d\right)\right\rangle + O(\beta_0^3)$$

where ω_p^* is the plasma diamagnetic frequency and ω_d the plasma drift frequency, defined by

$$\begin{aligned}
\omega_p^* &\equiv \frac{kc}{2N_0 e}\frac{\partial p}{\partial\psi} \\
\omega_d &\equiv \omega_\kappa - \frac{\beta_0\lambda B_0}{2B}\omega_p^* \\
\omega_\kappa &\equiv \frac{kcp}{2N_0 e B_0} w_c(\lambda,s) = \frac{kcp}{2N_0 e B_0}\left(\frac{2B_0}{B}-\lambda\right)\kappa.
\end{aligned}$$

A sufficient condition for flute stability is

$$\langle\overline{\omega}_\kappa\rangle\left\langle\frac{5}{2}\overline{\omega}_d - \omega_p^*\right\rangle > 0. \tag{8}$$

In the case of very high beta $\beta_0 > 1$, the magnetic perturbation $Q_L = Q_L^{(0)} + Q_L^{(1)} + \cdots$ can be expressed as a power series in $1/\beta_0$. The lowest order solution $Q_L^{(0)}$ is obtained by equating to zero the trapped particle contributions to the right-hand side of Eq. (6): $\int_{B_0/B_{\max}}^{B_0/B} d\lambda\, g(\lambda,B)\,\lambda\,\epsilon_2(\lambda,\omega)\{\lambda\,\overline{Q}_L^{(0)} + \overline{w_c(\lambda,s)\xi^\psi}\} = 0$. This equation must be satisfied independent of the field line variable s, and to lowest order in $1/\beta_0$, we therefore require:

$$\int ds\, g(\lambda,B)\,\lambda\, Q_L^{(0)} = -\int ds\, g(\lambda,B)\, w_c(\lambda,s)\,\xi^\psi \tag{9}$$

where we consider symmetric perturbations for which $\xi^\psi(s) = \xi^\psi(-s)$, $Q_L(s) = Q_L(-s)$. By Abel inversion, we obtain $Q_L^{(0)}$ in terms of ξ^ψ

$$Q_L^{(0)} = -\frac{\partial}{\partial s} B\int_0^s ds'\frac{\kappa\xi^\psi}{B}. \tag{10}$$

We assume $B_0/B_{\max} \ll 1$ where $B_{\max}$ is the magnetic field maximum, and negligibly small passing particles. To next order in $1/\beta_0$, we have $\frac{15pB}{8B_0^3}\int_0^{B_0/B} d\lambda\, g(\lambda,B)\,\lambda^2\,\overline{Q}_L^{(1)} = -Q^{(0)}/4\pi$, and by Abel inversion, we obtain

$$\overline{Q}_L^{(1)} = -\frac{2B_0^2}{15\pi^2\lambda^2 p}\frac{\partial}{\partial\lambda}\int_0^\lambda \frac{du\, u^{1/2}}{(\lambda-u)^{1/2}} Q_L^{(0)} \tag{11}$$

where $u = B_0/B$.

In the very high beta limit where $\omega_p^* \sim \omega_d$, we have approximately $w_c(\lambda,s) = 8\pi N_0 e/kcB_0(\lambda B_0\omega_p^*/B + \omega_d/\beta_0) \sim (4\pi\lambda/B)(\partial p/\partial\psi)$, and from Eq. (10) we obtain $Q_L^{(0)} \sim$

$-(4\pi/B)(\partial p/\partial\psi)\xi^\psi$. Substituting for $Q_L^{(0)}$ in Eq. (11), we then obtain the simple estimate for $\overline{Q}_L^{(1)} \sim (2B_0/5p\lambda)(\partial p/\partial\psi)\xi^\psi$ in the limit of flute perturbations $\partial\xi^\psi/\partial s = 0$.

Equation (5) for $\delta W^{\rm KO}$ can be written, in the high beta limit, as

$$\delta W_0(\xi^\psi) = \int \frac{ds}{4\pi B}\left(\frac{\partial\xi^\psi}{\partial s}\right)^2 - \frac{3}{4B_0^2}\left\langle w_c\xi^\psi\xi^\psi\frac{\partial p}{\partial\psi}\right\rangle + \int\frac{ds}{4\pi B}\left(Q_L^{(0)} + Q_L^{(0)}Q_L^{(1)}/2\right)^2 + O(1/\beta^2). \tag{12}$$

By a similar argument as above, $\delta W_0(\xi^\psi)$ for flute perturbations is bounded from below by

$$\delta W_0(\xi^\psi) \geq -\frac{9N_0 e}{80\pi kcpB_0^3}\frac{(\xi^\psi)^2}{\int\frac{ds}{B^3}}\langle w_c\rangle\left\langle\left(\omega_p^* - \frac{5}{2}\overline{\omega}_{dp}\right)\right\rangle.$$

Thus Eq. (8) is also a sufficient condition for flute stability in the high beta limit.

This inequality is identical to the MHD criterion Eq. (15) for flute stability, a result which is not surprising since the MHD energy functional $\delta W^{\rm MHD}$ [15] is a lower bound on $\delta W^{\rm KO}$:

$$\delta W^{\rm KO} \geq \delta W^{\rm MHD}(\xi^\psi) = \int\frac{ds}{B}\left(\frac{1}{4\pi}\frac{\partial\xi^\psi}{\partial s}\frac{\partial\xi^\psi}{\partial s} - 2\frac{\kappa}{B}\frac{\partial p}{\partial\psi}\xi^\psi\xi^\psi\right) + \frac{20p}{3}\frac{\left\{\int\frac{ds}{B^2}\kappa\xi^\psi\right\}^2}{\int\frac{ds}{B}\left(1+\frac{20\pi p}{3B^2}\right)} \tag{13}$$

where $\delta W^{\rm MHD}$ is the MHD energy functional minimised with respect to

$$Q_L = -\frac{40\pi p}{3B}\frac{\int\frac{ds}{B^2}\kappa\xi^\psi}{\left(\int\frac{ds}{B}(1+(20\pi p)/(3B^2))\right)}.$$

A sufficient stability condition for arbitrary plasma beta can therefore be determined from an analysis of the Euler-Lagrange equation obtained from the first variation of $\delta W^{\rm MHD}$ with respect to ξ^ψ

$$\frac{\omega^2 mN}{B^2}\xi^\psi + \frac{B}{4\pi}\frac{\partial}{\partial s}\frac{1}{B}\frac{\partial\xi^\psi}{\partial s} + 2\frac{\kappa}{B}\frac{\partial p}{\partial\psi}\xi^\psi - \frac{20p\kappa}{3B^2}\frac{\int\frac{ds}{B^2}\kappa\xi^\psi}{\int\frac{ds}{B}\left(1+\frac{20\pi p}{3B^2}\right)} = 0. \tag{14}$$

For stability of flute perturbations, we require

$$\left(\int\frac{ds}{B^2}\kappa\right)\left[\frac{20p}{3}\frac{\int\frac{ds}{B^2}\kappa}{\int\frac{ds}{B}\left(1+\frac{20\pi p}{3B^2}\right)} - 2\frac{\partial p}{\partial\psi}\right] > 0. \tag{15}$$

This inequality can also be written in terms of the flux tube volume $V = \int ds/B$ as follows:

$$\delta\left(pV^{5/3}\right) > 0. \tag{16}$$

The focus of the present investigation is the high bounce frequency limit. We should mention ,however, that in the limit where mode frequencies are larger than the particle bounce frequency, the orbit averaging of the interaction Hamiltonian is removed, and we have for δW:

$$\delta W(\xi^\psi) = \int \frac{ds}{B}\left(\frac{1}{4\pi}\frac{\partial \xi^\psi}{\partial s}\frac{\partial \xi^\psi}{\partial s} - 2\frac{\kappa}{B}\ \frac{\partial p}{\partial \psi}\xi^\psi\xi^\psi\right) + p\int \frac{ds}{B^3}\frac{\left(7+\frac{5\beta_0 B_0^2}{2B^2}\right)}{\left(1+\frac{\beta_0 B_0^2}{B^2}\right)}\kappa^2\left(\xi^\psi\right)^2 \tag{17}$$

where δW has been minimised with respect to $Q_L = -\frac{3\beta_0 B_0^2}{2B^2}\frac{\kappa\xi^\psi}{(1+\beta_0 B_0^2/B^2)}$. In deriving Eq. (17) , we assumed the ion beta to be larger than the electron beta ($T_i \gg T_e$) and we ignored the contributions of trapped electrons. In this "fast" MHD limit, the compressional stabilising term increases with β_0 and eventually becomes the dominant term. The physical interpretation is that there is not enough time for the perturbed plasma pressure to propagate along the field line and hence the energy required to compress the plasma increases with β_0. In a previous publication [11] we estimated the critical plasma beta for stabilisation to be $\beta_0 \sim 3$. Thus, unstable perturbations of very high beta equilibria cannot have growth rates exceeding the particle bounce frequency.

Note that a necessary condition for stability is that there are no drift reversed particles, $\overline{\omega}_d > 0$

At marginal stability, the growth rate is zero. Thus, for purely growing modes, our assumed frequency ordering, namely $|\omega| \gg |\omega_j^*|, |\omega_{Dj}|$, is not satisfied at marginal stability. However, in the very low frequency regime where $|\omega| \ll |\omega_j^*|, |\omega_{Dj}|$, we can again use the Schwartz inequality to derive a lower bound on the appropriate quadratic form [16], and we also find that flute perturbations are stable if there are no drift-reversed particles in the equilibrium.

IV. DRIFT FREQUENCY REGIME

In this section we discuss the drift frequency range in which mode frequencies are of the order of the particle diamagnetic frequency $\omega \sim \omega_j^*$. We consider perturbations for which the perturbed field $E_\parallel = -\boldsymbol{b}\cdot\nabla\phi \neq 0$ is finite. These modes have been extensively studied as drift waves or trapped particle modes in tokomak geometries. Kesner [17,18] has

investigated these modes in low beta, collisional plasmas in magnetic dipolar equilibria in the limit where the perturbations are considered to be purely electrostatic. In equilibria with high plasma beta, the perturbations will not be purely electrostatic, and we will here investigate the modifications introduced by the admixture of magnetic perturbations.

A. Low beta

In the low beta limit, the "line bending term" proportional to $d\xi^\psi/ds$ is dominant, so that perturbations of ξ^ψ will be flute-like with $d\xi^\psi/ds = 0$ to lowest order. It may then be verified that the quadratic form can be reexpressed as a functional of only two field variables, $\phi' = \phi + \omega\xi^\psi/kc$ and $Q'_L = Q_L + (4\pi/B)(\partial p/\psi)\xi^\psi$. This reduction of the quadratic form to two field variables can effectively be achieved by setting $\xi^\psi = 0$ in Eq. (1) and we will adopt this procedure to avoid introducing new notations. Thus we have

$$\mathcal{L}(\phi, Q_L) = \int \frac{ds}{4\pi}\frac{B_0}{B}Q_L^2 - \frac{N_0T_0}{B_0^2}\left\langle \sum_j \frac{N_jT_0}{2N_0T_j}\phi^2 - \epsilon_0\,\overline{\phi}^2 - 2\epsilon_1\,\lambda\overline{Q}_L\overline{\phi} - \,\epsilon_2\,\lambda^2\,\overline{Q}_L^2\right\rangle \tag{18}$$

where we have introduced dimensionless field variables: $Q_L/B_0 \to Q_L$, $e\phi/T_0 \to \phi$.

We will hereafter simplify the analysis by considering equilibria in which the ions and electrons have equal temperatures and equal spatial profiles $T_e = T_i = T_0$, $N_i = N_e = N_0$, $\omega^*_{Ne} = -\omega^*_{Ni}, \overline{\omega}_{de}(\lambda) = -\overline{\omega}_{di}(\lambda)$, where $\omega^*_{Ni} \equiv (kcT_i/N_ie)(\partial N_i/\partial\psi)$ and $\omega_{di} \equiv \frac{kcT_i}{N_ieB_0}\left(w_c(\lambda, s) - \frac{4\pi\lambda}{B}(\partial p/\partial\psi)\right)$. Note that $\omega^*_p = \omega^*_{Ni}(1+\eta_i)$ and $\omega_{Di} = (\mathcal{E}/T_i)\omega_{di}$, where $\eta_i \equiv \left(\frac{1}{T_i}\frac{\partial T_i}{\partial\psi}\right) / \left(\frac{1}{N_i}\frac{\partial N_i}{\partial\psi}\right)$ is the usual ion temperature gradient parameter.

In the limit of low but finite plasma beta, the magnetic perturbations $Q_L = Q_L^{(0)} + \cdots$ can be approximately expressed in terms of ϕ by

$$\frac{Q_L^{(0)}}{4\pi} = -\frac{N_0T_0}{B_0^3}B\int_0^{B_0/B} d\lambda\, g(\lambda, B)\,\lambda\,\epsilon_1(\lambda,\omega)\,\overline{\phi} + \cdots\ . \tag{19}$$

Substituting this expression for $Q_L^{(0)}$ in Eq. (18), we obtain the following quadratic functional in ϕ:

$$-\mathcal{L}(\phi,\omega) = \frac{N_0T_0}{B_0}\left\{\left\langle\phi^2 - \overline{\phi}^2\right\rangle + \left\langle(1-\epsilon_0(\lambda,\omega))\,\overline{\phi}^2\right\rangle \right.$$
$$\left. + \frac{3k^2r_{Li}^2}{4}\left\langle\frac{\lambda B_0}{B}\overline{\phi}^2\right\rangle + \frac{4\pi N_0T_0}{B_0^2}\int ds\frac{B_0}{B}\left(\int_0^{B_0/B} d\lambda g(\lambda,B)\,\lambda\,\epsilon_1(\lambda,\omega)\overline{\phi}\right)^2\right\} = 0 \tag{20}$$

where we have added a small ion Larmor radius correction proportional to $r_{Li}^2 = \frac{m_i T_i c^2}{e^2 B_0^2}$, and the fourth term of order β is the correction to the electrostatic approximation due to magnetic perturbations. Note that $\left\langle \phi^2 - \overline{\phi}^2 \right\rangle \geq 0$.

Let us first ignore the magnetic perturbations and the ion Larmor radius correction. From a Nyquist plot of $-\mathcal{L}(\phi, \omega)$, we can determine sufficient conditions for stability. The function $\epsilon_0(\lambda, \omega)$ is a complex function of real ω due to the singularity in the energy integration (see Eq. (2)). The imaginary part of $\epsilon_0(\lambda, \omega)$ for real ω can readily be evaluated and it is equal to zero at $|\omega| = \omega_0(\lambda)$ given by $\omega_0/\omega_{Ni}^* = (3\eta_i/2 - 1)/(\eta_i \omega_{Ni}^*/\omega_{di}(\lambda) - 1)$ provided that the following inequalities are satisfied $\eta_i > \mathrm{Max}\{2/3,\, \overline{\omega}_{di}(\lambda)/\omega_{Ni}^*\}$ or $\eta_i < \mathrm{Min}\{2/3,\, \overline{\omega}_{di}(\lambda)/\omega_{Ni}^*\}$. The magnitude of $1 - \epsilon_0(\lambda, \omega_0)$ is: $1 - \epsilon_0(\lambda, \omega_0) = \frac{2(\overline{\omega}_{di} - \eta_i \omega_{Ni}^*)}{\pi^{1/2}} \int_0^\infty d\mathcal{E}_i \, \frac{\mathcal{E}_i^{3/2} \exp(-\mathcal{E}_i)}{\omega_0 + \mathcal{E}_i \overline{\omega}_{di}}$ and it is positive if $\overline{\omega}_{di}/\omega_{Ni}^* > \eta_i$, negative if $\eta_i > \overline{\omega}_{di}/\omega_{Ni}^*$.

In addition, the imaginary part of $\epsilon_0(\lambda, \omega)$ is also zero at $\omega = 0$ and at $|\omega| = \infty$: note that $\{1 - \epsilon_0(\lambda, \omega \to 0)\} \to 1 - 2\omega_{Ni}^*(1 - \eta_i)/\omega_{di}(\lambda)$, and $\{1 - \epsilon_0(\lambda, \omega \to \infty)\} \to 0$.

The above properties of the function $1 - \epsilon_0(\lambda, \omega)$ can now be used to map out the stability boundaries in the parameter space $\{\eta_i, \overline{\omega}_{di}/\omega_{Ni}^*\}$ as shown in Fig. 1. The solid lines correspond to $\overline{\omega}_{di}/\omega_{Ni}^* = \eta_i$ and $\overline{\omega}_{di}/\omega_{Ni}^* = 2(1 - \eta_i)$, and they intersect at $\overline{\omega}_{di}/\omega_{Ni}^* = 2/3$, $\eta_i = 2/3$. If, for a given equilibrium, the orbit averaged particle drifts $\overline{\omega}_{di}$ are such that the values of $\overline{\omega}_{di}/\omega_{Ni}^*$ are in the region above the solid lines of Fig 1 for all values of the pitch angle, the following inequality is satisfied:

$$\overline{\omega}_{di}(\lambda)/\omega_{Ni}^* > \mathrm{Max}\left\{\eta_i,\, 2 - 2\eta_i\right\}. \tag{21}$$

It may then be verified that the Nyquist polar plot of $-\mathcal{L}(\phi, \omega)$ intersects the positive real axis when $\omega = 0, |\omega| = \infty$, and also when $|\omega| \sim \omega_0 > 0$. The Nyquist polar plot will not enclose the origin as ω encircles the upper half of the complex ω plane. Equation (21) is therefore a sufficient condition for the stability of electrostatic perturbations.

If the values of $\overline{\omega}_{di}/\omega_{Ni}^*$ are all in the region below the solid lines of Fig. 1, Eq. (21) is not satisfied, and the Nyquist polar plot of $-\mathcal{L}(\phi, \omega)$ will encircle the origin for flute perturbations where $\left\langle \phi^2 - \overline{\phi}^2 \right\rangle \to 0$. Flute perturbations would then be unstable.

In the intermediate case where $\overline{\omega}_{di}/\omega_{Ni}^*$ is above the solid lines for some values of λ and below for other values of λ, stability (or instability) will usually require solving the eigenmode

equation. It may be stated, however, that stability (or instability) will be favoured if there is a preponderance of particles with $\overline{\omega}_{di}/\omega^*_{Ni}$ above (or below) the solid lines, respectively.

When magnetic perturbations are included, the Nyquist polar plot of $-\mathcal{L}(\phi,\omega)$ intersects the real axis when $\omega \sim \omega_0$ at an algebraically larger coordinate, while the intersections when $\omega = 0$ and $|\omega| = \infty$ remain unchanged. Note that the function $\epsilon_1(\lambda,\omega)$ is equal to zero at $\omega = 0, |\omega| = \infty$, and that $Im(\epsilon_1) = 0$ at $\omega = \omega_0$. Thus, small but finite magnetic perturbations will broaden somewhat the stable regime of parameter space but will not otherwise modify significantly the Nyquist polar plots of $-\mathcal{L}(\phi,\omega)$.

Since Eq. (20) is a variational quadratic form, numerical estimates of the mode frequencies can be obtained by substitution of an appropriate trial function for ϕ.

Fig. 1 is useful not only in assessing drift mode stability but also MHD stability. A sufficient condition for MHD flute stability is $\overline{\omega}_{di}/\omega^*_{Ni} > 2\,(1+\eta_i)\,/5$ for all pitch angles λ, where $\omega^*_p = \omega^*_{Ni}\,(1+\eta_i)$. The dotted line in Fig. 1 corresponds to $\overline{\omega}_{di}/\omega^*_{Ni} = 2\,(1+\eta_i)\,/5$, and MHD stability (or instability) is favoured if there is a preponderance of particles with $\overline{\omega}_{di}/\omega^*_{Ni}$ above (or below) the dotted line, respectively. Note that we can use Fig. 1 to explore stability (or instability) for increasing values of plasma beta by determining whether the average value of $\overline{\omega}_{di}/\omega^*_{Ni}$ increases (or decreases) with increasing plasma beta. This strategy will be adopted in Sec. 5 to discuss the stability of the Earth's magnetosphere.

B. Very high beta

In the very high beta limit $\beta_0 > 1$, we express the magnetic field perturbation $Q_L = Q_L^{(0)} + Q_L^{(1)} + \cdots$ as a power series in $1/\beta_0$ (see Sec. 3). The lowest order perturbed field $Q_L^{(0)}$ is given by $\epsilon_2(\lambda,\omega)\{\lambda\overline{Q}_L^{(0)} + \overline{(2B_0/B-\lambda)(\kappa/\kappa_0)\xi^\psi}\} + \epsilon_1(\lambda,\omega)\overline{\phi} = 0$ where ξ^ψ is now a dimensionless field variable: $\kappa_0\xi^\psi/B_0 \to \xi^\psi$. Here $\kappa_0 = \kappa(s=0)$ is the field line curvature at $s = 0$.

As before, we determine $Q_L^{(0)}$ in terms of ϕ and ξ^ψ by Abel inversion, and we obtain:

$$Q_L^{(0)} = -\frac{\partial}{\partial s}\left\{B(s)\int_0^s \frac{ds'}{B(s')}\frac{\kappa(s')}{\kappa_0}\,\xi^\psi(s')\right\}$$
$$-\frac{1}{\pi u^{1/2}(s)}\frac{\partial}{\partial s}\int_0^s ds' u^{1/2}(s')\phi(s')\int_{u(s)}^{u(s')}\frac{d\lambda}{(\lambda-u(s))^{1/2}(u(s')-\lambda)^{1/2}}\,\frac{1}{\lambda}\frac{\epsilon_1(\lambda,\omega)}{\epsilon_2(\lambda,\omega)} \qquad (22)$$

where $u(s) = B_0/B(s,\psi)$.

To next order in $1/\beta$, $Q_L^{(1)}$ is determined in terms of $Q_L^{(0)}$ by: $\overline{Q}_L^{(1)} = -\frac{B_0^2}{4\pi^2 N_0 T_0}\frac{1}{\lambda^2\epsilon_2}\frac{\partial}{\partial\lambda}\int_0^\lambda \frac{du}{(\lambda-u)^{1/2}} u^{1/2} Q_L^{(0)}$.

Substituting for $Q_L^{(0)}$ in our quadratic form (Eq. (1)) we obtain

$$\mathcal{L}(\phi,\xi^\psi) = -\frac{N_0T_0}{B_0}\left[\left\langle\phi^2\right\rangle + \left\langle\left\{\epsilon_0(\lambda,\omega) - \frac{\epsilon_1^2(\lambda,\omega)}{\epsilon_2(\lambda,\omega)}\right\}\overline{\phi}^2\right\rangle\right.$$
$$\left. + \frac{B_0^3}{N_0T_0}\left\{\int\frac{ds}{4\pi B}\left(Q_L^{(0)}\right)^2 + \frac{1}{\kappa_0^2}\int\frac{ds}{4\pi B}\left(\left(\frac{\partial\xi^\psi}{\partial s}\right)^2 - \frac{8\pi\kappa}{B}\frac{\partial p}{\partial\psi}\left(\xi^\psi\right)^2\right) - \int\frac{ds}{B}\frac{mN\omega^2}{B^2\kappa_0^2}\left(\xi^\psi\right)^2\right\}\right]. \quad (23)$$

This yields eigenmode equations which couple the field variables ϕ and ξ^ψ.

We note that terms involving ξ^ψ are of order $1/\beta$ and we can therefore ignore the coupling of ξ^ψ to ϕ unless mode frequencies become comparable and the coupling is resonantly enhanced. We defer a discussion of this case to a later publication. We then have

$$-\mathcal{L}(\phi) = \frac{N_0T_0}{B_0}\left\{\left\langle\phi^2 - \overline{\phi}^2\right\rangle + \left\langle\left(1 - \epsilon_0(\lambda,\omega) + \frac{\epsilon_1^2(\lambda,\omega)}{\epsilon_2(\lambda,\omega)}\right)\overline{\phi}^2\right\rangle + \frac{3k^2r_{Li}^2}{4}\left\langle\frac{\lambda B_0}{B}\overline{\phi}^2\right\rangle\right\} \quad (24)$$

where the additional term proportional to ϵ_1^2/ϵ_2 represents the high beta modication due to magnetic compressional perturbations, and we have again added a small ion Larmor radius correction.

The function $\epsilon_2(\lambda,\omega)$ is finite and positive at $\omega = 0$ and $|\omega| = \infty$, and also at $\omega = \omega_0$: note that $\epsilon_2(\lambda,0) = \frac{3}{2}\frac{\omega_{Ni}^*}{\omega_{di}(\lambda)}(1+\eta_i)$ and $\epsilon_2(\lambda,\infty) = \frac{15p}{8N_0T_0}$. In a Nyquist polar plot of $-\mathcal{L}(\phi)$, the intersections of the real axis at $\omega = 0$ and $|\omega| = \infty$ remain unchanged while the intersection at $\omega \sim \omega_0$ is shifted to an algebraically larger coordinate. Thus modifications introduced by magnetic compressional perturbations broaden the stable regime of parameter space.

Equation (24) is a variational quadratic form and numerical estimates of the mode frequencies can be obtained by substitution of an appropriate trial function for ϕ.

V. STABILITY OF DIPOLAR MAGNETIC FLUX SURFACES

In this section, we discuss the local stability of a dipolar magnetic plasma equilibrium. We use the following flux function $\psi(x,z)$ to model locally the plasma equilibrium in the neighbourhood of a given flux surface $\psi_0 = \psi(x = x_0, z = 0)$ which intersects the equatorial plane $z = 0$ at a reference distance $x = x_0$:

$$\psi(x,z) = B_d \frac{x_0^2 x}{x^2+z^2} - B'_{1x}\frac{z^2}{2} - B_{1z}x. \tag{25}$$

The first term represents the magnetic flux generated by a magnetic dipole with strength proportional to B_d. The second is the flux due to a plasma diamagnetic current $\boldsymbol{J}_p = c/4\pi B B'_{1x}\nabla y$, where $B'_{1x} = 4\pi\partial p/\partial\psi$ and $p(\psi) = p_0\psi/\psi_0$ is the plasma pressure with magnitude p_0 on the surface $\psi = \psi_0$. The third term is the flux due to external currents. The relative amplitudes $B' \equiv (B'_{1x}x_0)/B_d$ and $B_n \equiv B_{1z}/B_d$ are parameters which determine the distortion of the equilibrium magnetic surfaces away from the vacuum magnetic dipole surfaces.

The components of the equilibrium magnetic field iare given by $\boldsymbol{B}(x,z) = \nabla y \times \nabla\psi$, and the field line curvature by $\boldsymbol{\kappa} = \frac{1}{B}\left[\frac{\partial}{\partial x}\frac{B_z}{B} - \frac{\partial}{\partial z}\frac{B_x}{B}\right]\nabla\psi$. We limit the range of parameters B' and B_n to values such that the minimum B_0 of the magnetic field magnitude on the flux surface $\psi = \psi_0$ is at the equatorial plane, $B_0 = |\boldsymbol{B}(x = x_0, z = 0) = B_d + B_{1z}$.

This flux function produces flux surfaces that are similar in shape to the flux surfaces generated by the experimentally based Tsyganenko model [19] of the Earth's magnetosphere during the quiescent phase in the absence of magnetic substorms. In Fig. 2, we plot the Tsyganenko flux surfaces, and for comparison, we also plot the flux surfaces generated by Eq. (25) for the values $B' \sim 7, B_n \sim 0.1$ optimised to reproduce the shape of the Tsyganenko flux surfaces.

Let $x_p \equiv (B_0/p_0)(\partial p/\partial\psi)^{-1}$ denote the scale length of the plasma pressure gradient and let $\beta_0 \equiv 8\pi p_0/B_0^2$ denote the plasma beta on the flux surface $\psi = \psi_0$ at the mid-plane. Then we have $x_p = \psi_0/B_0 = \frac{1-B_n}{1+B_n}x_0$ and $\beta_0 = 2B'(1-B_n)/(1+B_n)^2$ in terms of B' and B_n.

On the surface $\psi = \psi_0$ at $z = 0$, the magnitude κ_v of the vacuum field line curvature is $\kappa_v x_0 = 2/(1+B_n)$, and the magnitude of the field line curvature κ_0 for finite beta is $\kappa_0 x_0 = \kappa_v x_0(1 + B'/2) = \kappa_v x_0(1 + \beta_0/2\kappa_v x_p)$. Note that the curvature at the equatorial plane $\kappa_0 x_0$ increases above the vacuum field line curvature with increasing β_0. This behaviour is characteristic of magnetic dipole equilibria.

We established earlier that it is sufficient for MHD flute stability to have $\overline{\omega}_{di}/\omega_p^* > 2/5$ for all values of the pitch angle λ, and that drift frequency stability requires $\overline{\omega}_{di}/\omega_{Ni}^* > \mathrm{Max}\{\eta_i,\, 2(1-\eta_i)\}$ (see Fig. 1). Thus the ratio of the drift frequency to the diamagnetic

frequency ω_{di}/ω_p^* is an important stability parameter. In our model equilibrium, we have

$$\frac{\omega_{dp}}{\omega_p^*} = \kappa_v x_p \left\{ \left(1 + \frac{\beta_0}{2\kappa_v x_p}\right) \left(2\frac{B_0}{B} - \lambda\right) \frac{\kappa}{\kappa_0} - \frac{\lambda\beta_0}{2\kappa_v x_p} \frac{B_0}{B} \right\}. \tag{26}$$

Note that for well trapped particles with $\lambda \to 1$, we have

$$\frac{\overline{\omega}_{dp}}{\omega_p^*} \to \kappa_v x_p \tag{27}$$

and the frequency ratio for these particles, equal to the product of scale length of the plasma gardient and the vacuum field line curvature, remains constant with increasing plasma beta. For other particles with $\lambda \neq 1$, the frequency ratio $\overline{\omega}_{dp}/\omega_p^*$ can be less than or greater than $\kappa_v x_p$ depending on the details of the equilibrium field line structure and on the plasma beta.

If $\omega_{dp}(\lambda)/\omega_p^*$ has a minimum at $\lambda \to 1$ so that $\omega_{dp}(\lambda)/\omega_p^* \geq \omega_{dp}(\lambda = 1)/\omega_p^*$, and if this inequality persists as the plasma beta increases, then $\kappa_v x_p \geq 2/5$ is sufficient to satisfy the criterion for MHD flute stability. This appears to be the case for magnetic dipole equilibria whose stability to MHD modes has been discussed in references [20-22].

On the other hand, if $\omega_{dp}(\lambda)/\omega_p^*$ is a minimum at intermediate values of λ, the issue of stability is more complex , particularly if this minimum depends on β_0. Scenarios are now possible in which $[\omega_{dp}/\omega_p^*]_{\min} \geq 2/5$ at low values of β but not necessarily at larger values of β, and MHD instabilities may occur for β above some critical value β_{crit}. This is in fact the case for our equilibrium.

In Fig. 3, we plot the variation of the frequency ratio $\overline{\omega}_{dp}/\omega_p^*$ with the particle pitch angle λ for several values of the plasma beta. We consider the case of $\kappa_v x_p = 4/9$, slightly larger than the flute stability limit of 2/5, where the equilibrium is stable to flute modes in the limit of low beta.

The magnitude of $\overline{\omega}_{dp}/\omega_p^*$ decreases below 4/9 as the pitch angle λ decreases from unity, reaches a minimum, and thereafter increases as λ approaches zero. The minimum value reached decreases with increasing plasma beta β_0. At a high enough value of β_0, the fraction of particles with $\overline{\omega}_{dp}/\omega_p^* < 2/5$ becomes large enough so that the condition for flute stability (Eq. (8)) is violated.

This behaviour suggests that MHD modes would be destabilised at high plasma beta. The MHD ballooning mode equation given by Eq. (14) was solved numerically, with the

boundary condition $\partial\xi^{\psi}/\partial s = 0$ at $s = \pm L$, to determine the mode frequency. In Fig. 4 we plot ω^2 as a function of increasing β_0. MHD ballooning modes are unstable ($\omega^2 < 0$) if the plasma beta exceeds a critical beta of $\beta_{\rm crit} \sim 2$. From a numerical evaluation of $\left\langle \frac{5}{2}\overline{\omega}_{dp} - \omega_p^* \right\rangle$ with increasing plasma beta, we estimate the critical beta for flute stability to be $\beta_{\rm flute} \sim 1.8$. The proper MHD eigenfunction is not a flute perturbation but involves some line bending, and hence $\beta_{\rm crit} > \beta_{\rm flute}$.

MHD stability at a given value of β_0 can, however, be achieved if $\kappa_v x_p$ is sufficiently larger than 2/5. The optimised values of $B' \sim 7$ and $B_n \sim 0.1$ required to reproduce the Tsyganenko flux surfaces correspond to $\kappa_v x_p \sim 1.5$ and $\beta_0 \sim 10$, and the Earth's magnetosphere in its quiescent phase is stable to MHD modes.

Stability to MHD modes does not guarantee stability to drift modes. For the considered case of $\kappa_v x_p = 4/9$, we have $\overline{\omega}_{di}/\omega_{Ni}^* = 4(1+\eta_i)/9$ for particles with pitch angle $\lambda \to 1$ and the drift stability criterion is satisfied for these particles only if $4/5 > \eta_i > 7/11$ (see Fig. 1). At values of η_i outside this stability gap, the drift stability criterion is violated and drift instabilities are possible.

We have estimated the drift mode frequency by substituting the following trial function $\overline{\phi} = \phi_0\,(1+\alpha\lambda)$, where α is a variational parameter, in our low beta and high beta quadratic variational forms given by Eq. (20) and Eq. (24) . By Abel inversion, we obtain the field line variation: $\phi = \phi_0\{1 + \alpha(u + 1/2u^{1/2}\,\partial u/\partial s \int_0^s ds' u^{1/2}(s'))\}$. We again considered the case of $\kappa_v x_p = 4/9$, where MHD modes are stable at small plasma beta $\beta_0 < 1$, and we estimated the mode frequencies for values of $\beta_0 = 0.45$ and 1.8, and $k^2 r_{Li}^2 = 0.1$. At low beta $\beta_0 = 0.45$, we find that for $\eta_i < \eta_{\rm low} \sim 0.5$, drift modes are unstable with mode frequency pure imaginary, while for $\eta_i > \eta_{\rm high} \sim 3.5$, drift modes are unstable with the real part of the mode frequency finite. At higher values of β_0, the stability gap narrows as $\eta_{\rm low}$ and $\eta_{\rm high}$ approach each other. In Fig. 5 we plot the real and imaginary parts of ω as functions of η_i for $\beta_0 = 1.8$, and $k^2 r_{Li}^2 = 0.1$

We conclude that flux surface equilibria close to the marginal limit for low beta flute stability $\kappa_v x_p = 2/5$ are unstable to MHD ballooning modes if $\beta_0 > 2$ and that drift modes are unstable except within a gap in the neighbourhood of $\eta_i \sim 2/3$. However, plasma stability for a given value of plasma beta can be achieved if $\kappa_v x_p > 2/5$ is sufficiently large

and $\eta_i \sim 2/3$.

VI. SUMMARY

Although flute perturbations may not be proper eigenfunctions of the low frequency eigenmode equations, the condition for flute stability provides a simple and convenient criterion for assesing plasma stability. A sufficient condition for flute stability in the kinetic MHD frequency regime is the MHD flute stability criterion: $\left\langle \frac{5}{2}\overline{\omega}_d - \omega_p^* \right\rangle > 0$.

For magnetic dipole equilibria which can be modeled by the flux function of Eq. (25), the product of the vacuum field line curvature κ_v and the plasma scale length $x_p \equiv \left(\frac{B_0}{p_0}\frac{\partial p}{\partial \psi}\right)^{-1}$ at the equatorial plane must be of the order of or greater than 2/5 to satisfy the low beta MHD flute stability criterion. There exists a critical beta β_{crit} above which MHD ballooning modes are predicted to be unstable, with β_{crit} increasing as $\kappa_v x_p$ increases above 2/5.

A sufficient condition for flute stability of drift modes in a low beta plasma equilibrium is $\overline{\omega}_d(\lambda)/\omega_N^* > Max\{\eta, 2(1-\eta)\}$. The stability boundaries are displayed in Fig. 1. For equilibria close to the MHD flute stability limit, drift modes are unstable except for η_i values within a gap in the neighbourhood of $\eta_i \sim 2/3$. The magnitude of the stability gap decreases as the plasma beta increases. The admixture of compressional magnetic perturbations, finite for high beta plasmas, broadens somewhat the stability domain.

The Earth's magnetosphere, stable in its quiescent phase, is predicted to be unstable whenever the plasma gradient becomes sufficiently steep so that $\kappa_v x_p < 2/5$. This condition does not by itself single out flux surfaces located at ~ 8 Earth radii where $\beta_0 \sim 10$, unless the unstable modes are fast-growing with growth rates exceeding the particle bounce frequency. We pointed out in Sec. 3 that fast-growing instabilities can occur only at modest values of the plasma beta since stabilising compressional effects become dominant at higher plasma beta.

While the manifestations of the onset of the exponentially growing modes are varied and need more research, one scenario is described in some detail by Maynard etal. [23]. In that work with CRRES data, observations of 2-25 mHz growing oscillations of $\delta E_y(t)$ are made at satellite positions in the transitional region near the inner edge of the plasma sheet.

These oscillations typically occur two minutes before the sharp growth in the westward electrojet is seen with ground based magnetometers. At the last stages of the exponential growth the flux tube carrying the largest upward perturbed current will produce a substantial component of downward precipitating electrons. The perturbed upward field aligned current is large enough to produce a parallel potential drop that accelerates downward the magnetospheric electrons producing the auroral bright spot observed to correlate well with the expansion phase of the substorm. Clearly much nonlinear, interacting magnetosphere-ionosphere coupling-physics needs to be worked out to make a quantitative model. This sequence of events described by the Maynard scenario seems to fit well with the stability calculations described here. In this way the last e-folding of the unstable ballooning interchange instability gives the trigger for the substorm.

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FIGURE CAPTIONS

FIG. 1. Stability boundaries: Drift modes (1) $\overline{\omega}_{di}/\omega^*_{Ni} = \eta_i$; (2) $\overline{\omega}_{di}/\omega^*_{Ni} = 2\,(1-\eta_i)$; MHD flute (3) $\overline{\omega}_{di}/\omega^*_{Ni} = 2\,(1+\eta_i)\,/5$.

FIG. 2. Flux surfaces of Earth's magnetosphere: upper shows Tsganenko 1996; lower shows the model equilibrium (Constant Current + 2D dipole) of Eq. (25).

FIG. 3. Variation of frequency ratio $\overline{\omega}_{di}/\omega^*_{pi}$ as a function of the pitch angle λ for plasma beta values of $\beta_0 = 0.09, 0.9, 1.8, 3.6, 7.2$ and with $\kappa_v x_p = 4/9$.

FIG. 4. Solution of MHD ballooning mode equation, Eq. (14), showing ω^2 vs β_0 for $\kappa_v x_p = 4/9$.

FIG. 5. Solution of quadratic variational functional, Eq. (24), using trial function for ϕ, which gives Re(ω) and Im(ω) vs η_i for $\kappa_v x_p = 4/9$ and $\beta_0 = 1.8$, $k^2 r^2_{Li} = 0.1$.

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