The Proper Homogeneous Lorentz Transformation Operator

$$e^L = e^{-\boldsymbol{\omega}\cdot\boldsymbol{S}-\boldsymbol{\xi}\cdot\boldsymbol{K}},$$

Where's It Going, What's the Twist

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Abstract

A discussion of the proper homogeneous Lorentz transformation operator $e^L = \exp[-\boldsymbol{\omega}\cdot\boldsymbol{S}-\boldsymbol{\xi}\cdot\boldsymbol{K}]$ is given where e^L transforms coordinates of an observer $\mathcal{O}$ to those of an observer $\mathcal{O}'$. Two methods of evaluation are presented. The first is based on a dynamical analog. It is shown that the transformation can be evaluated from the set of equations that are identical to the set of equations that determine the 4-velocity of a charged particle in response to a combined spatially uniform and temporally constant electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, where $\boldsymbol{E}$ is parallel to $\boldsymbol{\xi}$ and $\boldsymbol{B}$ is antiparallel to $\boldsymbol{\omega}$, and $E/B = \xi/\omega$. The principal difference in the two problems is that in the dynamics problem, the initial conditions for the 4-velocity u must satisfy the constraint, $u \cdot u = 1$, whereas the inner product of the coordinates acted on by e^L can have any real value. In order to evaluate e^L, one can then apply the simplifying techniques of transforming to the frame where $\boldsymbol{E}$ is parallel or antiparallel to $\boldsymbol{B}$, whereupon the transformation e^L in this special frame is trivially evaluated. Then we transform back to the original frame. We determine the $\boldsymbol{\beta}$ and the rotation $\boldsymbol{\Omega}$ that results from a successive boost and rotation that the operator e^L produces. A second method is based on a direct summation of the power series of the matrix elements of e^L that

has been used in relativistic quantum theory. The summation is facilitated by observing that the operators, $\boldsymbol{J}_{\pm} \equiv \boldsymbol{K} \pm i\boldsymbol{S}$ commute with each other, and can be represented in terms of the Pauli spin matrices. Indeed, we can reduce the Lorentz transformation to the product of spinor operators to give a compact way to compute the elements of the Lorentz operator e^{L}.

I. INTRODUCTION

In a well-known textbook by Jackson (Ref. [1]) the most general form of a proper homogeneous Lorentz transformation used in classical special relativity is shown to have the form,

$$A(\boldsymbol{\omega}, \boldsymbol{\xi}) = e^{L} \tag{1}$$

with

$$L = -\boldsymbol{\omega} \cdot \boldsymbol{S} - \boldsymbol{\xi} \cdot \boldsymbol{K} \tag{2}$$

with $\boldsymbol{S}$ the generator for pure spatial rotations, and $\boldsymbol{K}$ the generator for pure boosts,

$$\begin{aligned} \boldsymbol{S} &= \widehat{\boldsymbol{x}} S_1 + \widehat{\boldsymbol{y}}\, S_2 + \widehat{\boldsymbol{z}}\, S_3, \\ \boldsymbol{K} &= \widehat{\boldsymbol{x}}\, K_1 + \widehat{\boldsymbol{y}} K_2 + \widehat{\boldsymbol{z}} K_3 \end{aligned} \tag{3}$$

where

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

and the other values of S_i and K_i are found in Ref. [1] (p. 546). They satisfy the commutation relations

$$[S_i, S_k] = \epsilon_{ijk} S_k, \qquad [S_i, K_j] = \epsilon_{ijk} K_k, \qquad [K_i, K_j] = -\epsilon_{ijk}\, S_k \tag{5}$$

where $[i = 1, 2, 3]$ and the repetition of indices imply summation. The first row and column will be referred as the 0-row or column, and the remaining will be labeled 1-3. Note that $i\hbar\boldsymbol{S}$ is a representation of an angular momentum operator in quantum mechanics and $-i\hbar\boldsymbol{K}$ is a representation of the boost operator found in relativistic quantum mechanics [2] (p. 39). The difference in the angular momentum and boost representation that we use and in the quantum system is a matter of bookkeeping. In a quantum system the generators are chosen so that rotation operators are Hermitian and boost operators are anti-Hermitian. However, the basic mathematical structure of the symmetry that is being described is really the same in the two systems. The commutation relations given by Eq. (5) give the structure constants, ϵ_{ijk} that define the Lie algebra [3,4] for the proper homogeneous Lorentz transformation. These relations are derived from symmetry arguments based on infinitesimal change of reference frame. The transformation operator $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$ is the formal solution to a finite change of the reference frame.

This exponential form does not directly answer the following questions for a given set of input parameters $\boldsymbol{\omega}$ and $\boldsymbol{\xi}$ in $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$: (a) what are expeditious ways to calculate the matrix elements of this Lorentz transformation; (b) what is the physical interpretation of the resulting Lorentz transformation. The answer is easy to obtain if either $\boldsymbol{\omega}$ or $\boldsymbol{\xi}$ vanish, but more difficult when both input parameters are nonzero (to aid the reader, the forms for pure rotation, $\exp(-\boldsymbol{\omega}\cdot\boldsymbol{S})$, and pure boost, $\exp(-\boldsymbol{\xi}\cdot\boldsymbol{K})$, is given in the Appendix). For the nontrivial case, where both $\boldsymbol{\omega}$ and $\boldsymbol{\xi}$ are nonzero, a method used in relativistic quantum mechanics can be applied which is based on the close correspondence of the $\mathrm{SL}(2, C)$ group and the proper homogeneous Lorentz group [2,5]. Here we present an alternative method of evaluation that generalizes a method used in the classical theory of special relativity for calculating the 4-velocity of a charged particle in spatially and temporally constant electric and magnetic fields. In this method, described in many textbooks on the classical theory of special relativity [e.g. in Refs. [1] and [6], the dynamical problem is solved by transforming to a frame where the electric and magnetic field are parallel (or anti-parallel)]. Here it is shown that very similar method can be applied to a different problem in special relativity; for the evaluation of the matrix elements of the operator e^{L}.

The results of the dynamical analog method are the same as the results that emerge in the quantum method described in Ref. [5]. However, a direct comparison to demonstrate the equivalency requires rather involved algebra. To make the comparison expeditiously, and self-contained within this paper, we present in detail a logical presentation of the relativistic quantum method that is somewhat different from that found in Ref. [5], but with the final results manifestly the same.

We have not found in the literature an answer to question (6) presented above, though the answer may exist in some publication. Hence to determine the answer we present an extensive discussion of interpreting the Lorentz transformation associated with the operator $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$.

To interpret the physical effect of the transformation we note that if a general homogeneous Lorentz transformation is applied to a set of coordinates of an observer $\mathcal{O}$ that is in an inertial frame, the cartesian coordinates are changed to those of an observer $\mathcal{O}'$ moving with a velocity $\boldsymbol{\beta} = \boldsymbol{v}/c$ with respect to $\mathcal{O}$. If the axes of $\mathcal{O}'$ are parallel to those of $\mathcal{O}$, the velocity of $\mathcal{O}$ with respect to $\mathcal{O}'$ will be $\boldsymbol{\beta}'$, where the components of $\boldsymbol{\beta}'$ are the negative of the components of $\boldsymbol{\beta}$. If the axes of $\mathcal{O}'$ are rotated with respect to $\mathcal{O}$, then the components of $\boldsymbol{\beta}'$ will differ from those of $\boldsymbol{\beta}$ (with the constraint that $|\boldsymbol{\beta}| = |\boldsymbol{\beta}'|$). The appropriate components of $\boldsymbol{\beta}'$ can be inferred by noting that $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$ can be written as a successive boost where there is no rotation of axes, followed by a rotation without a boost (or alternatively a successive rotation followed by a boost). Mathematically, we then have

$$e^{L(\boldsymbol{\omega},\boldsymbol{\xi})} = e^{L(\boldsymbol{\Omega},0)} e^{L(0,\boldsymbol{\Xi})} = e^{L(0,-\boldsymbol{\Xi}')} e^{L(\boldsymbol{\Omega},0)}. \tag{6}$$

where $\tanh |\boldsymbol{\Xi}| = \tanh |\boldsymbol{\Xi}'| = |\boldsymbol{\beta}|$, $\boldsymbol{\Xi}/|\boldsymbol{\Xi}| = \widehat{\boldsymbol{\beta}}$ and $\boldsymbol{\Xi}'/|\boldsymbol{\Xi}'| = \widehat{\boldsymbol{\beta}}'$. The determination of $\boldsymbol{\Omega}$ and $\boldsymbol{\Xi}$ as a function of $\boldsymbol{\omega}$ and $\boldsymbol{\xi}$ gives the physical meaning of the transformation $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$ as $\widehat{\boldsymbol{\Xi}}$ determines the magnitude of the boost, and $\boldsymbol{\Omega} = \Omega\widehat{\boldsymbol{\Omega}}$ determines the direction of the axis of rotation and angle Ω of the rotation.

We show in Sec. II that the transformation, $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$, is related to a set of differential equations that are identical to the equations of motion of the 4-velocity, u, of a charged particle of charge q and mass m in an electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ which are constant in space and time, but arbitrary in magnitude and direction. The equations of motion for u are, $du^i/d\tau = qF^i{}_k\, u^k/mc$, where $F^i{}_k$ is the mixed electromagnetic field strength tensor for $\boldsymbol{E}$ and $\boldsymbol{B}$ [obtained from Eqs. (11.137) or (11.138) in Ref. [1]], τ is the proper time, and

u^i a contravariant 4-vector. In the dynamics problem the inner product $u^i u_i$ is constrained to satisfy $u^i u_i = 1$, but in the Lorentz transformation problem u^i is a contravariant vector that can have an arbitrary inner product. We identify $\boldsymbol{E} = \lambda\boldsymbol{\xi} = \lambda\xi\widehat{\boldsymbol{\xi}}$, $\boldsymbol{B} = -\lambda\boldsymbol{\omega} = -\lambda\omega\widehat{\boldsymbol{\omega}}$ and $\omega/\xi = E/B \equiv \alpha$ (λ is a constant that is found in Sec. IV and α is a specified ratio for each specific transformation). That there is a relationship between rotation and boosts to the electric and magnetic fields has also been noted in the tract by Synge [7] (p. 94) with the comment "*the algebra of the infinitesimal transformation is essentially the same as that of the electromagnetic field.*" This means that the generators of the Lorentz group and electromagnetic field are in one to one correspondence with each other [8]. However, we have not found in the literature a calculation where this algebra is employed to evaluate e^L.

In the calculation presented here, we explicitly show that the mathematical techniques used to calculate the 4-velocity of a charged particle in an electric and magnetic field, is directly generalized to calculate the specific 4×4 matrix that the Lorentz transformation $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$ corresponds to. Specifically, the differential equation for an arbitrary 4-vector u^i can be solved by first transforming to a frame where $\boldsymbol{E}$ and $\boldsymbol{B}$ (or equivalently $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$) are parallel or antiparallel. In this frame the solution is extremely simple. Then, transforming back to the original frame produces the desired transformation $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$. The details of the calculation are presented in Sec. II. From this solution one easily determines the value of $\boldsymbol{\beta}$ that the transformation produces.

The evaluation of $\boldsymbol{\Omega}$ is not directly obvious even after the matrix elements of $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$ are known. In the latter part of Sec. II we discuss an interesting method by which $\boldsymbol{\Omega}$ can be determined, which is based on the symmetry found in the matrix elements of $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$. For fixed directions of $\boldsymbol{\omega}$ and $\boldsymbol{\xi}$, and fixed $\alpha \equiv |\boldsymbol{\omega}|/|\boldsymbol{\xi}|$, we find as $\xi \equiv |\boldsymbol{\xi}|$ varies that the rotation axis is along $\widehat{\boldsymbol{\omega}}$ as $\boldsymbol{\xi} \to 0$, and generally the axes of rotation continually precesses in the 3-plane defined by the 3-vectors $\widehat{\boldsymbol{\omega}}$ and $\widehat{\boldsymbol{\xi}}$, with the precession frequency a quasi-periodic function of ξ. The only exception arises to this precession is if $\boldsymbol{\xi} \cdot \boldsymbol{\omega} = 0$ or $\boldsymbol{\xi} \times \boldsymbol{\omega} = 0$, and then the axis of rotation is along $\widehat{\boldsymbol{\omega}}$.

In Sec. III we present the entirely different technique used in relativistic quantum theory for evaluating the matrix elements of $\exp[-\boldsymbol{\omega}\cdot\boldsymbol{S}-\boldsymbol{\xi}\cdot\boldsymbol{K}]$. A direct power series summation of the matrix elements for $\exp[-\boldsymbol{\omega}\cdot\boldsymbol{S}-\boldsymbol{\xi}\cdot\boldsymbol{K}]$ for arbitrary $\boldsymbol{\omega}$ and $\boldsymbol{\xi}$ appears at first sight to be quite complicated due to the multiplicity of commutation relations given by Eq. (5) for the 4×4 matrices $\boldsymbol{S}$ and $\boldsymbol{K}$. However, in Sec. III we show that the power series representation for the matrix elements of e^L can be summed straightforwardly when the generators are expressed in terms of $\boldsymbol{J}_\pm = \boldsymbol{K}\pm i\boldsymbol{S}$. We use that $[\boldsymbol{J}_+,\boldsymbol{J}_-]=0$ and that $J_{\pm i}^2 = 1$. We show that $\boldsymbol{J}_\pm$ can be expressed in terms of the Pauli spin matrix $\boldsymbol{\sigma}$ and then further reduced to obtain the Lorentz transformation in a form expressed solely in terms of spinor operators as it is found in Refs. [5] and [9]. Further analysis then enables us to replicate the results obtained by the previous method to find $\boldsymbol{\beta}$ and $\boldsymbol{\Omega}=\Omega\widehat{\boldsymbol{\Omega}}$.

II. DYNAMICAL ANALOG SOLUTION FOR $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$

In this section we solve and interpret the specific form of the Lorentz transformation operator by the dynamical analog methods discussed in the introduction.

A. Posing the Problem

We observe that if we take $\omega = \alpha\xi$ then e^L can be written in the form,

$$e^{L(\boldsymbol{\omega},\boldsymbol{\xi})} = \exp\left[-\xi\left(\alpha\widehat{\boldsymbol{\omega}}\cdot\boldsymbol{S}+\widehat{\boldsymbol{\xi}}\cdot\boldsymbol{K}\right)\right].$$

If we now take the derivative with respect to ξ, we find [we use that $AF(\lambda A)=F(\lambda A)A$ with λ a scalar, A an operator, and $F(x)$ an analytic function],

$$\frac{d}{d\xi}\,e^{L(\xi)} = \frac{\partial}{\partial\xi}\,\exp\left[-\xi\left(\alpha\widehat{\boldsymbol{\omega}}\cdot\boldsymbol{S}+\widehat{\boldsymbol{\xi}}\cdot\boldsymbol{\kappa}\right)\right] = -\left(\alpha\widehat{\boldsymbol{\omega}}\cdot\boldsymbol{S}+\widehat{\boldsymbol{\xi}}\cdot\boldsymbol{K}\right)e^{L(\xi)}. \tag{7}$$

Equation (7) is directly related to the equations of motion for the 4-velocity of a charged particle in a spatially and temporally constant electric and magnetic field. To see this, let the operators in Eq. (7) act on an arbitrary 4-vector

$$x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

and we obtain the equation,

$$\frac{dy}{d\xi} = -\left(\alpha\widehat{\boldsymbol{\omega}} \cdot \boldsymbol{S} + \widehat{\boldsymbol{\xi}} \cdot \boldsymbol{K}\right) y \tag{8}$$

where $y = e^{L(\xi)}x$, or more explicitly,

$$y_i(\xi) = \sum_{i=0}^{3} \left(e^{L(\xi)}\right)_{ij} \; x_j \tag{9}$$

(note that $y_i(0) = x_i$ and $\left(e^{L(\xi)}\right)_{ij}$ are the matrix elements we are seeking).

Now let us compare Eq. (8) to the response of the 4-velocity of a particle of charge q, and mass m to a uniform electric field $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$. The contravariant 4-velocity $u = (u_0, u_1, u_2, u_3) = (\gamma, \gamma\boldsymbol{v}/c)$ can be written as

$$\frac{du}{d\tau} = \frac{q}{mc}\left[-\boldsymbol{B} \cdot \boldsymbol{S} + \boldsymbol{E} \cdot \boldsymbol{K}\right] u \tag{10}$$

or in tensor notation

$$\frac{du^i}{d\tau} = \frac{q}{mc} F^i{}_k u^k, \tag{11}$$

with τ the local time of the accelerating particle, $\boldsymbol{E}$ and $\boldsymbol{B}$ are electric and magnetic fields that are constant in space and time,

$$F^i{}_k = F^{i\ell} g_{\ell k} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{12}$$

where $F^{i\ell}$ is given by Eq. (11.138) in Ref. [1] and g_{ij} is the usual metric tensor $g_{00} = 1,\ g_{ii} = -1\ (i = 1-3),\ g_{ij} = 0 (i \neq j)$.

If we take

$$\boldsymbol{B} = -\alpha E \hat{\boldsymbol{\omega}}, \qquad \boldsymbol{E} = E\hat{\boldsymbol{\xi}}, \qquad \xi = -q\tau E/mc \tag{13}$$

then Eqs. (10) and (11) are identical to Eq. (8), and thus the solution of the equation that determines the operator e^L is the same as the general solution of the equation that determines the particle 4-velocity in a constant electric and magnetic field. We note that the general solution allows a larger class of initial conditions than the physical dynamics problem. The initial conditions for the dynamics problem has a 4-velocity that must satisfy $u \cdot u \equiv u^i u_i \equiv u_0^2 - \boldsymbol{u} \cdot \boldsymbol{u} = 1$, while in Eq. (8) $x_0^2 - \boldsymbol{r} \cdot \boldsymbol{r} =$ const where the constant is real and can have any sign. Further, the antisymmetric nature of the covariant electromagnetic field strength tensor $F_{ik} = g_{i\ell} F^{\ell}{}_{i}$ guarantees that the solution $y^i y_i =$ const is independent of τ (or equivalently, ξ). Thus $u_i(\tau) u^i(\tau) = u_i(0) u^i(0) \equiv 1$ and $y_i(\xi) y^i(\xi) = x_i x^i$ (with $x^i = y^i(\xi = 0)$).

In principle Eq. (10) can be solved by straightforward techniques for solving a set of four coupled first-order linear equations. But as the equation that determines e^L is the same as the equation for the evolution of a 4-velocity in a constant electric and magnetic field, one can expedite the solution by first transforming to a reference frame $\mathcal{O}''$ where $\boldsymbol{E}''$ and $\boldsymbol{B}''$ are parallel (or antiparallel) to each other. The integrals that need to be performed are then indeed trivial, and one can then transform back to the initial reference frames. The procedure will be given below. The case when both $E = B$ and $\boldsymbol{E} \cdot \boldsymbol{B} = 0$ can be found as limiting case of the general problem that is solved.

B. Detailed Solution for $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$

Equations (10) [or (8)] determine a 2D Euclidean plane in which the 3-vectors $\hat{\boldsymbol{\omega}}$ and $\hat{\boldsymbol{\xi}}$ lie. Let us choose this plane to be the x-y plane. Thus, without loss of generality ω_3

and ξ_3 are set to zero. Further trivial simplification is achieved by choosing the x-axis so that $E_2 = 0$. Then the effective electric field is $\boldsymbol{E} = E\widehat{\boldsymbol{x}}$ and the effective magnetic field is $\boldsymbol{B} = -\alpha E\widehat{\boldsymbol{\omega}}$ and $\widehat{\boldsymbol{\omega}} = \cos\chi\widehat{\boldsymbol{x}} + \sin\chi\widehat{\boldsymbol{y}}$.

The effective fields $\boldsymbol{E}$ and $\boldsymbol{B}$ can be made antiparallel (or parallel) to each other by considering the equation in a frame $\mathcal{O}''$, moving in the z-direction, with a relative speed β''. In this intermediate frame, the electric and magnetic fields will be $\boldsymbol{E}'' = E''(\cos\sigma\widehat{\boldsymbol{x}} + \sin\sigma\widehat{\boldsymbol{y}})$ and $\boldsymbol{B}'' = \mp B''(\cos\sigma\widehat{\boldsymbol{x}} + \sin\sigma\widehat{\boldsymbol{y}})$ where E'' and B'' are related to $\boldsymbol{E}$ and $\boldsymbol{B}$ by (see Eq. (11.149) of Ref. [1]),

$$
\begin{aligned}
E''_x &= E''\cos\sigma = \gamma''(E_x - \beta'' B_y); \quad E''_y = E''\sin\sigma = \gamma''(E_y + \beta'' B_x) \\
B''_x &= \mp B''\cos\sigma = \gamma''(B_x + \beta'' E_y); \quad B''_y = \mp B''\sin\sigma = \gamma''(B_y - \beta'' E_x)
\end{aligned}
\tag{14}
$$

with $\gamma'' = 1/(1-\beta''^2)^{1/2}$, and $\begin{pmatrix} E_x \\ E_y \end{pmatrix} = E \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} B_x \\ B_y \end{pmatrix} = -\alpha E \begin{pmatrix} \cos\chi \\ \sin\chi \end{pmatrix}$. By setting $E''_x/B''_x = E''_y/B''_y$, we eliminate σ and we find the relation (found in Ref. [6], p. 65),

$$
\frac{\beta''}{1+\beta''^2} \equiv \frac{(\boldsymbol{E}\times\boldsymbol{B})_3}{|E|^2 + |B|^2} \equiv \frac{S_3}{W} \equiv \lambda = \frac{-\alpha\sin\chi}{(1+\alpha^2)}
\tag{15}
$$

where cS_3 is the Poynting Flux in the z-direction and W the field energy of our analogous effective fields.

Note $|\lambda| < 1/2$. The solution for β'' is

$$
\beta'' = \frac{1}{2\lambda}\left[1 - (1-4\lambda^2)^{1/2}\right]
$$

$$
\gamma'' = \frac{1}{(1-\beta''^2)^{1/2}} = \frac{\left[1 + (1-4\lambda^2)^{1/2}\right]^{1/2}}{\sqrt{2}(1-4\lambda^2)^{1/4}}
$$

$$
\gamma''\beta'' = \frac{\left[(1 - (1-4\lambda^2)^{1/2}\right]^{1/2}\lambda}{\sqrt{2}(1-4\lambda^2)^{1/4}|\lambda|}.
\tag{16}
$$

Observe that $\beta''(\alpha) = \beta''(1/\alpha)$.

In the intermediate frame, given by

$$
x'' = \exp\left[-\tanh^{-1}\beta'' K_3\right] x \equiv Nx
$$

where

$$N = \begin{pmatrix} \gamma'' & 0 & 0 & -\beta''\gamma'' \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta''\gamma'' & 0 & 0 & \gamma'' \end{pmatrix} \tag{17}$$

the electric and magnetic fields, will be antiparallel if $-\pi/2 < \chi < \pi/2$ or parallel if $\pi/2 < \chi < 3\pi/2$.

We note that the magnitude of the transformed electric $\boldsymbol{E}'' = E''(\widehat{\boldsymbol{x}} \cos\sigma + \widehat{\boldsymbol{y}} \sin\sigma)$ and magnetic fields $\boldsymbol{B}'' = \mp B''(\widehat{\boldsymbol{x}} \cos\sigma + \widehat{\boldsymbol{y}} \sin\sigma)$ and its orientation angle σ, are given by,

$$\begin{aligned} \frac{E''}{E} &= \gamma'' \left[1 + \alpha^2\beta''^2 + 2\alpha\beta'' \sin\chi\right]^{1/2} = \frac{1}{\sqrt{2}} \left\{\left[(1-\alpha^2)^2 + 4\alpha^2\cos^2\chi\right]^{1/2} + 1 - \alpha^2\right\}^{1/2} \equiv \lambda_1 \\ \frac{B''}{E} &= \gamma'' \left[\alpha^2 + \beta''^2 + 2\alpha\beta'' \sin\chi\right]^{1/2} = \frac{1}{\sqrt{2}} \left\{\left[(1-\alpha^2)^2 + 4\alpha^2\cos^2\chi\right]^{1/2} + \alpha^2 - 1\right\}^{1/2} \equiv \lambda_2 \end{aligned} \tag{18}$$

$$\begin{aligned} \begin{pmatrix} \cos\sigma \\ \sin\sigma \end{pmatrix} &= \begin{pmatrix} 1 + \alpha\beta'' \sin\chi \\ -\beta''\alpha\cos\chi \end{pmatrix} \frac{1}{[1 + \alpha^2\beta''^2 + 2\alpha\beta'' \sin\chi]^{1/2}} \\ &= \pm \begin{pmatrix} \alpha\cos\chi \\ \alpha\sin\chi + \beta'' \end{pmatrix} \frac{1}{[\alpha^2 + \beta''^2 + 2\alpha\beta'' \sin\chi]^{1/2}}. \end{aligned} \tag{19}$$

Here the upper sign is chosen for the $-\pi/2 < \chi < \pi/2$ ($\boldsymbol{B}''$ antiparallel to $\boldsymbol{E}''$) and the lower sign for $\pi/2 < \chi < 3\pi/2$ ($\boldsymbol{B}''$ parallel to $\boldsymbol{E}''$). Unless otherwise stated, this sign choice holds throughout this paper. Observe that:

$$\lambda_1(\alpha) = \alpha\lambda_2(1/\alpha) \tag{20}$$

and that we may also determine σ from the last relation in Eq. (19),

$$\tan\sigma = (\tan\chi + \beta''/\alpha\cos\chi). \tag{21}$$

We note that if χ is in the first quadrant, then by using Eqs. (15), (16), and (21), we find

$$\frac{\tan\sigma}{\tan\chi} = 1 - \frac{1}{2(1+\alpha^2)\lambda^2}\left[1 - (1-4\lambda^2)^{1/2}\right] \equiv 1 - R\Big(\alpha, \lambda(\alpha,\chi)\Big).$$

We then observe that $f(x) \equiv \frac{1}{x}\left[1-(1-x)^{1/2}\right]$ is a positive monotonically increasing function of x in the regime $0 < x < 1$. Hence

$$0 < R\Big(\alpha, \lambda(\alpha,\chi)\Big) < R\Big(\alpha, \lambda(\alpha,\pi/2)\Big) = \frac{1}{2\alpha^2}\left[(1+\alpha^2) - |1-\alpha^2|\right]$$
$$= \begin{cases} 1, & \text{if } \alpha < 1 \\ \\ 1/\alpha^2, & \text{if } \alpha > 1. \end{cases} \tag{22}$$

Thus, $0 \leq R\Big(\alpha, \lambda(\alpha,\chi)\Big) \leq 1$, and $0 \leq \tan\sigma/\tan\chi \leq 1$. It then follows that for χ in the first quadrant, $0 \leq \sigma \leq \chi$. More generally, it readily follows for the various quadrants, (taking $-\pi < \chi < \pi$)

$$\begin{array}{ll} \chi \text{ in first quadrant} & 0 \leq \sigma \leq \chi \\ \chi \text{ second quadrant} & \pi \geq \sigma + \pi \geq \chi \\ \chi \text{ third quadrant} & -\pi \leq \sigma - \pi \leq \chi \\ \chi \text{ fourth quadrant} & 0 \geq \sigma \geq \chi. \end{array} \tag{23}$$

One can further ascertain from Eq. (21) that if $\sigma(\chi)$ is known for χ in the first quadrant, we can express $\sigma(\chi)$ in any quadrant,

$$\begin{aligned} \sigma(\alpha, -\chi) &= -\sigma(\alpha,\chi) \\ \sigma(\alpha, \pi-\chi) &= -\sigma(\alpha,\chi) \\ \sigma(\alpha, \chi-\pi) &= \sigma(\alpha,\chi). \end{aligned} \tag{24}$$

In Figs. 1-3, we present several figures for the parameters we have introduced. In Fig. 1 we plot $-\beta''$ vs. $\chi' \equiv 2\chi/\pi$ $(0 < \chi' < 1)$ for the values $\alpha = .1, .25, .5, .75$, and 1, with the curves for the smaller values of α lying below the curves with a larger value of α. These curves can be applied for $\alpha > 1$, as $\beta''(1/\alpha) = \beta''(\alpha)$. In Fig. 2 we plot $\chi' - \sigma' \equiv 2(\chi-\sigma)/\pi$ vs. χ' for $\alpha = .25, .5, .75, 1., 1.25, 2, 4$. The curves for larger α lie below the curves with smaller α. In Fig. 3a we plot λ_1 vs. χ for $\alpha = .1, .25, .5, .75$, and 1, and the curves with larger α lie below the curves with smaller α. In Fig. 3b we plot $\Delta\lambda_1(\alpha) \equiv \lambda_1(1) - \lambda_1(\alpha)$ vs. χ' for $\alpha = 1.25, 2.5, 2$, and 4. The curves for λ_2 can be inferred from these graphs, using Eq. (20). Note that $\lambda_1(\alpha, \chi)$ decreases monotonically as a function of χ from $\chi = 0$ to $\chi = \pi/2$, ranging from unity when $\chi = 0$, to $(1 - \alpha^2)^{1/2}$ if $\alpha < 1$ or zero if $\alpha > 1$, when $\chi = \pi/2$.

Continuing in the construction of e^L, it is convenient to make an additional rotation transformation so that $\boldsymbol{E}''$ is along an intermediate x-axis. Hence, with $x''' = Hx''$, where $H = e^{-\sigma S_3}$, we need to evaluate the dynamical equation,

$$\frac{mc}{qE}\frac{dx'''}{d\tau} = -\frac{dx'''}{d\xi} = HNFN^{-1}H^{-1}x''' \tag{25}$$

where $HNFN^{-1}H^{-1} \equiv R$ is explicitly given by

$$R = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp\lambda_2 \\ 0 & 0 & \pm\lambda_2 & 0 \end{pmatrix}. \tag{26}$$

Note that equations for x''' have been reduced to two uncoupled sets of 2×2 linear equations. We readily find the screw transformation defined by Eq. (127), Chap. IV, in Ref. [7] gives the solution for $x'''(\xi)$ in terms of $x'''(0)$,

$$\begin{pmatrix} x_0'''(\xi) \\ x_1'''(\xi) \\ x_2'''(\xi) \\ x_3'''(\xi) \end{pmatrix} = \begin{pmatrix} \cosh\lambda_1\xi & -\sinh\lambda_1\xi & 0 & 0 \\ -\sinh\lambda_1\xi & \cosh\lambda_1\xi & 0 & 0 \\ 0 & 0 & \cos\lambda_2\xi & \pm\sin\lambda_2\xi \\ 0 & 0 & \mp\sin\lambda_2\xi & \cos\lambda_2\xi \end{pmatrix} \begin{pmatrix} x_0'''(0) \\ x_1'''(0) \\ x_2'''(0) \\ x_3'''(0) \end{pmatrix} \tag{27}$$

In compact notation, we write Eq. (27) as

$$x'''(\xi) = Q(\xi)x'''(0) \tag{28}$$

where $Q(\xi)$ is the square matrix shown in Eq. (27).

Returning to our original reference frame we have

$$x(\xi) = N^{-1}H^{-1}QHN\, x(0). \tag{29}$$

Thus the desired solution for the transformation is

$$\exp\left[-\xi(\alpha\cos\chi S_1+\alpha\sin\chi S_2+K_1)\right]=e^L=N^{-1}H^{-1}QHN. \tag{30}$$

To facilitate the matrix multiplication note that

$$HN=\begin{pmatrix}\gamma'' & 0 & 0 & -\gamma''\beta'' \\ 0 & \cos\sigma & \sin\sigma & 0 \\ 0 & -\sin\sigma & \cos\sigma & 0 \\ -\gamma''\beta'' & 0 & 0 & \gamma''\end{pmatrix} \tag{31}$$

and $N^{-1}H^{-1}=(HN)^{-1}$ is obtained by changing the sign of β'' and σ in Eq. (31). By performing the matrix multiplications, the matrix form of e^L is obtained,

$$\begin{pmatrix}
\gamma''^2\Big[\cosh(\lambda_1\xi)-\beta''^2\cos(\lambda_2\xi)\Big], & -\gamma''\Big[\sinh(\lambda_1\xi)\cos\sigma\mp\beta''\sin(\lambda_2\xi)\sin\sigma\Big], & -\gamma''\Big[\sinh(\lambda_1\xi)\sin\sigma\pm\beta''\sin(\lambda_2\xi)\cos\sigma\Big], & -\gamma''^2\beta''\Big[\cosh(\lambda_1\xi)-\cos(\lambda_2\xi)\Big], \\
-\gamma''\Big[\sinh(\lambda_1\xi)\cos\sigma\mp\beta''\sin(\lambda_2\xi)\sin\sigma\Big], & \cos^2\sigma\cosh(\lambda_1\xi)+\sin^2\sigma\cos(\lambda_2\xi), & \cos\sigma\sin\sigma\Big[\cosh(\lambda_1\xi)-\cos(\lambda_2\xi)\Big], & \gamma''\Big[\beta''\sinh(\lambda_1\xi)\cos\sigma\mp\sin(\lambda_2\xi)\sin\sigma\Big], \\
-\gamma''\Big[\sinh(\lambda_1\xi)\sin\sigma\pm\beta''\cos\sigma\sin(\lambda_2\xi)\Big], & \cos\sigma\sin\sigma\Big[\cosh(\lambda_1\xi)-\cos(\lambda_2\xi)\Big], & \cosh(\lambda_1\xi)\sin^2\sigma+\cos(\lambda_2\xi)\cos^2\sigma, & \gamma''\Big[\beta''\sinh(\lambda_1\xi)\sin\sigma\pm\sin(\lambda_2\xi)\cos\sigma\Big], \\
\gamma''^2\beta''\Big[\cosh(\lambda_1\xi)-\cos(\lambda_2\xi)\Big], & -\gamma''\Big[\beta''\sinh(\lambda_1\xi)\cos\sigma\mp\sin(\lambda_2\xi)\sin\sigma\Big], & -\gamma''\Big[\beta''\sinh(\lambda_1\xi)\sin\sigma\pm\sin(\lambda_2\xi)\cos\sigma\Big], & \gamma''^2\Big[-\beta''^2\cosh(\lambda_1\xi)+\cos(\lambda_2\xi)\Big],
\end{pmatrix}. \tag{32}$$

We have confirmed with spot numerical checks that $\partial\exp[-\xi(\alpha\hat{\boldsymbol{\omega}}\cdot\boldsymbol{S}+K_1)]/\partial\xi = N^{-1}H^{-1}\,\partial Q/\partial\xi\,HN=-(\alpha\hat{\boldsymbol{\omega}}\cdot\boldsymbol{S}+K_1)e^L$. Note that the choice of sign changes just when χ passes through $\pm\pi/2$. One can show that Eq. (32) is continuous and smooth as χ goes through $\pm\pi/2$. Observe that Eq. (32) has the property that

$$L_{00}+L_{33}=L_{11}+L_{22}=\cosh\lambda_1\xi+\cos\lambda_2\xi. \tag{33}$$

The solution we have obtained is for a conveniently chosen coordinate system where $\hat{\boldsymbol{z}}$ is perpendicular to $\hat{\boldsymbol{\omega}}$ and $\hat{\boldsymbol{\xi}}$ (or equivalently $\boldsymbol{E}$ and $\boldsymbol{B}$) and $\hat{\boldsymbol{\xi}} = \hat{\boldsymbol{x}}$. With additional straightforward rotation operations the solution can easily be made arbitrarily general.

The value of the boost is obtained fairly straightforwardly from Eq. (33). By using an initial value of $x = (1, 0, 0, 0)$ which is the 4-velocity in frame $\mathcal{O}$, then $y(\xi) = e^L x$ gives this 4-velocity $\gamma[1, \beta_1', \beta_2', \beta_3']$ in the $\mathcal{O}'$ system. This is the first column of Eq. (32). To obtain β_i $(i = 1 - 3)$ [the components of the relative velocity of $\mathcal{O}'$ in the $\mathcal{O}$ frame] we need to obtain the first column of the matrix that is inverse to Eq. (32). This is achieved by setting $\xi \to -\xi$ in Eq. (32). We then infer that γ and $\boldsymbol{\beta}$ are given by,

$$
\begin{aligned}
\gamma &= \gamma''^2 \left[\cosh(\lambda_1\xi) - \beta''^2 \cos(\lambda_2\xi)\right] \\
\beta_1 &= \frac{[\sinh(\lambda_1\xi)\cos\sigma \mp \beta''\sin(\lambda_2\xi)\sin\sigma]}{\gamma''\,[\cosh(\lambda_1\xi) - \beta''^2\cos(\lambda_2\xi)]} \\
\beta_2 &= \frac{[\sinh(\lambda_1\xi)\sin\sigma \pm \beta''\sin(\lambda_2\xi)\cos\sigma]}{\gamma''\,[\cosh(\lambda_1\xi) - \beta''^2\cos(\lambda_2\xi)]} \\
\beta_3 &= \frac{\beta''\,[\cosh(\lambda_1\xi) - \cos(\lambda_2\xi)]}{[\cosh(\lambda_1\xi) - \beta''^2\cos(\lambda_2\xi)]}.
\end{aligned}
\tag{34}
$$

We note that there is an indeterminacy in Eq. (32) when $\boldsymbol{\xi} \perp \boldsymbol{\omega}$ and $\alpha = 1$ where $|\beta''| = 1$. To resolve the result we take the limit $\alpha \to 1$ and we find,

$$
e^L = \begin{pmatrix}
1 + \dfrac{\xi^2}{2} & -\xi & 0 & \pm\dfrac{\xi^2}{2} \\
-\xi & 1 & 0 & \mp\xi \\
0 & 0 & 1 & 0 \\
\mp\dfrac{\xi^2}{2} & \pm\xi & 0 & 1 - \dfrac{\xi^2}{2}
\end{pmatrix}.
\tag{35}
$$

This particular transformation is denoted in the text by Hamermesh [4] on p. 494 as $T_1(-\xi)$ (note in Ref. [4] the ordering of the axes are 3, 2, 1, 0 rather than 0, 1, 2, 3).

C. Evaluation of Orientation Axes of $\mathcal{O}'$

There is a striking symmetry in the result in Eq. (32); the off-diagonal elements when neither i or j is three, satisfy $\left(e^L\right)_{ij} = \left(e^L\right)_{ji}$, while if either i or j is three, these off-diagonal elements satisfy $\left(e^L\right)_{ij} = -\left(e^L\right)_{ij}$. We will develop a method for the evaluation of $\boldsymbol{\Omega}$, that makes use of the constraint that the Lorentz transformation for $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$, which can be expressed as a successive boost, followed by a pure rotation, must exhibit this symmetry, as well as reproduce the relation given by Eq. (33).

In general, a successive boost followed by a rotation does not exhibit the symmetry found in Eq. (32). For example, if we choose the rotation axis as the x-axis then we find

$$\exp(-\Omega S_1)\exp(-\boldsymbol{\Xi}\cdot\boldsymbol{K}) = \begin{pmatrix} \gamma & -\gamma\boldsymbol{\beta} \\ -\gamma\boldsymbol{\beta}'^T & \dfrac{-\gamma^2\boldsymbol{\beta}'^T\boldsymbol{\beta}}{\gamma+1} + \exp_3(-\Omega S_1) \end{pmatrix} \tag{36}$$

where row 3-vectors are $\boldsymbol{\beta} \equiv (\beta_1, \beta_2, \beta_3)$, $\boldsymbol{\beta}' = (\beta_1, \beta_2\cos\Omega + \beta_3\sin\Omega, -\beta_2\sin\Omega + \beta_3\cos\Omega)$, and $\exp_3(-\Omega S_1)$ is a 3×3 rotation matrix without the 0-row and 0-column. The superscript "T" denotes transpose that converts a row vector to a column vector, and $\boldsymbol{\beta}'^T\boldsymbol{\beta}$ is a direct product of the column and row vectors.

Clearly Eq. (36) does not exhibit the required symmetry unless there is a special relation between the components of $\boldsymbol{\beta}'$ and $\boldsymbol{\beta}$. As an example where this relation occurs is when there is a rotation about the x-axis, which leaves β_1 untouched, but takes $\beta_2 \to \beta_2$ and β_3 to $-\beta_3$. This rotation is shown in Fig. 4, which clearly gives the relation

$$\tan\frac{\Omega}{2} = \frac{\beta_3}{\beta_2}. \tag{37}$$

One can verify algebraically that Eq. (37) implies

$$\begin{aligned} \beta_2\cos\Omega + \beta_3\sin\Omega &= \beta_2 \\ -\beta_2\sin\Omega + \beta_3\cos\Omega &= -\beta_3. \end{aligned} \tag{38}$$

Denoting $\mathcal{L}_{ij}$ as the matrix elements of Eq. (36) we also have

$$\mathcal{L}_{00} + \mathcal{L}_{33} = \mathcal{L}_{11} + \mathcal{L}_{22} = \gamma + \cos\Omega - (\gamma - 1)\ \beta_3^2/\beta^2. \tag{39}$$

Because of matrix element symmetry observed in Eq. (32), we call this the synchronism between rotation and boost required to achieve the synchronous rotational boost symmetry (SRB).

More generally, if the rotation axis in the x-y plane is along $\widehat{\boldsymbol{\phi}} = \cos\phi\widehat{\boldsymbol{x}} + \sin\phi\widehat{\boldsymbol{y}}$, then the Lorentz transformation

$$\exp\left[-\Omega\widehat{\boldsymbol{\phi}}\cdot\boldsymbol{S}\right]\exp[-\boldsymbol{\Xi}\cdot\boldsymbol{K}],$$

exhibits SRB symmetry, if

$$\begin{aligned} \widetilde{\beta}_2\cos\Omega + \beta_3\sin\Omega &= \widetilde{\beta}_2 \\ -\widetilde{\beta}_2\sin\Omega + \beta_3\cos\Omega &= -\beta_3. \end{aligned} \tag{40}$$

where $\widetilde{\beta}_1 = \beta_1\cos\phi + \beta_2\sin\phi$ is the component of $\boldsymbol{\beta}$ along the $\widehat{\boldsymbol{\phi}}$ axis and $\widetilde{\beta}_2 = -\beta_1\sin\phi + \beta_2\cos\phi$ is the component of $\boldsymbol{\beta}$ along the $\widehat{\boldsymbol{z}}\times\widehat{\boldsymbol{\phi}}$ axis. It can be shown that if $\boldsymbol{\omega}$ and $\boldsymbol{\xi}$ lie in the x-y plane, Eq. (40), with $\boldsymbol{\phi}$ only in the x-y plane, is the most general relation needed to achieve SRB symmetry.

The form of the Lorentz transformation, when Eq. (40) is satisfied is then found to be,

$$e^{-\Omega(\cos\phi S_1+\sin\phi S_2)}\cdot e^{-\boldsymbol{\xi}\cdot\boldsymbol{K}} = \begin{pmatrix} \gamma, & -\gamma\boldsymbol{\beta}_a \\ -\gamma\boldsymbol{\beta}_b^T & \dfrac{\gamma^2}{\gamma-1}\ \boldsymbol{\beta}_b^T\boldsymbol{\beta}_a + \exp_3[-\Omega\widehat{\boldsymbol{\phi}}\cdot\boldsymbol{S}] \end{pmatrix} \tag{41}$$

where $\boldsymbol{\beta}_a = (\widetilde{\beta}_1, \widetilde{\beta}_2, \beta_3)$ and $\boldsymbol{\beta}_b = (\widetilde{\beta}_1, \widetilde{\beta}_2, -\beta_3)$. Further, Eq. (39) is found to still be correct independent of $\boldsymbol{\phi}$. Now, we can obtain the value $\Omega\widehat{\boldsymbol{\phi}}$ by matching the elements $\mathcal{L}_{ij}$ of Eq. (41) term by term to be elements of Eq. (32). With the help of Eq. (A3) of the Appendix we also establish that the diagonal matrix elements of Eq. (41) satisfy the same relation as given by Eq. (39) which does not depend on ϕ.

We now determine the direction of the axis of rotation $\widehat{\boldsymbol{\Omega}} = \cos\phi\widehat{\boldsymbol{x}} + \sin\phi\widehat{\boldsymbol{y}}$ and the amount of rotation Ω about the axis for the Lorentz transformation $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$. From Eq. (34)

we know the value of $\boldsymbol{\beta}$ of $\mathcal{O}'$ with respect to $\mathcal{O}$ in terms of the input parameters α, χ and ξ. Then by equating the 3-3 elements of Eqs. (32) and (41), we find, after some algebra, that Ω is determined by,

$$\cos\Omega = \frac{\gamma^2}{\gamma+1}\,\beta_3^2 + \gamma''^2\left[-\beta''^2\cosh\lambda_1\xi + \cos\lambda_2\xi\right] = -1 + G(\xi) \tag{42}$$

with

$$G(\xi) = \frac{(1+\cos\lambda_2\xi)}{\gamma''^2\left[1-\beta''^2(1+\cos\lambda_2\xi)/(\cosh\lambda_1\xi+1)\right]}. \tag{43}$$

Note that the minimum value of $\cos\Omega$ is -1, and unless either $\beta''=0$, $\lambda_1=0$, or $\xi=0$, the maximum value of $\cos\Omega<1$. This follows because $0\le G(\xi)<\frac{1+\cos\lambda_2\xi}{\gamma''^2(1-\beta''^2)}=1+\cos\lambda_2\xi\le 2$, and the right-sided equality cannot be achieved unless either β'', λ_1 or ξ vanish. Except for these special cases (which, if $\xi\neq 0$, turn out to be equivalent to $\chi=p\pi/2$, with p an integer), then as ξ increases, $\cos\Omega=1$ only for $\xi=0$, and Ω remains less than 2π. As a result Ω oscillates. The maximum value of $\cos\Omega$ occurs close to $1+\cos\lambda_2\xi=2$ (this condition becomes more accurate the larger $\cosh\lambda_1\xi$ becomes). For $\cosh\lambda_1\xi\gg 1$, Ω oscillates between

$$-\cos^{-1}(1-2\beta''^2)<\Omega<\cos^{-1}(1-2\beta''^2). \tag{44}$$

The value of $\sin\Omega$ is determined to within a sign, $\sin\Omega=\pm[1-\cos^2\Omega]^{1/2}$. For sufficiently small $\xi, \sin\Omega>0$, and $\sin\Omega$ changes sign when $1+\cos\Omega$ vanishes, or equivalently when Ω passes through π. From Eqs. (42) and (43) this occurs when $\cos\lambda_2\xi=-1$. Hence $\sin\Omega>0$ for $2n\pi<\lambda_2\xi<(2n+1)\pi$, for integer n, and $\sin\Omega<0$ for $(2n+1)\pi<\lambda_2\xi<2(n+1)\pi$. Also note that when $\Omega\to(2n+1)\pi$, that $\frac{1+\cos\Omega}{\sin\Omega}\to 0$ because $1+\cos\Omega$ vanishes quadratically in $\lambda_2\xi-(2n+1)\pi$, while $\sin\Omega$ vanishes linearly in $\lambda_2\xi-(2n+1)\pi$.

To obtain the axis of rotation, we use the relation in Eq. (40),

$$(\beta_1\sin\phi-\beta_2\cos\phi)\sin\Omega+\beta_3\cos\Omega=-\beta_3$$

and we substitute for β_1, β_2, and β_3 the equations given by Eq. (34). After some algebra we find for the equation determining ϕ,

$$\gamma''\beta''(\cosh\lambda_1\xi - \cos\lambda_2\xi)\frac{(1+\cos\Omega)}{\sin\Omega}$$

$$= -\frac{\left[\cos\left(\frac{\lambda_2\xi}{2}\right)\sinh(\lambda_1\xi) + \sin\left(\frac{\lambda_2\xi}{2}\right)\sin\lambda_2\xi\left(1+\frac{2}{(\cosh\lambda_1\xi-1)}\right)^{1/2}\right]}{\left\{1+\frac{\sin^2(\lambda_2\xi/2)}{\beta''^2}\left[\frac{1}{\gamma''^2}+\frac{2}{\cosh\lambda_1\xi-1}\right]\right\}^{1/2}}$$

$$= -\sinh\lambda_1\xi\sin(\phi-\sigma) \pm \beta''\sin\lambda_2\xi\cos(\phi-\sigma). \tag{45}$$

The solution to this equation is,

$$\sin(\phi-\sigma) =$$

$$\frac{\sinh\lambda_1\xi\left(\sinh\lambda_1\xi\,\cos\left(\frac{\lambda_2\xi}{2}\right) + \sin(\lambda_2\xi)\sin\left(\frac{\lambda_2\xi}{2}\right)\left(1+\frac{2}{\cosh\lambda_1\xi-1}\right)^{1/2}\right) - 2\sin^2\left(\frac{\lambda_2\xi}{2}\right)\cos\left(\frac{\lambda_2\xi}{2}\right)\sqrt{M^2}}{\left(\sinh^2(\lambda_1\xi)+\beta''^2\sin^2\lambda_2\xi\right)\left[1+\frac{\sin^2\left(\frac{\lambda_2\xi}{2}\right)}{\beta''^2}\left(\frac{1}{\gamma''^2}+\frac{2}{\cosh\lambda_1\xi-1}\right)\right]^{1/2}} \tag{46}$$

$$\xrightarrow[\lambda_1\xi\gg 1]{} \sin\left(\frac{\pi}{2}-\frac{\lambda_2\xi}{2}\right)\Big/\left[1+\sin^2\left(\frac{\lambda_2\xi}{2}\right)/(\gamma''\beta'')^2\right]^{1/2} \tag{47}$$

with

$$\sqrt{M^2} = (\cosh\lambda_1\xi - 1) + \left[2-\beta''^2\left(1+\cos\lambda_2\xi\right)\right]. \tag{48}$$

Note that one can show that this solution only gives precession without oscillation of ϕ as ξ changes as long as $|\chi| \neq \pi/2$, $\chi \neq 0$ or $\chi \neq \pi$. For the cases that are exceptions, the orientation of the rotation axis remains fixed as ξ changes.

We also observe that if a solution for ϕ is obtained for a given χ, then for $\chi \to -\chi$, the solution for $\widehat{\boldsymbol{\phi}}$ is $\widehat{\boldsymbol{\phi}} = \cos\phi\widehat{\boldsymbol{x}} - \sin\phi\widehat{\boldsymbol{y}}$; for $\chi \to \pi - \chi$, the solution for $\widehat{\boldsymbol{\phi}}$ is $\widehat{\boldsymbol{\phi}} = -\cos\phi\widehat{\boldsymbol{x}} + \sin\phi\widehat{\boldsymbol{y}}$; and $\chi \to \chi + \pi$ the solution for $\widehat{\boldsymbol{\phi}}$ is $\widehat{\boldsymbol{\phi}} = -\cos\phi\widehat{\boldsymbol{x}} - \sin\phi\widehat{\boldsymbol{y}}$. These symmetry conditions allow us to describe all solutions in terms of χ in the first quadrant. One can show that precession is clockwise for χ in the first quadrant, and thus it follows that the precession is clockwise for χ in the third quadrant, and counterclockwise for χ in the second and fourth quadrants.

Now consider the special cases $\chi = 0, \pi$ ($\beta'' = 0, \gamma'' = 1$) and $\chi = \pm\pi/2$. For the first cases ($\chi = 0, \pi$), where $\sigma = 0$, we obtain from Eqs. (42) and (45),

$$\cos\Omega = \cos\lambda_2\xi \ , \qquad \sin\Omega = \sin\lambda_2\xi, \qquad \sin(\phi - \sigma) = 0. \tag{49}$$

Thus, $\phi = \chi$ and $\chi = 0$ or $\chi = \pi$. Hence in this case there is no precession, but a fixed axis of rotation, and Ω changes without bound as ξ increases.

If $\chi = \pm\pi/2$, we have for $\alpha < 1$; $\beta'' = \mp\alpha$, $\gamma'' = 1/(1-\alpha^2)^{1/2}$, $\lambda_1 = (1-\alpha^2)^{1/2}$, $\lambda_2 = 0$, and $\sigma = 0$. Then, from Eq. (45) we find the solution $\sin(\phi - \sigma) = 1$. Thus, the solution has a fixed rotation axis at $\chi = \pm\pi/2$. From Eq. (42) we have,

$$\cos\Omega = -1 + \frac{2(1-\alpha^2)}{\{1 - 2\alpha^2/\left[\cosh\left(1-\alpha^2\right)^{1/2}\xi\right) + 1]\}}, \tag{50}$$

or equivalently,

$$\sin\Omega = \frac{2\alpha(1-\alpha^2)^{1/2}\sinh[(1-\alpha^2)^{1/2}\xi]}{\cosh[(1-\alpha^2)^{1/2}\xi] + 1 - 2\alpha^2} \xrightarrow[(1-\alpha^2)^{1/2}\xi \gg 1]{} 2\alpha(1-\alpha^2)^{1/2}. \tag{51}$$

Note from Eq. (51) that Ω rotates from zero to a maximum value of $\Omega_{\rm max} = \sin^{-1}\left(2\alpha(1-\alpha^2)^{1/2}\right) < \pi$, where $\Omega_{\rm max}$ is approached as $[(1-\alpha^2)^{1/2}\xi]^{-1} \to 0$.

For the case $\alpha = 1$, $\chi = \pm\pi/2$ we need to expand in $1 - \alpha^2$, and take the limit as this quantity vanishes. We find,

$$\phi = \frac{\pi}{2}, \qquad \sin\Omega = \frac{4\xi}{4 + \xi^2}. \tag{52}$$

Thus $\Omega \to 0$ as $\xi \to 0$, $\Omega \to \pi$ for $\xi \to \infty$ and $\Omega = \pi/2$ for $\xi = 2$.

When $\alpha > 1$, $\chi = \pm\pi/2$ algebraic manipulation gives [using $\lambda_2 = (\alpha^2-1)^{1/2}$, $\lambda_1 = 0$, $\beta'' = \mp 1/\alpha$, $\sigma = \pi/2$],

$$\phi = \pi/2, \quad \sin\Omega = \frac{2\alpha(\alpha^2-1)^{1/2}\sin[(\alpha^2-1)^{1/2}\xi]}{2\alpha^2 - 1 - \cos[(\alpha^2-1)^{1/2}\xi]} \xrightarrow[\alpha \gg 1]{} \sin[\alpha^2 - 1)^{1/2}\xi]. \tag{53}$$

Note that maximum and minimum values of $\sin\Omega$ occur when $\sin[(\alpha^2-1)^{1/2}\xi] = \pm 2\alpha(\alpha^2-1)^{1/2}/(2\alpha^2-1)$, whereupon $\sin\Omega = \pm 1$. Thus as ξ increases, Ω can rotate without bound, with the rotation axis fixed at $\phi = \chi = \pm\pi/2$.

It is interesting to note how the precessional solution blends in smoothly with the pure rotational solution as $\chi \to 0$ where $|\beta''| \ll 1$. Then from Eq. (46), we see that $\phi - \sigma$ hardly changes until the vicinity of $\lambda_2\xi/2 = n\pi$, whereupon $\lambda_2\xi/2$ goes from $n\pi - \epsilon$ to $n\pi + \epsilon$, for

$1 \gg \epsilon \gg |\beta''|$. Then the axis has flipped direction, but one can also infer that $d\Omega/d\xi$ also flips sign as $\lambda_2\xi/2$ passes through $n\pi$. Note that if the axis changes direction and $d\Omega/d\xi$ changes sign, it is essentially the same rotational effect as having the axis fixed and $d\Omega/d\xi$ maintaining the same sign. Hence except for the small region $|\lambda_2\xi/2 - n\pi| \approx |\beta''|$, the transformation for small χ is almost the same as the case $\chi = 0$.

III. SPINOR ALGEBRAIC DESCRIPTION OF LORENTZ TRANSFORMATIONS

In this section, we solve the problem discussed in the preceding sections by a different method based on spin algebra used in relativistic quantum mechanics. In this treatment, it is not possible to relate the problem to a dynamical analog considered before, but we are able to exploit spinor algebra to sum a power series of an exponential matrix in a rather straightforward manner.

A. Reduction of the Four-Dimensional Lorentz Transformation to Spinor Representation Form

We first demonstrate how the general 4-dimensional Lorentz transformation of Eq. (9) can be directly reduced to a well-known spinor representation of the form [5,9];

$$Y = LXL^{\dagger}, \tag{54}$$

where $Y(X)$ is a 2×2 matrix constructed from the space-time coordinate (y_0, y_1, y_2, y_3) $((x_0, x_1, x_2, x_3))$ as

$$Y = \begin{pmatrix} y_0 + y_3 & y_1 - iy_2 \\ y_1 + iy_2 & y_0 - y_3 \end{pmatrix} = y_0 + \boldsymbol{y} \cdot \boldsymbol{\sigma} \tag{55}$$

(similarly for X), where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli spin matrix vector and

$$L(\boldsymbol{\omega}, \boldsymbol{\xi}) = e^{(-\boldsymbol{\xi} + i\boldsymbol{\omega}) \cdot \boldsymbol{\sigma}/2}. \tag{56}$$

In order to derive Eq. (54) from Eq. (9), we first rewrite $e^L \equiv A(\boldsymbol{\omega}, \boldsymbol{\xi}) = e^{-\boldsymbol{\xi}\cdot\boldsymbol{K}-\boldsymbol{\omega}\cdot\boldsymbol{S}}$ thereby using $\boldsymbol{J}_\pm$ defined as

$$\boldsymbol{J}_\pm = \boldsymbol{K} \pm i\boldsymbol{S}, \tag{57}$$

where $\boldsymbol{K}$ and $\boldsymbol{S}$ are the same as those inferred in Eq. (4) [compare with Eqs. (2.70) of Ref. [2] and Eqs. (5.6.7) and (5.6.8) of Ref. [3]]. $\boldsymbol{J}_\pm$ satisfy the following relations;

$$[J_{\pm i}, J_{\pm j}] = \pm\epsilon_{ijk}J_{\pm k}, \quad [J_{+i}, J_{-j}] = 0, \quad \text{and} \quad J_{\pm i}^2 = 1. \tag{58}$$

Because of the commutativity between $\boldsymbol{J}_+$ and $\boldsymbol{J}_-$, one can express $\mathcal{A}$ as a product of two terms, each involving $\boldsymbol{J}_+$ or $\boldsymbol{J}_-$ as

$$A = \mathcal{A}\mathcal{A}^*, \tag{59}$$

where

$$\mathcal{A} = e^{\Lambda\widehat{\boldsymbol{n}}\cdot\boldsymbol{J}_+/2}, \tag{60}$$

with

$$\Lambda = [(-\boldsymbol{\xi} + i\boldsymbol{\omega})\cdot(-\boldsymbol{\xi} + i\boldsymbol{\omega})]^{1/2} \equiv \Lambda_R + i\Lambda_I = \xi(\lambda_1 \mp i\lambda_2), \tag{61}$$

$$\widehat{\boldsymbol{n}} = \frac{1}{\Lambda}(-\boldsymbol{\xi} + i\boldsymbol{\omega}). \tag{62}$$

In Eq. (61), the branch of Λ is chosen in the first or fourth quadrant and therefore $\Lambda_R > 0$. Note that λ_1 and λ_2 are just the functions defined in Eq. (18), with the upper sign chosen for $-\pi/2 < \chi < \pi/2$ and the lower sign for $\pi/2 < \chi < 3\pi/2$. The sign choice follows from a relation that readily finds from Eq. (61)

$$\lambda_1\lambda_2 = \alpha[\cos^2\chi]^{1/2} = \pm\alpha\cos\chi, \tag{63}$$

since λ_1 and λ_2 are by construction in Eq. (18) non-negative. Further, using Eqs. (19) and (61), one can show that λ_1 and λ_2 can be expressed in terms of the angle σ introduced in Eq. (14), together with γ'' and β'' of Eq. (16), as

$$\lambda_1 = \frac{1}{\gamma''}\cos\sigma \quad \text{and} \quad \lambda_2 = \mp\frac{1}{\gamma''\beta''}\sin\sigma. \tag{64}$$

We now transform the 4-vectors and matrices involved by means of the following unitary transformation R;

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \tag{65}$$

R transforms a 4-vector such as x as

$$x_t \equiv R\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} x_0 + x_3 \\ x_1 + ix_2 \\ x_1 - ix_2 \\ x_0 - x_3 \end{pmatrix}, \tag{66}$$

and an operator, such as $\widehat{\boldsymbol{n}} \cdot \boldsymbol{J}_+$ in $\mathcal{A}$, as

$$R\widehat{\boldsymbol{n}} \cdot \boldsymbol{J}_+ R^{-1} = \begin{pmatrix} \widehat{\boldsymbol{n}} \cdot \boldsymbol{\sigma} & 0 \\ 0 & \widehat{\boldsymbol{n}} \cdot \boldsymbol{\sigma} \end{pmatrix}. \tag{67}$$

Eq. (84) implies that $\mathcal{A}$ transforms as

$$\mathcal{A}_t \equiv R\mathcal{A}R^{-1} = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \equiv \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \tag{68}$$

where L is the spinor matrix of Eq. (56). The product $A = \mathcal{A}\mathcal{A}^*$ is then transformed as

$$A_t \equiv RAR^{-1} = R\mathcal{A}R^{-1}RR^{*-1}(R\mathcal{A}R^{-1})^* R^* R^{-1} = \mathcal{A}_t E \mathcal{A}_t^* E^*, \tag{69}$$

where

$$E \equiv RR^{*-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = E^*. \tag{70}$$

Transforming the 4-dimensional Lorentz transformation $y = Ax$ by multiplying both side by $\sqrt{2}R$, we obtain

$$\sqrt{2}Ry = \begin{pmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{pmatrix} = A_t \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{pmatrix} = \mathcal{A}_t E \mathcal{A}_t^* E \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{pmatrix}, \tag{71}$$

where

$$\mathcal{Y}_i \equiv \begin{pmatrix} y_0 \pm y_3 \\ y_1 \pm iy_2 \end{pmatrix} \quad \text{and} \quad \mathcal{X}_i \equiv \begin{pmatrix} x_0 \pm x_3 \\ x_1 \pm ix_2 \end{pmatrix} \quad \text{with} \quad \begin{cases} + & \text{for} \quad i = 1, \\ - & \text{for} \quad i = 2\ . \end{cases} \tag{72}$$

Inserting Eqs. (68) and (70) into (71), one obtains

$$\begin{pmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} L_{11}^* & L_{12}^* \\ L_{21}^* & L_{22}^* \end{pmatrix} \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{pmatrix}. \tag{73}$$

Using the relations

$$(L_{ij})_{k\ell} = \delta_{ij} L_{k\ell} \quad \text{with} \quad (E_{ij})_{k\ell} = \delta_{kj}\delta_{\ell i}$$

that follow from the definition of the matrices L_{ij} and E_{ij} and also noting that $(\mathcal{Y}_i)_\alpha$ and $(\mathcal{X}_i)_\alpha$ are the αi components of the matrices Y and X in Eq. (54), i.e., $Y_{\alpha i} = (\mathcal{Y}_i)_\alpha$ and $X_{\alpha i} = (\mathcal{X}_i)_\alpha$, respectively, it is straightforward to obtain our desired result. Specifically we have

$$\begin{aligned} Y_{\alpha i} = (Y_i)_\alpha &= (L_{ij})_{\alpha\beta}(E_{jk})_{\beta\gamma}(L_{k\ell}^*)_{\gamma\delta}(E_{\ell m})_{\delta\varepsilon}(\mathcal{X}_m)_\varepsilon \\ &= \delta_{ij} L_{\alpha\beta}\delta_{j\gamma}\delta_{k\beta}\delta_{k\ell}L_{\gamma\delta}^*\delta_{\ell\varepsilon}\delta_{m\delta}(A_m)_\varepsilon \\ &= L_{\alpha\beta}L_{i\delta}^*(\mathcal{X}_\delta)_\beta = L_{\alpha\beta}(X)_{\beta\delta}L_{\delta i}^\dagger = (LXL^\dagger)_{\alpha i}. \end{aligned} \tag{74}$$

Eq. (54) is thus proved.

B. Spinor Calculation of the Lorentz Transformation Matrix.

The merit of use of the spinor representation Eq. (54) of the Lorentz transformation is that it is possible to directly reduce the exponential form of $L(\boldsymbol{\omega}, \boldsymbol{\xi})$ to an explicit 2×2 matrix form in a very straight-forward manner by first expanding in a power series and then use that $(\boldsymbol{\sigma}\cdot\widehat{\boldsymbol{n}})^2 = 1$ for any 3-unit vector $\widehat{\boldsymbol{n}}$ (i.e. $\sum_i n_i^2 = 1$) even if n_i is complex. The resulting expression can be reduced to a form

$$L = L_0 + \boldsymbol{L}_1\cdot\boldsymbol{\sigma}, \tag{75}$$

where L_0 and $\boldsymbol{L}_1$ are independent of $\boldsymbol{\sigma}$. The series expansions for L_0 and $\boldsymbol{L}_1$ can be straightforwardly summed and we find for any complex function Λ, and any 3-unit vector $\widehat{\boldsymbol{n}}$,

$$e^{\Lambda\widehat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}/2} = \cosh\left(\frac{\Lambda}{2}\right) + \widehat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\Lambda}{2}\right). \tag{76}$$

In our case, Λ and $\widehat{\boldsymbol{n}}$ are given by Eqs. (61) and (62), respectively.

Thus,

$$L_0 = \cosh\left(\frac{\Lambda}{2}\right) = \cosh\left(\frac{\Lambda_R}{2}\right)\cos\left(\frac{\Lambda_I}{2}\right) + i\sinh\left(\frac{\Lambda_R}{2}\right)\sin\left(\frac{\Lambda_I}{2}\right), \tag{77}$$

$$\boldsymbol{L}_1 = \frac{1}{|\Lambda|^2}(\boldsymbol{A} + i\boldsymbol{C}), \tag{78}$$

where

$$\boldsymbol{A} = \mathrm{Re}[\Lambda^*(-\boldsymbol{\xi} + i\boldsymbol{\omega})\sinh\left(\frac{\Lambda}{2}\right)] = -a\boldsymbol{\omega} - b\boldsymbol{\xi}, \tag{79}$$

$$\boldsymbol{C} = \mathrm{Im}\left[\Lambda^*(-\boldsymbol{\xi} + i\boldsymbol{\omega})\sinh\left(\frac{\Lambda}{2}\right)\right] = -a\boldsymbol{\xi} + b\boldsymbol{\omega}, \tag{80}$$

with

$$a = \Lambda_R\cosh\left(\frac{\Lambda_R}{2}\right)\sin\left(\frac{\Lambda_I}{2}\right) - \Lambda_I\sinh\left(\frac{\Lambda_R}{2}\right)\cos\left(\frac{\Lambda_I}{2}\right), \tag{81}$$

$$b = \Lambda_R\sinh\left(\frac{\Lambda_R}{2}\right)\cos\left(\frac{\Lambda_I}{2}\right) + \Lambda_I\cosh\left(\frac{\Lambda_R}{2}\right)\sin\left(\frac{\Lambda_I}{2}\right). \tag{82}$$

The explicit form of the transformation matrix $A(\boldsymbol{\omega},\boldsymbol{\xi})$ may then be obtained by establishing the relations between $(x_0,\boldsymbol{x})$ and $(y_0,\boldsymbol{y})$ by using Eq. (54). Inserting Eq. (75) into (54), we obtain

$$Y' = y_0 + \boldsymbol{y}\cdot\boldsymbol{\sigma} = (L_0 + \boldsymbol{L}_1\cdot\boldsymbol{\sigma})(x_0 + \boldsymbol{x}\cdot\boldsymbol{\sigma})(L_0^* + \boldsymbol{L}_1^*\cdot\boldsymbol{\sigma}). \tag{83}$$

The evaluation of the right most term in the above relation may be carried out with the help of the well-known relation; $(\boldsymbol{a}\cdot\boldsymbol{\sigma})(\boldsymbol{b}\cdot\boldsymbol{\sigma}) = \boldsymbol{a}\cdot\boldsymbol{b} + i\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{\sigma}$. Equating, separately, the time and space components of the resulting equation, we find

$$y_0 = (L_0 L_0^* + \boldsymbol{L}_1 \cdot \boldsymbol{L}_1^*) x_0 + (L_0 \boldsymbol{L}_1^* + L_0^* \boldsymbol{L}_1 - i \boldsymbol{L}_1 \times \boldsymbol{L}_1^*) \cdot \boldsymbol{x}, \tag{84}$$

$$\begin{aligned} \boldsymbol{y} &= (L_0 \boldsymbol{L}_1^* + L_0^* \boldsymbol{L}_1 + i \boldsymbol{L}_1 \times \boldsymbol{L}_1^*) x_0 \\ &\quad + (L_0 L_0^* - \boldsymbol{L}_1 \cdot \boldsymbol{L}_1^*) \boldsymbol{x} + i (L_0^* \boldsymbol{L}_1 - L_0 \boldsymbol{L}_1^*) \times \boldsymbol{x} + (\boldsymbol{x} \cdot \boldsymbol{L}_1^*) \boldsymbol{L}_1 + (\boldsymbol{x} \cdot \boldsymbol{L}_1) \boldsymbol{L}_1^*. \end{aligned} \tag{85}$$

One can then easily identify that the components $A_{\mu\nu}$ can be given in general as

$$\begin{aligned} A_{00} &= L_0 L_0^* + \boldsymbol{L}_1 \cdot \boldsymbol{L}_1^*, \\ A_{0i} &= L_0 L_{1i}^* + L_0^* L_{1i} - i \epsilon_{ijk} L_{1j} L_{1k}^*, \\ A_{i0} &= L_0 L_{1i}^* + L_0^* L_{1i} + i \epsilon_{ijk} L_{1j} L_{1k}^*, \\ A_{ij} &= (L_0 L_0^* - \boldsymbol{L}_1 \cdot \boldsymbol{L}_1^*) \delta_{ij} - i \epsilon_{ijk} (L_0^* L_{1k} - L_0 L_{1k}^*) + L_{1i} L_{1j}^* + L_{1i}^* L_{1j}, \end{aligned} \tag{86}$$

where ϵ_{ijk} represents the component of the antisymmetric third rank tensor.

Explicit expressions of $A_{\mu\nu}$ may be determined by using L_0 and $\boldsymbol{L}_1$ given by Eqs. (77)-(78). In carrying out the calculations, use is made of the same coordinate system as used before; the x-axis was chosen to be the direction of $\boldsymbol{\xi}$, while the y-axis is set in the plane defined by $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$. It is then possible to show after some algebra that the resultant Lorentz transformation matrix agrees with Eq. (32). We give here the results in a form that is directly obtained from the present spinor algebra;

$$\begin{aligned} &A_{00} = \gamma, \qquad A_{0i} = -\gamma \beta_i, \qquad A_{i0} = -\gamma(\beta_i - 2\beta_3 \delta_{i3}), \\ &A_{ij} = \delta_{ij} [f_i \cosh(\lambda_1 \xi) + (1 - f_i) \cos(\lambda_2 \xi)] + \\ &\qquad (1 - \delta_{ij}) \frac{1}{(1 + \alpha^2)\sqrt{1 - 4\lambda^2}} [(\cosh(\lambda_1 \xi) - \cos(\lambda_2 \xi)(\widehat{\xi}_i \widehat{\xi}_j + \alpha^2 \widehat{\omega}_i \widehat{\omega}_j) \\ &\mp (\lambda_2 \sinh(\lambda_1 \xi) - \lambda_1 \sin(\lambda_2 \xi)) \epsilon_{ijk} \widehat{\xi}_k + (\lambda_1 \sinh(\lambda_1 \xi) - \lambda_2 \sin(\lambda_2 \xi)) \alpha \epsilon_{ijk} \widehat{\omega}_k], \end{aligned} \tag{87}$$

where γ and β_i ($i = 1, 2,$ and 3) are the same as those of Eq. (34). Further,

$$(f_1, f_2, f_3) = (\cos^2 \sigma, \sin^2 \sigma, -\gamma''^2 \beta''^2). \tag{88}$$

C. Successive Boost and Rotation

The Lorentz transformation matrix L', corresponding to the successive boost and rotation, $\mathcal{L}(\Omega \widehat{\boldsymbol{\Omega}}, \Xi \widehat{\boldsymbol{\Xi}}) = e^{-\boldsymbol{\Omega} \cdot \boldsymbol{S}} e^{-\boldsymbol{\Xi} \cdot \boldsymbol{K}}$, introduced in Sec. III, may also be obtained as

$$L' = e^{i\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}/2} e^{-\boldsymbol{\Xi}\cdot\boldsymbol{\sigma}/2}, \tag{89}$$

with $\boldsymbol{\Xi} = \widehat{\boldsymbol{\beta}} \tanh^{-1}\beta$, and we note that $\gamma = \cosh\Xi$ and $\gamma\beta = \sinh\Xi$. The reduced form of L' can be obtained by performing the reduction for $e^{i\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}}$ and $e^{-\boldsymbol{\Xi}\cdot\boldsymbol{\sigma}}$ separately and then taking the product. (We note that Ref. [9] used $L' = L_1 L_2$ indicating that the final transformation is the result of successive transformation of L_2 and L_1, but it did not emphasize the special interpretative case where L_2 is a pure boost and L_1 is a pure rotation.)

The resultant reduced form of L' can still be cast in the general form of Eq. (75) as

$$L' = L'_0 + \boldsymbol{L}'_1 \cdot \boldsymbol{\sigma}, \tag{90}$$

where L'_0 and $\boldsymbol{L}'_1$ are found to be

$$L'_0 = \cos\left(\frac{\Omega}{2}\right)\cosh\left(\frac{\Xi}{2}\right) - i\sin\left(\frac{\Omega}{2}\right)\sinh\left(\frac{\Xi}{2}\right)\left(\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{\Xi}}\right), \tag{91}$$

$$\boldsymbol{L}'_1 = -\cos\left(\frac{\Omega}{2}\right)\sinh\left(\frac{\Xi}{2}\right)\widehat{\boldsymbol{\Xi}} + \sin\left(\frac{\Omega}{2}\right)\sinh\left(\frac{\Xi}{2}\right)\left(\widehat{\boldsymbol{\Omega}}\times\widehat{\boldsymbol{\Xi}}\right) + i\sin\left(\frac{\Omega}{2}\right)\cosh\left(\frac{\Xi}{2}\right)\widehat{\boldsymbol{\Omega}}. \tag{92}$$

D. Relations Between Two Lorentz Transformations

By equating Eq. (75) with (90), it is possible to establish the direct relations between $(\boldsymbol{\Omega}, \boldsymbol{\Xi})$ and $(\boldsymbol{\omega}, \boldsymbol{\xi})$ and simplify some of the relations derived in Sec. V. Since both L_0 and $\boldsymbol{L}_1$ are complex, the equivalence condition ($L_0 = L'_0$ and $\boldsymbol{L}_1 = \boldsymbol{L}'_1$) leads to the following four equations;

$$\cos\left(\frac{\Omega}{2}\right)\cosh\left(\frac{\Xi}{2}\right) = \cos\left(\frac{\Lambda_I}{2}\right)\cosh\left(\frac{\Lambda_R}{2}\right), \tag{93}$$

$$-\sin\left(\frac{\Omega}{2}\right)\sinh\left(\frac{\Xi}{2}\right)\left(\widehat{\boldsymbol{\Omega}}\cdot\widehat{\boldsymbol{\Xi}}\right) = \sin\left(\frac{\Lambda_I}{2}\right)\sinh\left(\frac{\Lambda_R}{2}\right), \tag{94}$$

$$-\cos\left(\frac{\Omega}{2}\right)\sinh\left(\frac{\Xi}{2}\right)\widehat{\boldsymbol{\Xi}} + \sin\left(\frac{\Omega}{2}\right)\sinh\left(\frac{\Xi}{2}\right)\left(\widehat{\boldsymbol{\Omega}}\times\widehat{\boldsymbol{\Xi}}\right) = \frac{1}{|\Lambda|^2}\boldsymbol{A}, \tag{95}$$

$$\sin\left(\frac{\Omega}{2}\right)\cosh\left(\frac{\Xi}{2}\right)\widehat{\boldsymbol{\Omega}} = \frac{1}{|\Lambda|^2}\boldsymbol{C}. \tag{96}$$

As seen in Eq. (96), $\widehat{\boldsymbol{\Omega}}$ is proportional to $\boldsymbol{C}$. Noting that $\boldsymbol{C}$ is given as a linear combination of $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$, the axis of the Ω-rotation is directly seen to be in the plane defined by the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$ in agreement with the relation obtained in earlier sections.

To obtain $\boldsymbol{\Omega}$, we may first deduce $\cos^2 \frac{\Omega}{2}$ by taking the square of both sides of Eq. (93) and then by dividing by $\cosh^2 \frac{\Xi}{2}$. The resultant expression reads

$$2\cos^2 \frac{\Omega}{2} = 4\cos^2 \frac{\Lambda_I}{2} \cosh^2 \frac{\Lambda_R}{2} / 2\cosh^2 \frac{\Xi}{2}. \tag{97}$$

Since $2\cosh^2 \frac{\Xi}{2} = 1 + \cosh \Xi$ can be calculated from $\cosh \Xi = \gamma$, the above establishes the relation between Ω and $(\boldsymbol{\xi}\ , \boldsymbol{\omega})$. Furthermore, it is possible to show after some algebra that the r.h.s. of the above equation reduces precisely to G defined in Eq. (43) and thus the result that determines Ω in Eq. (42) is reproduced. One can also determine Ω directly from the expression for $\cos \frac{\Omega}{2}$ obtained from Eq. (97);

$$\cos \frac{\Omega}{2} = \cos \left(\frac{\lambda_2 \xi}{2} \right) \cosh \left(\frac{\lambda_1 \xi}{2} \right) \sqrt{\frac{2}{\gamma + 1}}, \tag{98}$$

where use was made of $\cosh(\Xi/2) = \sqrt{(\gamma+1)/2}$. Since $0 \leq \Omega \leq 2\pi$, the above equation uniquely determines $\Omega/2$ and hence Ω.

In Sec. II, use was made of Eq. (40) derived on the basis of SRB symmetry in order to obtain the axis of rotation, $\widehat{\boldsymbol{\Omega}} = \cos\phi \widehat{\boldsymbol{x}} + \sin\phi \widehat{\boldsymbol{y}}$. Equation (40) can be rewritten as

$$\left[\sin \Omega \boldsymbol{\beta} \times \widehat{\boldsymbol{\Omega}} + (1 + \cos \Omega) \boldsymbol{\beta} \right] \cdot \widehat{\boldsymbol{z}} = 0. \tag{99}$$

In the present approach, the above equation follows from Eq. (95) if one multiplies both sides by $4\cos \frac{\Omega}{2} \cosh \frac{\Xi}{2}$ and then takes the scalar product with $\widehat{\boldsymbol{z}}$ using that $(\boldsymbol{A} \cdot \widehat{\boldsymbol{z}}) = 0$. As in Sec. III we can then derive Eq. (45) for the orientation angle ϕ between the axis of rotation $\widehat{\boldsymbol{\Omega}}$ and the direction $\widehat{\boldsymbol{\xi}}$.

One can obtain a compact closed form expression for $\widehat{\boldsymbol{\Omega}}$, which simply follows from Eq. (96);

$$\widehat{\boldsymbol{\Omega}} = \frac{1}{|\Lambda|^2} \boldsymbol{C} / \left(\sin \frac{\Omega}{2} \cosh \frac{\Xi}{2} \right). \tag{100}$$

If one takes the x and y components of the above equation, one finds after some algebra

$$\cos\phi = \pm \frac{\gamma''^2}{\sin \frac{\Omega}{2}} \left[\lambda_1 \sin \frac{\lambda_2 \xi}{2} \cosh \frac{\lambda_1 \xi}{2} - \lambda_2 \beta''^2 \cos \frac{\lambda_2 \xi}{2} \sinh \frac{\lambda_1 \xi}{2} \right] \sqrt{\frac{2}{\gamma - 1}},$$

$$\sin\phi = -\frac{\gamma''^2 \beta''}{\sin \frac{\Omega}{2}} \left[\lambda_2 \sin \frac{\lambda_2 \xi}{2} \cosh \frac{\lambda_1 \xi}{2} + \lambda_1 \cos \frac{\lambda_2 \xi}{2} \sinh \frac{\lambda_1 \xi}{2} \right] \sqrt{\frac{2}{\gamma - 1}}. \tag{101}$$

Equations (98) and (101) now establish the relation between $\boldsymbol{\Omega}$ and $(\boldsymbol{\xi}, \boldsymbol{\omega})$ and with the help of Eq. (93) one can show that Eq. (101) satisfies Eq. (45).

IV. SUMMARY

We have shown that the evaluation of the Lorentz transformation $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$ can be cast in terms of the evolution equation of the 4-velocity of a particle in an electric field, $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ that is uniform in space and time. The mathematical difference of the two problems is that the 4-velocity u is constrained to have as an initial condition, an inner product $u \cdot u = 1$ (which therefore leads to a set of particular solutions for the equations of motion), whereas the inner product of a general 4-vector, such as the coordinates of an event $(x_0, \boldsymbol{x})$ has an inner product of arbitrary value, for which we need a general solution to the "equations of motion" to describe an arbitrary event in a different Lorentz frame. To solve for the transformation, we can still use the well-known procedure of transforming to a frame where $\boldsymbol{\omega}$ and $\boldsymbol{\xi}$ (or equivalently $\boldsymbol{E}$ and $\boldsymbol{B}$) are parallel (or antiparallel), then solving an easy equation, and transforming back to the original frame. The equations solved have the intrinsic property that the inner product, once initially chosen, is conserved.

Having obtained the solution for $e^{L(\boldsymbol{\omega},\boldsymbol{\xi})}$, we succeeded in interpreting it in terms of the boost $\boldsymbol{\beta}$ and the rotation $\boldsymbol{\Omega}$ ($\boldsymbol{\Omega} = \Omega\widehat{\boldsymbol{\Omega}}$) where $\widehat{\boldsymbol{\Omega}}$ is the direction of the rotation axis and Ω the angle, defined by the conventional right-hand rule, of rotation about an axis. Specific formulas, such as Eqs. (14), (42), and (46) determine $\boldsymbol{\beta}$ and $\boldsymbol{\Omega}$. It is found that $\widehat{\boldsymbol{\Omega}}$ lies in the 3-plane of $\boldsymbol{\omega}$ and $\boldsymbol{\xi}$, and if $\boldsymbol{\omega} \cdot \boldsymbol{\xi} \neq 0$ and $\boldsymbol{\omega} \times \boldsymbol{\xi} \neq 0$ the axis of rotation varies only in this plane. For sufficiently small ξ the axis of rotation is directed along $\widehat{\boldsymbol{\omega}}$, but with increasing ξ, the orientation of the rotation axis with respect to $\widehat{\boldsymbol{\xi}}$ is given in Eq. (46). As a function of ξ (a pseudo-proper time coordinate), the rotation axis generally precesses in the 3-plane of $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$. We find that the mean precession frequency is the Larmor frequency [10], $qB''/2\gamma''\,mc$ in the intermediate frame where $\boldsymbol{E}$ and $\boldsymbol{B}$ are parallel. The only exceptions arise if $\boldsymbol{\xi} \cdot \boldsymbol{\omega} = 0$ or $\boldsymbol{\xi} \times \boldsymbol{\omega} = 0$ where then the rotation axis is fixed as ξ changes.

We have also shown how the exponential representation can be directly summed, and the results of the two methods of calculation are fully consistent with each other.

Acknowledgments

The authors are particularly grateful for the interest, encouragement, and comments of Prof. J.D. Jackson, and acknowledge important discussions with Prof. C. Morette Dewitt, Prof. E.C.G. Sudarshan, Dr. Boris Breizman, Mark Mims, and Prof. Paul Cox, who first noted to us that $\boldsymbol{J}_+$ and $\boldsymbol{J}_-$ commute.

This work was supported by the U.S. Dept. of Energy Contract No. DE-FG03-96ER-54346.

Appendix: Specific Representations for Pure Rotation and Boost Operators

It is well known that from the representations in Eq. (4) and the commutation relations in Eq. (5) one obtains

$$e^{-\Omega S_1}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos\Omega&\sin\Omega\\0&0&-\sin\Omega&\sin\Omega\end{pmatrix},\quad e^{-\Omega S_2}=\begin{pmatrix}1&0&0&0\\0&\cos\Omega&0&\sin\Omega\\0&0&1&0\\0&\sin\Omega&0&\cos\Omega\end{pmatrix},\quad e^{-\Omega S_3}=\begin{pmatrix}1&0&0&0\\0&\cos\Omega&\sin\Omega&0\\0&-\sin\Omega&\cos\Omega&0\\0&0&0&1\end{pmatrix}. \tag{A1}$$

$$e^{-\xi K_1}=\begin{pmatrix}\cosh\xi&-\sinh\xi&0&0\\0\sinh\xi&\cosh\xi&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad e^{-\xi K_2}=\begin{pmatrix}\cosh\xi&0&-\sinh\xi&0\\0&1&0&0\\\sin\xi&0&1&0\\0&0&0&1\end{pmatrix},\quad e^{-\xi K_3}=\begin{pmatrix}\cosh\xi&0&0&-\sinh\xi\\0&1&0&0\\0&0&1&0\\-\sinh\xi&0&0&1\end{pmatrix}. \tag{A2}$$

The general pure rotation and pure boost operators can be obtained from straightforward group theoretic relations. If $\widehat{\boldsymbol{\Omega}}=\cos\theta\widehat{\boldsymbol{z}}+\sin\theta\cos\phi\widehat{\boldsymbol{x}}+\sin\theta\sin\widehat{\phi}\widehat{y}\equiv\widehat{\Omega}_1\widehat{\boldsymbol{x}}+\widehat{\Omega}_2\widehat{\boldsymbol{y}}+\widehat{\Omega}_3\widehat{\boldsymbol{z}}$ we have

$$e^{-\boldsymbol{\Omega}\cdot\boldsymbol{S}}=e^{\phi S_3}e^{\theta S_2}e^{-\Omega S_3}e^{-\theta S_2}e^{-\phi S_2}$$
$$=\begin{pmatrix}1&0&0&0\\0&1-(\widehat{\Omega}_2^2+\widehat{\Omega}_3^2)f(\Omega)&\widehat{\Omega}_3\sin\Omega+\widehat{\Omega}_1\widehat{\Omega}_2f(\Omega)&-\widehat{\Omega}_2\sin\Omega+\widehat{\Omega}_2\widehat{\Omega}_3f(\Omega)\\0&-\widehat{\Omega}_3\sin\Omega+\widehat{\Omega}_1\widehat{\Omega}_2f(\Omega)&1-(\widehat{\Omega}_1^2+\widehat{\Omega}_3^2)f(\Omega)&\widehat{\Omega}_1\sin\Omega+\widehat{\Omega}_2\widehat{\Omega}_3f(\Omega)\\0&\widehat{\Omega}_2\sin\Omega+\widehat{\Omega}_1\widehat{\Omega}_3f(\Omega)&-\widehat{\Omega}_1\sin\Omega+\widehat{\Omega}_2\widehat{\Omega}_3f(\Omega)&1-(\widehat{\Omega}_1^2+\widehat{\Omega}_2^2)f(\Omega)\end{pmatrix} \tag{A3}$$

where $f(\Omega)=1-\cos\Omega$. If $\boldsymbol{\Xi}=\Xi\widehat{\boldsymbol{\beta}}$, and $\widehat{\boldsymbol{\beta}}=\cos\theta'\widehat{\boldsymbol{x}}+\sin\theta'\cos\phi\widehat{\boldsymbol{y}}+\sin\theta'\sin\phi\widehat{\boldsymbol{z}}=\widehat{\beta}_1\widehat{x}+\widehat{\beta}_2\widehat{y}+\widehat{\beta}_3\widehat{z}$, then

$$e^{\boldsymbol{\Xi}\cdot\boldsymbol{K}} = e^{\phi' S_1} e^{\theta' S_3} e^{-\Xi K_1} e^{-\theta' S_3} e^{-\phi' S_1} = \begin{pmatrix} \gamma & -\gamma\boldsymbol{\beta} \\ & \\ -\gamma\boldsymbol{\beta}^T & \boldsymbol{I} + \dfrac{\gamma - 1}{\beta^2}\,\boldsymbol{\beta}^T\boldsymbol{\beta} \end{pmatrix} \tag{A4}$$

where $\gamma = \cosh\Xi$, $\gamma\boldsymbol{\beta} = \widehat{\boldsymbol{\beta}}\sinh\Xi$, and see Eq. (36) for standard definitions of the vector and direct product notations.

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FIGURE CAPTIONS

FIG. 1. Curves of $-\beta''(\alpha)$ vs. $\chi' \equiv 2\chi/\pi$ for $\alpha = (.1,\ .25,\ .5,\ .75,\ 1)$. The curves for large α lie above the curves for smaller α.

FIG. 2. Plot of $\chi' - \sigma' \equiv 2(\chi - \sigma)/\pi$ vs. $\chi' \equiv 2\chi/\pi$ for $\alpha = (.25,\ .5,\ .75,\ 1,\ 1.25,\ 2,\ 4)$. The curves with larger α lie below the curves with smaller α.

FIG. 3. (a) Plot of $\lambda_1(\alpha) = \alpha\lambda_2(1/\alpha)$, vs. $\chi' \equiv 2\chi/\pi$ for $\alpha = (.1,\ .25,\ .5,\ .75,\ 1)$ with curves for larger α lying below curves of smaller α. (b) Plot of $\Delta\lambda_1 = \lambda_1(1) - \lambda_1(\alpha)$ vs. $\chi' = 2\chi/\pi$ for $\alpha = (1.25,\ 1.5,\ 2,\ 4)$ with curves of larger α lying above curves of smaller α.

FIG. 4. A rotation operation about the x-axis that leaves β_2 invariant and changes the sign of β_3. Here $\boldsymbol{\beta}_\perp = \boldsymbol{\beta} - \beta_1\hat{\boldsymbol{x}}$.

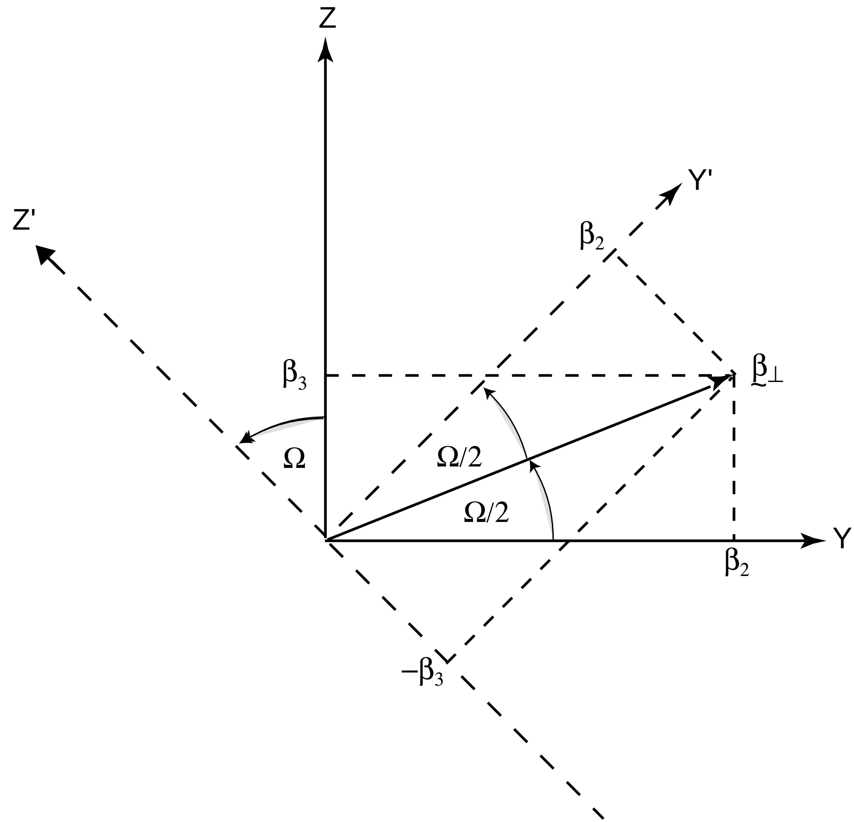

Fig.1

Fig.2

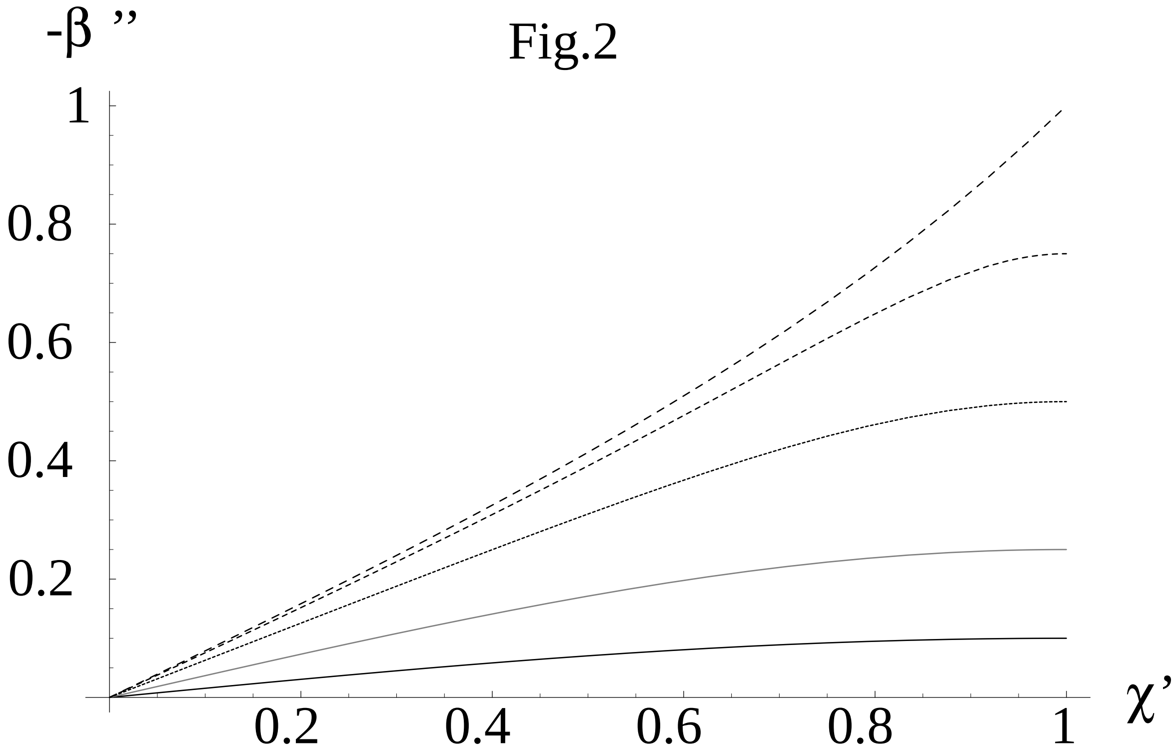

Fig.3

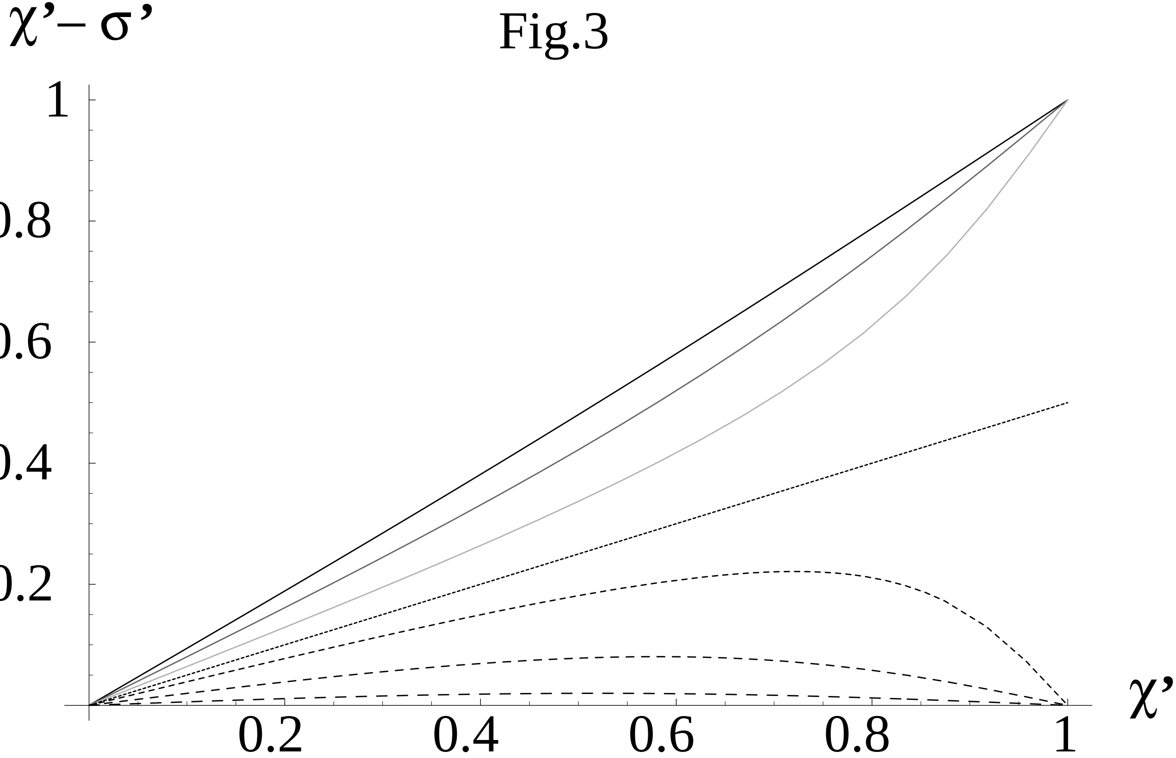

Fig.4a

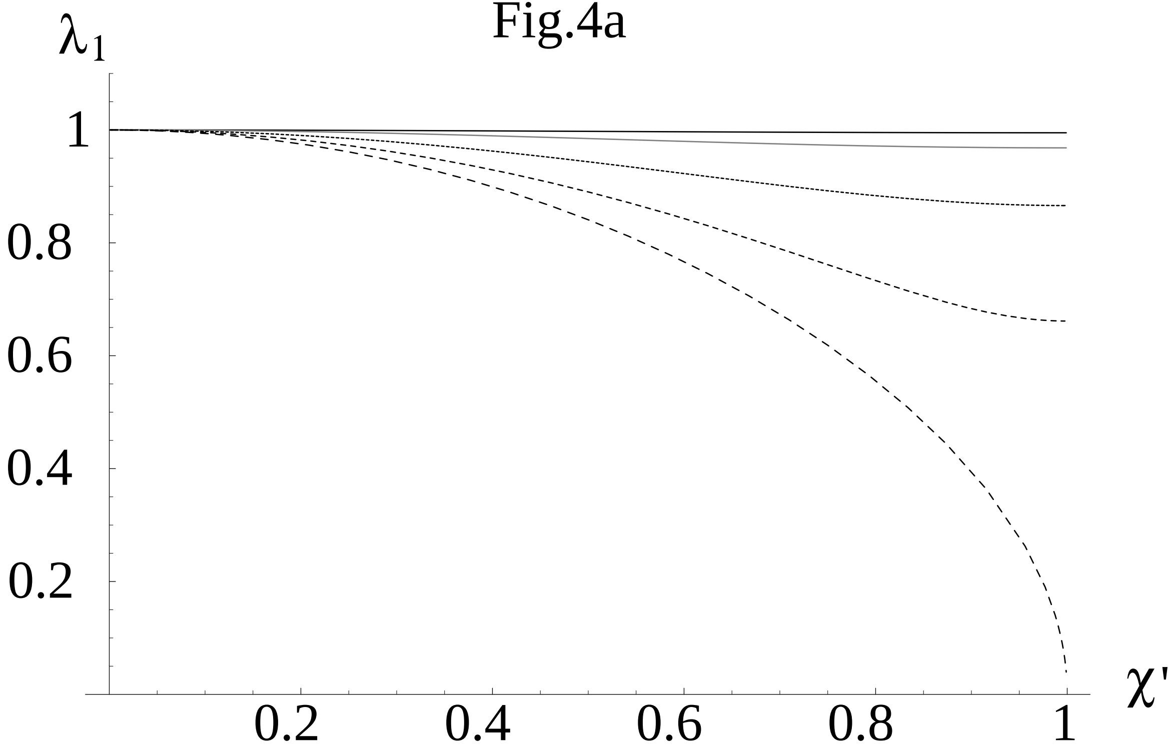

Fig.4b