Potato, Banana, Local and Non-Local Transport

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Abstract

The relationship between potato and banana transport theories is addressed using the solution of the drift kinetic equation. It is shown that they are two limits of a complete theory. The potato theory is the $\psi \to 0$ limit and the banana theory is the $\psi \to \infty$ limit. Here, ψ is the poloidal flux function. These local transport theories are valid even in the steep gradient situations because the real orbit width is usually smaller than the gradient scale length when the appropriate orbit squeezing effects are taken into account. The characteristic feature of a non-local theory is also discussed. It is shown that the equilibrium distribution in such a theory must be non-Maxwellian and non-expandable.

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I. INTRODUCTION

Since the discovery of the core confinement improvement modes, it becomes necessary to deal with collisionless transport problems with the equilibrium scale length L_p of the order of the nominal orbit width Δ_n. It is usually not realistic by just considering nominal orbit width in transport problems and the conclusions, based on that consideration, are often misleading. (Here, the nominal orbit width here is defined as the orbit width without the effects of orbit squeezing, and the orbit width and the length scale are measured in terms of the poloidal flux function ψ.) Because L_p is of the order of Δ_n, it is argued that transport problem is non-local and non-diffusive. However, this conclusion is based on a concept that the width of the orbit remains the same when L_p reduces to a magnitude of the order of Δ_n. This concept is not really true. In reality, when L_p is of the order of Δ_n, the realistic orbit width is squeezed so that the real orbit width Δ_r is still less than L_p. The transport problem is again local and diffusive. The solution to transport problem with L_p of the order of Δ_n, in our opinion, is to calculate diffusive transport coefficients by considering the realistic orbits with the effects of orbit squeezing included. Similar problems seem to have occurred in the near-axis transport problem. There, the non-squeezed potato transport theory is sometimes applied incorrectly to the situation where L_p is of the order of potato width Δ_p which is a nominal orbit width in our definition and it is concluded that near-axis transport problem is non-local and non-diffusive. In reality, of course, when L_p is of the order of Δ_p the realistic potato orbit width is also squeezed and becomes less than L_p, The transport is again local and diffusive in the near-axis region. The proper transport coefficients are those with the effects of squeezed potato width included.

The definition of a local transport theory here is that transport coefficients can be calculated from a linearized drift kinetic equation. The radial extension of the orbits can be finite. A non-local theory is that transport coefficients can not be obtained from a linearized drift kinetic equation because the width of the orbit is of the order of L_p.

One of the purposes of this paper is to emphasize the fact that the realistic orbit width is usually less than the gradient scale length in tokamaks and local diffusive transport theory is applicable when the proper orbit squeezing effects are taken into account.

It is also not appropriate to think of the potato transport theory as a theory different from the bananas transport theory. Any local transport theory must have at least three separate scales. One is the orbit width, another is L_p, and the other is an intermediate scale over which a local transport coefficient can be defined. This intermediate scale length is also the region over which one performs radial average to obtain the local transport coefficients. In a collisionless tokamak, it is obvious that the transport coefficient coefficients in the outer region are dominated by the banana transport because the potato effects, which are the effects that are resulted from the variation of the inverse aspect ratio ε over the width of the banana orbit, are weak there. In the region closer to the magnetic axis, the potato effects becomes stronger. It is clear then that the potato transport theory is the theory that is valid in the limit where the radial variable ψ approaches zero. It is then obvious that banana and potato transport theories are the two limits of the same theory.

We would also like to emphasize that the equilibrium distribution function in a non-local transport theory is non-Maxwellian, a fact that is known for quiet sometime. The fact that the equilibrium distribution function is non-Maxwellian indicates that the governing equation for a non-local transport theory is an integral-differential equation, a formidable problem to solve. Fortunately, local diffusive transport theory is adequate for most of the transport problems in tokamaks when the realistic orbit width is taken into account.

The rest of the paper is organized as follows. In Sec. II, we demonstrate that when L_p is of the order of the nominal orbit width Δ_n the realistic orbit width is squeezed for both banana and potato orbits. In Sec. III, we outline how to obtain the perturbed distribution function in both the collisionless and the plateau regimes for the banana and the potato orbits to illustrate the fact that they are two limits of the same transport theory. The issue of the source term is discussed in Sec. IV. We illustrate that the equilibrium distribution function for the non-local transport problem is non-Maxwellian and non-expandable in Sec. V. The concluding remarks are given in Sec. VI.

II. ORBIT SQUEEZING

We demonstrate that when L_p is of the order of Δ_n the real orbit width is squeezed by the equilibrium radial electric field.

Recall that the orbit squeezing factor is[1,2]

$$S = 1 + \left(\frac{I}{\Omega}\right)^2 \left(\frac{e\Phi''}{M}\right), \tag{1}$$

where $I = RB_t, R$ is the major radius, B_t is the toroidal magnetic field, Ω is the ion gyrofrequency, e is the ion charge, Φ is the electrostatic potential, M is the ion mass, and prime denotes $d/d\Psi$. From both experimental observations and theoretical considerations, Φ' is related to pressure gradient approximately

$$\frac{P'}{P} \approx \frac{e\Phi'}{T}, \tag{2}$$

where P is the ion pressure, and T is the ion temperature. Equation (2) is a valid neoclassical formula if the ion temperature gradient is ignored and the toroidal flow is assumed to be small.[3,4] Because the detailed relation between Φ' and the pressure and the temperature gradients is not important to our argument, we adopt Eq. (2) for our purpose. Since we assume L_p is of the order of Δ_n

$$e\Phi'' \approx \frac{T}{(\Delta_n)^2}. \tag{3}$$

The sign of Φ'' is not crucial because $S \gg 1$. Substituting Eq. (3) into Eq. (1) we obtain

$$S \approx 1 + \left(\frac{I v_t}{(\sqrt{2}\Omega\Delta_n)}\right)^2, \tag{4}$$

where v_t is the ion thermal speed. When Δ_n is of the order of the banana width $(\sqrt{\varepsilon}\,\rho_p\, RB_p)$, where ρ_p is the ion poloidal gyroradius and B_p is poloidal magnetic field strength,

$$S \approx 1 + \left(\frac{1}{2\varepsilon}\right). \tag{5}$$

In large aspect ratio tokamaks, $S \gg 1$ and banana orbit is squeezed by a factor of $\sqrt{|S|}$.[1,2] The real orbit width is smaller than the nominal banana width, the assumed equilibrium scale length. From this simple argument, we conclude that when the equilibrium scale length becomes shorter, the orbit width also becomes smaller to keep the local transport theory valid.

It is interesting to note that S is inversely proportional to ε. As ε becomes smaller, S becomes larger. One expects, then, that orbit squeezing becomes more effective in the near axis region. Indeed, when L_p is of the order of the potato width, S is even larger. To see this, recall that the potato width Δ_p is[5–7]

$$\Delta_p \approx \left(\frac{C_1 I^2 v_t^2}{2\Omega^2}\right)^{2/3}, \tag{6}$$

where $C_1 = \varepsilon/\sqrt{\psi}$. Substituting Eq. (6) into Eq. (4), we find

$$S \approx 1 + \frac{1}{(2^{1/3} f_t^2)}, \tag{7}$$

where $f_t = (I v_t C_1^2/\Omega)^{1/3}$ is proportional to the fraction of the potato orbits. For typical tokamaks, f_t is of the order of 15%. Potato orbit width measured in terms of the poloidal flux ψ is squeezed by a factor of $|S|^{2/3} \gg 1$.[8] We conclude that realistic potato orbits are strongly squeezed if the equilibrium gradient scale length is of the order of the nominal potato width and the width of the realistic orbit is smaller than L_p.

We have illustrated using both the potato and the banana orbits that when the equilibrium gradient scale length L_p is of the order of the nominal orbit width Δ_n, the orbit squeezing effects becomes important to keep the real orbit width Δ_r less than L_p. This naturally occurred orbit squeezing effect upholds the integrity of the local transport theory. The transport process is, therefore, local and diffusive.

We see that orbit squeezing transport theory is an integral part of a complete local transport theory. It becomes important when L_p is comparable to Δ_n.

III. POTATO AND BANANA TRANSPORT THEORIES: TWO LIMITS OF A COMPLETE THEORY

It is likely that potato transport theory is viewed as a completely different theory from the banana transport theory. We would like to clarify this misconception and emphasize that potato and banana transport theories are two limits of a complete theory. The banana theory corresponds to the $\psi \to \infty$ limit and the potato theory is the $\psi \to 0$ limit.

For a local transport theory to be valid, there are at least three separate scales. The width of the orbit, the equilibrium scale length, and an intermediate scale length over which a local transport coefficient can be defined. The intermediate scale length is also the region where a radial average procedure can be defined. Because of the orbit squeezing effects, intermediate scale length usually exists and a valid local transport theory can be defined.

One might argue that there is no intermediate scale length in the potato transport theory. The argument is that because the potato orbit characteristics disappear quickly when an orbit is couple potato orbit width away from the magnetic axis, the equilibrium gradient scale length must be of the order of Δ_p. It follows then that potato transport theory must be non-local. This argument is false in several aspects. First, there is no direct correlation between how fast the potato characteristics vanish in radius and the equilibrium gradient scale length. The proper gradient scale length is determined from the transport equation. Second, as we discussed in Sec. II, the orbit squeezing effects make the real orbit width smaller than the gradient scale length even if the equilibrium gradient scale length is of the order of the potato width. One can define an intermediate scale length. Third, because orbits make natural transition from potato to banana and vice versa, one should not separate artificially a potato transport region and a banana transport region. As one performs the radial average over the intermediate scale length, in the outer region of the minor radius the transport coefficients have almost pure banana transport characteristics and in the $\psi \to 0$ limit, more of the potato transport characteristics.

Here, we outline a procedure that can obtain perturbed particle distribution which leads to the two limits, namely, the banana and potato transport theories. For simplicity, we do not include the effects of orbit squeezing to illustrate the procedure.

The drift kinetic equation for ions in tokamaks is[3]

$$v_\parallel \widehat{n} \cdot \boldsymbol{\nabla} f + \boldsymbol{v}_d \cdot \boldsymbol{\nabla}\theta \, \frac{\partial f}{\partial \theta} + \boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi \, \frac{\partial f}{\partial \psi} = C(f), \tag{8}$$

where f is the particle distribution, $\widehat{n} = \boldsymbol{B}/B$, $v_\parallel$ is the parallel particle speed, $C(f)$ is the Coulomb collision operator, and $\boldsymbol{v}_d$ is the drift velocity which for low-β (β is the ratio of the plasma pressure to the magnetic field pressure) plasma

$$\boldsymbol{v}_d = -v_\parallel \widehat{n} \times \boldsymbol{\nabla} \left(\frac{v_\parallel}{\Omega} \right). \tag{9}$$

The independent variables in Eq. (8) are (E, μ, ψ, θ) where $E = v^2/2$ is the particle energy, $\mu = v_\perp^2/2B$ is the magnetic moment, v is particle speed, $v_\perp$ is the perpendicular (to $\boldsymbol{B}$) particle speed, and θ is the poloidal angle. The components of $\boldsymbol{v}_d$ are

$$\boldsymbol{v}_d \cdot \boldsymbol{\nabla}\theta = -v_\parallel \widehat{n} \cdot \boldsymbol{\nabla}\theta \, \frac{\partial}{\partial \psi} \left(\frac{I v_\parallel}{\Omega} \right), \tag{10}$$

and

$$\boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi = v_\parallel \widehat{n} \cdot \boldsymbol{\nabla}\theta \, \frac{\partial}{\partial \theta} \left(\frac{I v_\parallel}{\Omega} \right). \tag{11}$$

Following the linearization procedure developed in Refs. 7,9, we obtain the linearized equation for perturbed particle distribution function f_1

$$\left(v_\parallel \widehat{n} + \boldsymbol{v}_d \right) \cdot \boldsymbol{\nabla}\theta \, \frac{\partial f_1}{\partial \theta} + \boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi \, \frac{\partial f_1}{\partial \psi} + \boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi \, \frac{\partial f_0}{\partial \psi} = C(f_1) \tag{12}$$

where f_0 is a Maxwellian distribution function f_M. We like to emphasize that the linearization procedure in Refs. 7,9 is applicable to any local transport problems with finite orbit width. Thus, Eq. (12) is applicable not only to the potato orbits but also to the banana orbits.

The solution to Eq. (12) can be expressed as [7]

$$f_1 = -\frac{I v_\parallel}{\Omega} \, f_M \left(\frac{p'}{p} + \frac{e\Phi'}{T} + y \, \frac{T'}{T} \right) + g, \tag{13}$$

where y is a parameter to be determined and the function g is a localized (in pitch angle) distribution function. It satisfies

$$(v_\parallel \widehat{n} + \boldsymbol{v}_d) \cdot \boldsymbol{\nabla}\theta \, \frac{\partial g}{\partial \theta} + \boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi \, \frac{\partial g}{\partial \psi}$$

$$+(\boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi) f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T} = C(g), \tag{14}$$

where $x = v^2/v_t^2$, and v is the particle speed.

To solve Eq. (14), we transform independent variables of Eq. (14) from (E, μ, ψ, θ) to $(E, \mu, p_\zeta, \theta)$ to obtain

$$(v_\parallel \widehat{n} + \boldsymbol{v}_d) \cdot \boldsymbol{\nabla}\theta \, \frac{\partial g}{\partial \theta} + (\boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi) f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T} = C(g), \tag{15}$$

where $p_\zeta = I v_\parallel / \Omega - \psi$ is the toroidal canonical momentum. In (E, μ, p_ζ) space,

$$(v_\parallel \widehat{n} + \boldsymbol{v}_d) \cdot \boldsymbol{\nabla}\theta = -\frac{I}{\Omega} \frac{\partial p_\zeta / \partial \psi}{\partial p_\zeta / \partial E} \, \widehat{n} \cdot \boldsymbol{\nabla}\theta, \tag{16}$$

and

$$\boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi = \frac{I}{\Omega} \frac{\partial p_\zeta / \partial \theta}{\partial p_\zeta / \partial E} \, \widehat{n} \cdot \boldsymbol{\nabla}\theta. \tag{17}$$

We are interested in the collisionless regime where the effective collision frequency ν_{eff} is less than the trapped particle bounce frequency ω_b. In this limit, the function g can be expanded in terms of the small parameter $\nu_{\text{eff}}/\omega_b$ to obtain $g = g_0 + g_1 + \cdots$ where g_0 and g_1 satisfy

$$(v_\parallel \widehat{n} + \boldsymbol{v}_d) \cdot \boldsymbol{\nabla}\theta \, \frac{\partial g_0}{\partial \theta} + \boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T} = 0, \tag{18}$$

and

$$(v_\parallel \widehat{n} + \boldsymbol{v}_d) \cdot \boldsymbol{\nabla}\theta \, \frac{\partial g_1}{\partial \theta} = C(g_0). \tag{19}$$

The function g_0 is then

$$g_0 = -(\psi - \psi_0) f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T} + h, \tag{20}$$

where h is an integration constant, and ψ_0 is the reference radial position for local transport coefficients. Note that Eq. (20) is valid for any shape of particles in tokamaks. If we had a general expression for $(\psi - \psi_0)$, Eq. (20) is the basis for a complete transport theory. Unfortunately, we only have simple analytic expressions for two limits. One is the $\psi \to \infty$, the banana limit and the other is $\psi \to 0$ limit, the potato limit. In the banana limit, g_0 becomes

$$g_0 = -\frac{Iv_\|}{\Omega} f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T} + h_b, \tag{21}$$

which is the well-known banana regime solution.[3,4] In the potato limit,

$$g_0 = -\frac{4}{3}\frac{I\omega}{\Omega} f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T} + h_p, \tag{22}$$

as given in Ref. 7, where $\omega = (v_\| \hat{n} + \boldsymbol{v}_d) \cdot \boldsymbol{\nabla}\theta / (\hat{n} \cdot \boldsymbol{\nabla}\theta)$.

It is obvious then that potato and banana transport theories are two limits of a complete theory. The complete analytical theory is not available because a simple yet general analytical expression for the orbit width has not been found. One should not view potato transport theory as an isolated theory.

It is interesting to note that the spatial ψ dependence in Eqs. (20) is converted to the velocity space dependence through the conservation of the toroidal canonical momentum p_ζ in Eqs. (21) and (22). The ω calculated in Ref. 7 do not have explicit p_ζ dependence. Furthermore, if we neglect the radial variation of B and Φ, $v_\|$ also has no explicit p_ζ dependence. Because h_b and h_p are driven by $(Iv_\|/\Omega)$ and $(I\omega/\Omega)$ respectively, h_b and h_p should not depend on p_ζ explicitly as well. Thus, it is not necessary to keep $\partial/\partial p_\zeta$ term in the pitch angle scattering operator to determine h_b and h_p through the standard procedure. (The $\partial/\partial p_\zeta$ term occurs because of the change of the independent variables.) If one wishes to include the radial variation of B and Φ in $v_\|$ in the calculation, one needs to calculate $(\psi - \psi_0)$ to include these effects. A simple example is shown in Appendix A.

It is obvious that the solutions in Eqs. (21) and (22) lead to the standard banana transport coefficients and potato transport coefficients respectively. However, we have shown that standard banana transport coefficients are also valid when the width of the banana orbits is taken into account. The width of the banana orbits is included in the $\boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi\ \partial f_1/\partial\psi$ term in Eq. (12). Of course, we still assume that the width of the banana orbits is less than the gradient scale length.

One of the differences between the banana theory and the potato theory is that in the banana theory $\boldsymbol{v}_d \cdot \boldsymbol{\nabla}\theta \ll v_\| \widehat{n} \cdot \boldsymbol{\nabla}\theta$ and can be neglected, while in the potato theory $\boldsymbol{v}_d \cdot \boldsymbol{\nabla}\theta \approx v_\| \widehat{n} \cdot \boldsymbol{\nabla}\theta$. One can easily estimate the condition that will make the approximation $v_\| \widehat{n} \cdot \boldsymbol{\nabla}\theta \gg \boldsymbol{v}_d \cdot \boldsymbol{\nabla}\theta$ valid. Since the slowest moving particles are trapped particles, $v_\| \approx v_t \sqrt{\epsilon}$. Noting that $\boldsymbol{v}_d \cdot \boldsymbol{\nabla}\theta / \widehat{n} \cdot \boldsymbol{\nabla}\theta \approx [(v_\|^2 + \mu B) I C_1 B_0 / (2\sqrt{\psi}\,\Omega B)] \cos\theta$ and $\epsilon = C_1 \sqrt{\psi}$, we conclude when

$$\psi > (\rho^2 I^2 C_1)^{2/3} \tag{23}$$

$v_\| \widehat{n} \cdot \boldsymbol{\nabla}\theta \gg \boldsymbol{v}_d \cdot \boldsymbol{\nabla}\theta$. Here, ρ is the ion gyroradius. The right-hand side of Eq. (23) is the width of the potato orbits. The meaning of the condition in Eq. (23) is simply that in the region greater than a potato width $v_\| \widehat{n} \cdot \boldsymbol{\nabla}\theta > \boldsymbol{v}_d \cdot \boldsymbol{\nabla}\theta$.

Similarly, potato-plateau and banana-plateau regimes are the two limits of a general plateau theory. To see this we transform independent variables of Eq. (14) from (E, μ, ψ, θ) to $(E, \mu, p_\zeta, \theta)$ to obtain

$$\omega \widehat{n} \cdot \boldsymbol{\nabla}\theta \, \frac{\partial g_1}{\partial \theta} - C(g) = -(\boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi) f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T}. \tag{24}$$

where the explicit expression of ω is $\omega = (v_\| \widehat{n} + \boldsymbol{v}_d) \cdot \boldsymbol{\nabla}\theta / \widehat{n} \cdot \boldsymbol{\nabla}\theta$. In the plateau regime, it is convenient to adopt ω as an independent variable instead of μ. In $(E, \omega, p_\zeta, \theta)$ coordinates, Eq. (24) has the form[10]

$$\omega \, \frac{\partial g_1}{\partial \theta} + \left(\frac{\partial \omega}{\partial \theta}\Big|_{E,\mu,p_\zeta} \right) \frac{\partial g}{\partial \omega} - C(g) = -(\boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi) f_M \left(x - \frac{5}{2} - y \right) \frac{T'}{T}. \tag{25}$$

The $(\partial\omega/\partial\theta)$ term is the 'mirror force'-like term. It describes the force that pulls trapped particles back from the turning points. In the plateau regime, the mechanism that removes the singularity at $\omega \approx 0$ in Eq. (25) is collisions. The mirror-force-like term can be neglected by the definition of the plateau regime. In the collisionless regime, the mechanism that removes the $\omega \approx 0$ singularity is the mirror force-like term to complete the collisionless orbit trajectory. Since collisions are only used to remove $\omega \approx 0$ singularity, a Krook model for $C(g)$ is adequate. Note that even though particles cannot complete their collisionless trajectory, in the plateau regime the explicit form of ω is needed to obtain the condition upon which mirror force-like term can be neglected.

Of course, a general solution to Eq. (25) can be found once the mirror force-like term is neglected. In two limits one can obtain explicit simple solutions. One is the $\psi \to 0$, the potato-plateau limit. The other is the $\psi \to \infty$, the banana-plateau limit. The transition between the two is the condition given in Eq. (23). When Eq. (23) is satisfied $\omega \approx v_{\|}$, the familiar form in the banana-plateau theory. It is also obvious then, that the region over which potato-plateau theory valid is of the order of a potato width.

IV. SOURCES

It is well-known that the fueling rate and the heating rate in local transport theory is of the same order as the diffusion rate.[3] Thus, the source rate is second order in Δ_r/L_p, in other words, $(\Delta_r/L_p)^2$ order. The perturbed distribution function is, on the other hand, calculated from the first order, i.e., (Δ_r/L_p) order, drift kinetic equation. It is consistent then to neglect the source term in calculating perturbed distribution for both potato and banana transport theories. It is also obvious that it is perfectly consistent to include source terms in the particle and energy transport equations with potato and banana transport coefficients.

Note that even in neutral particle beam, or fusion alpha particle, or radio-frequency wave heated tokamak plasmas, the transport coefficients calculated from a Maxwellian equilibrium distribution are still valid. The reason is that bulk plasmas are still close to a Maxwellian distribution for these plasmas. Only the tail part of the distribution are modified by the heating or fueling sources. The transport processes are usually dominated by the bulk plasmas. It is for this reason that standard banana transport coefficients are frequently employed in simulating high power heated tokamak plasmas. We expect the potato transport coefficients will be coupled to the banana transport coefficients in simulating tokamak transport behavior in the entire plasma column.

If the heating source is localized in ψ, there can only be two consequences. One is that L_p becomes narrow. In this case, orbit squeezing theory takes over as discussed in Sec. II. The other is L_p is not affected. In that case, the standard theory applies. In either case, the orderings discussed here hold.

V. EQUILIBRIUM DISTRIBUTION FUNCTION AND NON-LOCAL TRANSPORT

As discussed in Sec. II, because of the orbit squeezing effects the real orbit width is almost always less than the gradient scale length, the local transport theory is valid for tokamak plasmas. Here, we simply examine qualitatively the necessary ingredients for a non-local transport theory.

In a non-local transport theory, all the terms on the left-side of the steady state drift kinetic equation, i.e., Eq. (8), are of the same order. In the collisionless regime where collision frequency ν is less than the particle bounce frequency ω_b, we can expand Eq. (8) as

$$(v_\parallel \widehat{n} + \boldsymbol{v}_d) \cdot \nabla\theta \, \frac{\partial f_0}{\partial \theta} + \boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi \, \frac{\partial f_0}{\partial \psi} = 0, \tag{26}$$

and

$$\Big(v_\parallel \widehat{n} + \boldsymbol{v}_d\Big) \cdot \boldsymbol{\nabla}\theta \, \frac{\partial f_1}{\partial \theta} + \boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi \, \frac{\partial f_1}{\partial \psi} = C(f_0). \tag{27}$$

From Eq. (26), we conclude the equilibrium distribution is constant on the particle drift trajectory. The equilibrium distribution function f_0 is calculated from the orbit averaged collision operator $\langle C(f_0)\rangle_b = 0$, which is the constraint condition for Eq. (27). It is then obvious that the equilibrium distribution is non-Maxwellian in nature. The $\langle \quad \rangle_b$, here, denote orbit averaging. The 'correction' to the Maxwellian distribution is of the order of unity and is non-expandable. Physically this is because when the radial diffusion rate is of the order of the collision frequency, the particles have no time to relax to a Maxwellian distribution before they transport to a different radial position. We conclude then that equilibrium distribution function in a non-local transport theory must be non-Maxwellian. Note that this Non-Maxwellian nature of the equilibrium distribution function is the basic feature of the non-local transport theory. It has nothing to do with the source term. Whether the ordered set of equations, i.e., Eq. (26) and (27) has any relation to those in Ref. [11] remains to be seen.

Because the equilibrium distribution is non-Maxwellian, the orbit averaged collision operator is in general an integral-differential equation. It is difficult to solve even numerically. Fortunately, local transport theory is valid in most of the tokamak transport applications as discussed in previous sections.

We would like to note that solution of drift kinetic equation that can be approximated by a Maxwellian distribution function with a small perturbation always leads to the local transport theory that has already been developed.

VI. CONCLUDING REMARKS

We have discussed some transport issues in tokamak transport applications related especially to improved confined modes. Based on the discussions, we conclude: (1) Because of the orbit squeezing effects, the real orbit width is almost always less than the gradient scale length, local transport theory is valid even when the nominal orbit width is comparable to the gradient scale length. It is clear that orbit squeezing theory is an integral part of the local transport theory. (2) Potato transport theory and banana transport theory are two limits of a complete theory. The potato transport theory is the $\psi \to 0$ limit and banana is the $\psi \to \infty$ limit; (3) Local transport coefficients are defined over a distance larger than the width of the realistic orbits. Because potato orbits smoothly transform to the banana orbits and vice versa depending on the radial position at where the trapped particles located, one can always find an intermediate scale to define the transport coefficients. As $\psi \to 0$ the transport coefficients have more and more potato transport coefficient characteristics and as $\psi \rightarrow \infty$ more and more banana transport coefficient characteristics. (4) The standard banana transport coefficients are also valid when the width of the banana orbits is taken into account. (5) Because the source rate is of the order of $(\Delta_r/L_p)^2$ it is consistent to calculate perturbed distribution function from the first order drift kinetic equation without the source term. This is true even in high power heated plasmas because bulk plasmas which usually dominate the transport processes are still close to a Maxwellian distribution. (6) The equilibrium distribution function for a non-local transport theory is non-Maxwellian because particles have no time to relax to a Maxwellian before they transport away to a different position. The equation that governs the equilibrium distribution function is an integral-differential equation.

Appendix A: Banana Orbit

Banana orbit trajectory is well known. The purpose of this appendix is to illustrate that perturbed distribution function can be cast in a form that does not depend on p_ζ.

From p_ζ conservation, we have

$$\frac{Iv_\parallel}{\Omega} - \psi = \frac{Iv_{\parallel 0}}{\Omega_0} - \psi_0, \tag{A1}$$

where the subscript "0" indicates quantities are evaluated at (ψ_0, θ_0). For convenience, we choose $\theta_0 = 0$, the outside of the torus. From energy conversation,

$$v_\parallel^2 = 2\left(E - \mu B - \frac{e\Phi}{M}\right). \tag{A2}$$

We are interested in the case that Φ has a gradient. In the vicinity of ψ_0, we expand

$$\Phi = \Phi_0 - \Phi_0'(\psi - \psi_0). \tag{A3}$$

For simplicity, we neglect Φ_0'' which, if included, leads to orbit squeezing. Following the standard procedure, and assuming $\epsilon \ll 1$, we obtain, from Eqs. (A1)-(A3),

$$\psi - \psi_0 \cong -\frac{I}{\Omega}\left(v_{\parallel 0} + \frac{Ie\Phi_0'}{\Omega_0 M}\right) \pm \left\{\left[\frac{I}{\Omega_0}\left(v_{\parallel 0} + \frac{Ie\Phi_0'}{\Omega_0 M}\right)\right]^2 - 4\left(\frac{I}{\Omega_0}\right)^2 (v_{\parallel 0}^2 + \mu B_0)\epsilon \sin^2\frac{\theta}{2}\right\}^{1/2}. \tag{A4}$$

From Eq. (A4), we can define an effective parallel particle speed $v_{\parallel\mathrm{eff}}$ as

$$v_{\parallel\mathrm{eff}} = \pm\left|v_{\parallel 0} + \left(\frac{Ie\phi_0'}{\Omega_0 M}\right)\right|\left(1 - \kappa\,\sin^2\frac{\theta}{2}\right)^{1/2}, \tag{A5}$$

where

$$\kappa = 4\frac{(v_{\parallel 0}^2 + \mu B_0)\epsilon}{\left(v_{\parallel 0} + \frac{Ie\Phi_0'}{\Omega_0 M}\right)^2}. \tag{A6}$$

The circulating particles have $0 < \kappa < 1$, and trapped particles $1 < \kappa < \infty$.

The g_0 function is then

$$g_0 = -\frac{Iv_{\parallel\mathrm{eff}}}{\Omega} f_M \left(\chi - \frac{5}{2} - y\right)\frac{T'}{T} + h_{b'}. \tag{A7}$$

We see that $v_{\|\mathrm{eff}}$ has no explicitly p_ζ dependence. Standard neoclassical transport coefficients can be obtained by setting $S = 1$ in Ref. [12].

One can also take the radial gradient of B into account. In this case, the radial dependence ϵ on ψ must be kept in deriving the $(\psi - \psi_0)$ equation. This equation has already been derived in Ref. [7]. It is

$$(\psi - \psi_0)^2 + 2\,\frac{Iv_{\|0}}{\Omega_0}\,(1 + \epsilon_0 \cos\theta_0)\,(\psi - \psi_0) + \frac{2I^2}{\Omega_0^2}\left(v_{\|0}^2 + \mu B_0\right)(\epsilon_0 \cos\theta_0 - \epsilon \cos\theta) = 0. \quad \text{(A8)}$$

Note that because $\epsilon = C_1\sqrt{\psi}$, Eq. (A8) is, in general, a quartic equation of ψ. Equation (A8) describes both banana orbits, potato orbits, and the transition between these two classes of orbits. It is obvious from Eq. (A8) that $(\psi - \psi_0)$ is a function of (E, μ).

From this simple example and examples elsewhere, we see that there is no explicit p_ζ dependence in g_0, once ψ_0 is specified in a local transport theory. We would like to emphasize that the 'spatial' variation of g_0 is weak, when it is expressed in terms of $\omega, v_\|$, or $v_{\|\mathrm{eff}}$. It has strong spatial variation when it is expressed in terms of $(\psi - \psi_0)$.

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