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SEARCHING FOR INTEGRABLE SYSTEMS

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Lack of integrability leads to undesirable consequences in a number of physical systems. The lack of integrability of the magnetic field leads to enhanced particle transport in stellarators[1] and tokamaks[2] with tearing mode turbulence. Limitations of the luminosity of colliding beams may be due to the onset of stochasticity.[3] Enhanced radial transport in mirror machines caused by the lack of integrability and/or the presence of resonances may be a significant problem in future devices.[4]

To improve such systems one needs a systematic method for finding integrable systems. Of course, it is easy to find integrable systems if no restrictions are imposed; textbooks are full of such examples. The problem is to find integrable systems given a set of constraints.

An example of this type of problem is that of finding integrable vacuum magnetic fields with rotational transform. The solution to this problem is relevant to the magnetic confinement program.[5-7]

In vacuum Ampere's law dictates that magnetic fields be curl-free in addition to being divergence free. Thus,

$$\underline{B} = \nabla\Phi \ , \tag{1}$$

where

$$\nabla^2\Phi = 0 \ . \tag{2}$$

In the coordinate system (ξ,η,ϕ), where

$$R = R_0(1-\xi^2)^{1/2}/(1-\xi \cos\eta) \ , \tag{3a}$$

$$Z = -R_0\xi \sin\eta/(1-\xi \cos\eta) \ , \tag{3b}$$

and (R,ϕ,z) are the usual cylindrical coordinates, the general solution to (2) allowing for a current on the z-axis is

$$\Phi = I\phi + (\xi^{-1} - \cos\eta)^{1/2} \sum_{\ell,m} \alpha_{\ell m} Q^m_{\ell-1/2}(\xi^{-1})e^{i\ell\eta+im\phi} \ , \tag{4}$$

in which the coefficients I and $\alpha_{\ell m}$ are arbitrary, within the reality restriction $\alpha_{\ell m} = \alpha^*_{-\ell,m}$, and $Q^\mu_\nu(x)$ is the modified Legendre function of the second kind.

An integrable magnetic field is one for which the magnetic fields lie in nested toroidal "flux surfaces". The condition of nonzero rotational transform is dictated by physical requirements imposed by particle orbit theory. This condition also eliminates the trivial solutions, in which all field lines are closed, such the field due to a single current on the z-axis.

A method for obtaining integrable vacuum magnetic fields has recently been published.[8,9] This method is iterative. The first iteration has been shown to significantly decrease stochasticity. A complete discussion of this method has been presented in Refs. 8 and 9. In this paper a discussion of one of the ideas is presented.

A model stochastic system is that of the flow caused by the Hamiltonian

$$h = h_0(p) + \varepsilon[f_1(p)\cos(2\pi q) + f_2(p)\cos(2\pi q - 2\pi t)] , \quad (5)$$

in which h_0, f_1, and f_2 are particular functions to be specified shortly, and ε is a constant. For the choices

$$h_0 = \frac{1}{2} p^2 , \quad (6a)$$

$$f_1 = 1 , \quad (6b)$$

and

$$f_2 = 1 , \quad (6c)$$

this system has been studied in detail.[10,11] In general this system is not integrable because two resonances are present. This system would be integrable if either f_1 or f_2 vanished.

The two resonances are illustrated in the surface of section of Fig. 1, which is for the case of Eqs. (5) and (6) with $\varepsilon = 0.01$. The calculation of the widths is standard.[12] The half-width for resonance i(=1 or 2, corresponding to f_i) is

$$\Delta p_i = 2\sqrt{\varepsilon f_i(v_i)} \quad (7a)$$

$$= 2\sqrt{\varepsilon} \quad (7b)$$

where

$$v_i = \begin{cases} 0 & i=1 \\ 1 & i=2 \end{cases} \quad (8)$$

is the velocity of resonance i.

As ε is increased, the resonances overlap. The result is stochastic motion. This is illustrated in Fig. (2), a surface of section for the case $\varepsilon = 0.05$. One notes that the stochastic orbit covers a range in the momentum variable of $\Delta p = 1.9$.

Suppose instead one is given the Hamiltonian

$$H = h_0(p) + \varepsilon[f_1(p)\cos(2\pi q) + f_2(p)\cos(2\pi q-2\pi t)]$$

$$+ \alpha_1 g_1(p)\cos(2\pi q) + \alpha_2 g_2(p)\cos(2\pi q-2\pi t) \; , \tag{8}$$

with g_1 and f_1, and g_2 and f_2 being linearly independent functions. Suppose furthermore that ε is fixed as before, but one is allowed to choose the values of α_1 and α_2 with the goal being to have a system as integrable as possible. How are α_1 and α_2 to be chosen? Obviously, no choice of α_1 and α_2 will completely eliminate stochasticity, because there will always be two resonances.

Since stochasticity increases with island size, it is natural to use α_1 and α_2 to diminish the island size. According to Eq. (7), this prescription gives

$$\alpha_i = -\varepsilon f_i(v_i)/g_i(v_i) \; . \tag{9}$$

Of course, in this case the formula (7a) is no longer accurate, since its derivation relied on the resonance amplitude being a slowly varying function of p.

To determine the island width for the case when Eq. (9) is imposed, we consider the island structure for the Hamiltonian,

$$H = \frac{1}{2} p^2 + \varepsilon p \cos(2\pi q) \ , \tag{10}$$

which contains just one of the resonances. Being time independent, the flow due to the Hamiltonian (10) conserves the Hamiltonian. The level curves are shown in Fig. 3. One notes that there are two islands for this resonance. The separatrix is now defined by the two curves, $p=0$ and $p = -2\varepsilon \cos(q)$. Most important is the fact that the island half width is now

$$\Delta p = 2\varepsilon \ . \tag{11}$$

This implies that given a small value of ε, the island is much smaller than before (7b). Fig. 4 is a surface of section for the Hamiltonian

$$H = \frac{1}{2} p^2 + \varepsilon \{p \cos(2\pi q) + (p-1)\cos[2\pi(q-t)]\} \ , \tag{12}$$

with $\varepsilon = 0.05$. This is a model Hamiltonian of the type (5), but for which the resonance amplitudes $f_i(p)$ vanish at the resonance, yet their derivatives are $\mathcal{O}(1)$. As one can see, in comparison with Fig. 2, the islands are much smaller and the stochastic region is greatly diminished.

Another way of looking at the problem is to consider perturbation theory for a Hamiltonian of the form,

$$h = h_0(p) + \varepsilon \sum_{mn} f_{mn}(p) e^{i2\pi(mq-nt)} , \tag{13}$$

which includes all systems which are periodic in q and t. Suppose one now applies perturbation theory to attempt to determine a quantity P which is conserved to $\mathcal{O}(\varepsilon)$. Standard analysis[12] shows that the generating function

$$F(q,P,t) = qP + \varepsilon \sum_{mn} F_{mn}(P) e^{i2\pi(mq-nt)} \tag{14}$$

must satisfy

$$F_{mn}(P) = (i/2\pi) f_{mn}(P)/(m \frac{\partial h_0}{\partial p}(P) - n) . \tag{15}$$

Thus, the analysis is invalid unless each resonance amplitude vanishes at its resonant surface:

$$f_{mn}(P_{mn}) = 0 ,$$

where (16)

$$\frac{\partial h_0}{\partial p}(P_{mn}) = n/m .$$

The perturbation theory indicates how one might proceed to higher order. One has to be able to choose the Hamiltonian to make the resonance amplitude vanish at the resonant surface to each order. Thus, higher order calculations may slightly modify the values of coefficients like α_i as given by lower order theory. Of course, like any perturbation technique this method likely relies on the smallness of the stochasticity.

Application of these ideas to the problem of improving the integrability of vacuum magnetic fields has been successful.[8,9] In that case it seems, intuitively, that sufficient freedom of choice is available. However, a mathematical proof has not been given. This is the subject of further research.

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Figure Captions

Fig. 1 -

Surface of section for the Hamiltonian of Eqs. (5) and (6) with $\varepsilon = 0.01$.

Fig. 2 -

Surface of section for the Hamiltonian of Eqs. (5) and (6) with $\varepsilon = 0.05$.

Fig. 3 -

Structure of the resonance of the Hamiltonian of Eq. (10).

Fig. 4 -

Surface of section for the Hamiltonian of Eq. (12) for $\varepsilon = 0.05$.

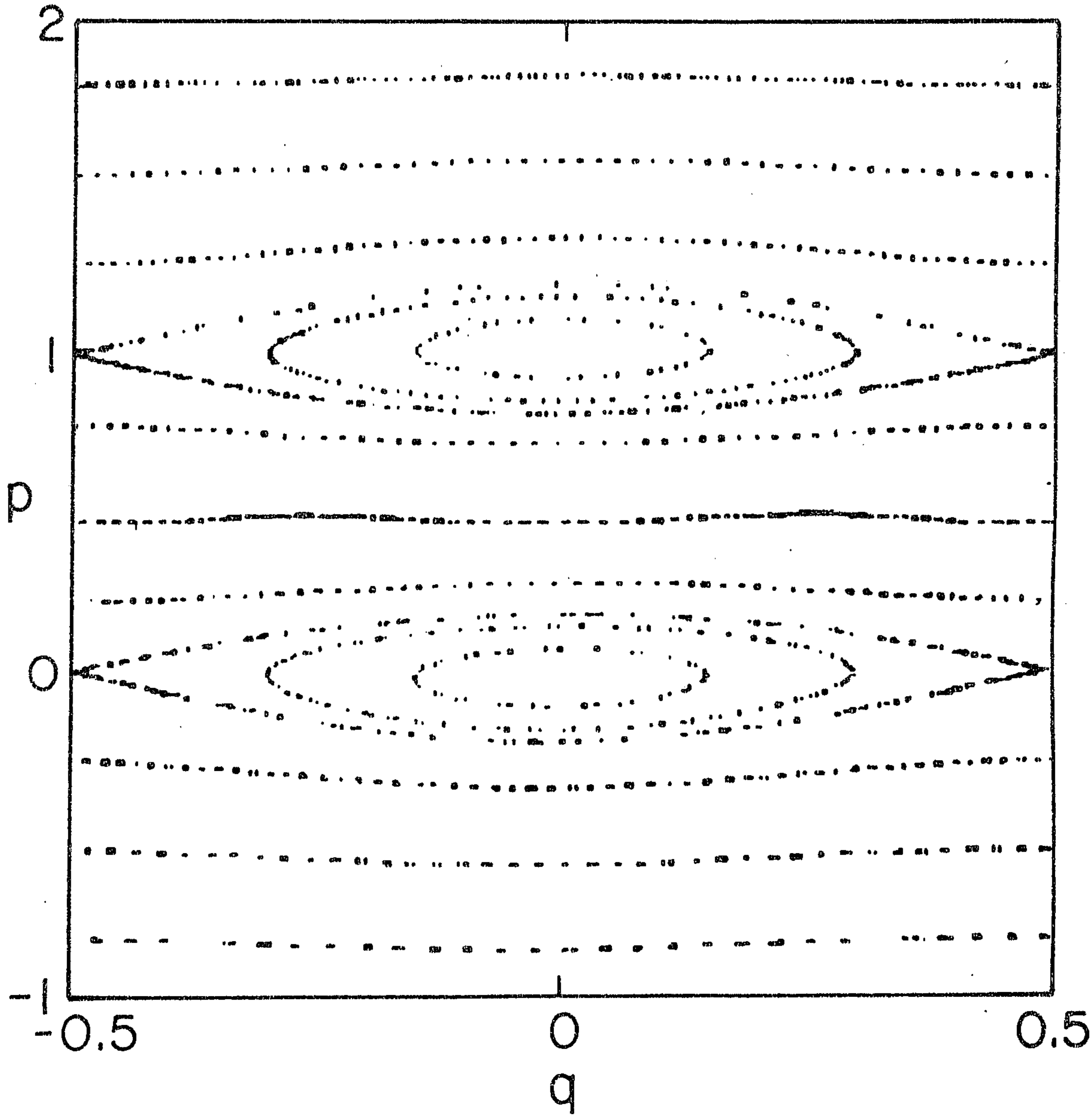

FIG. 1

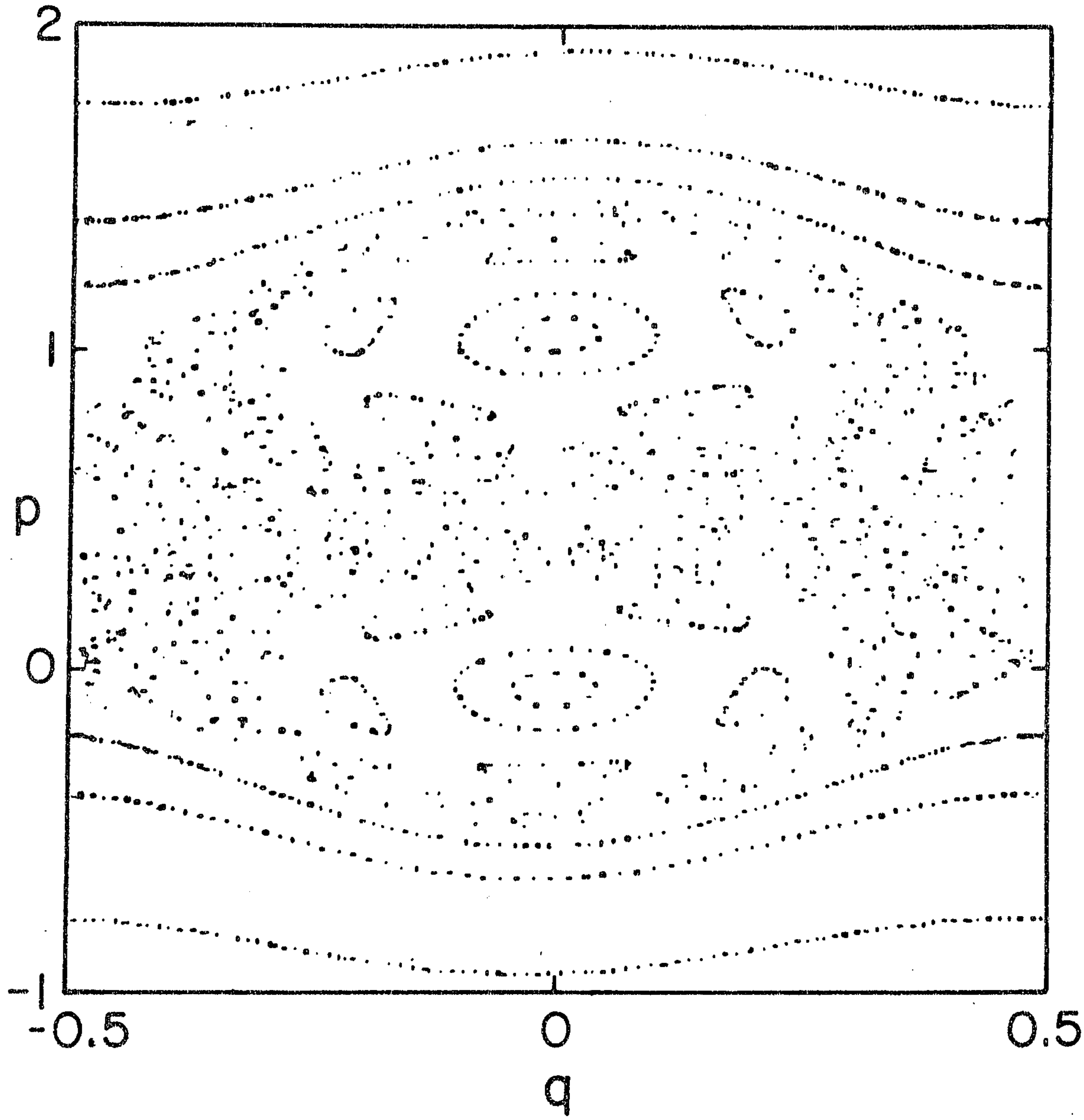

FIG. 2

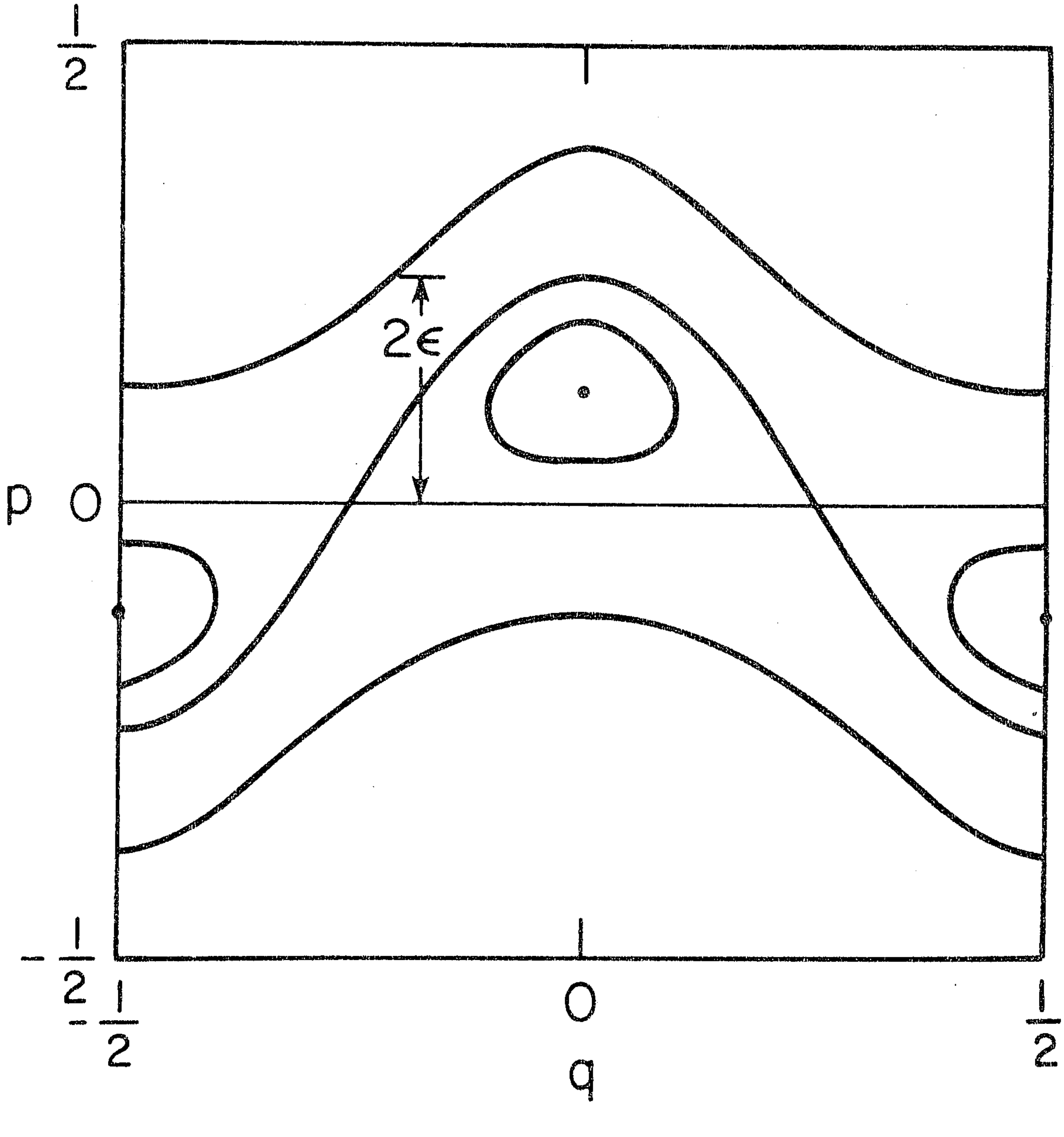

FIG. 3

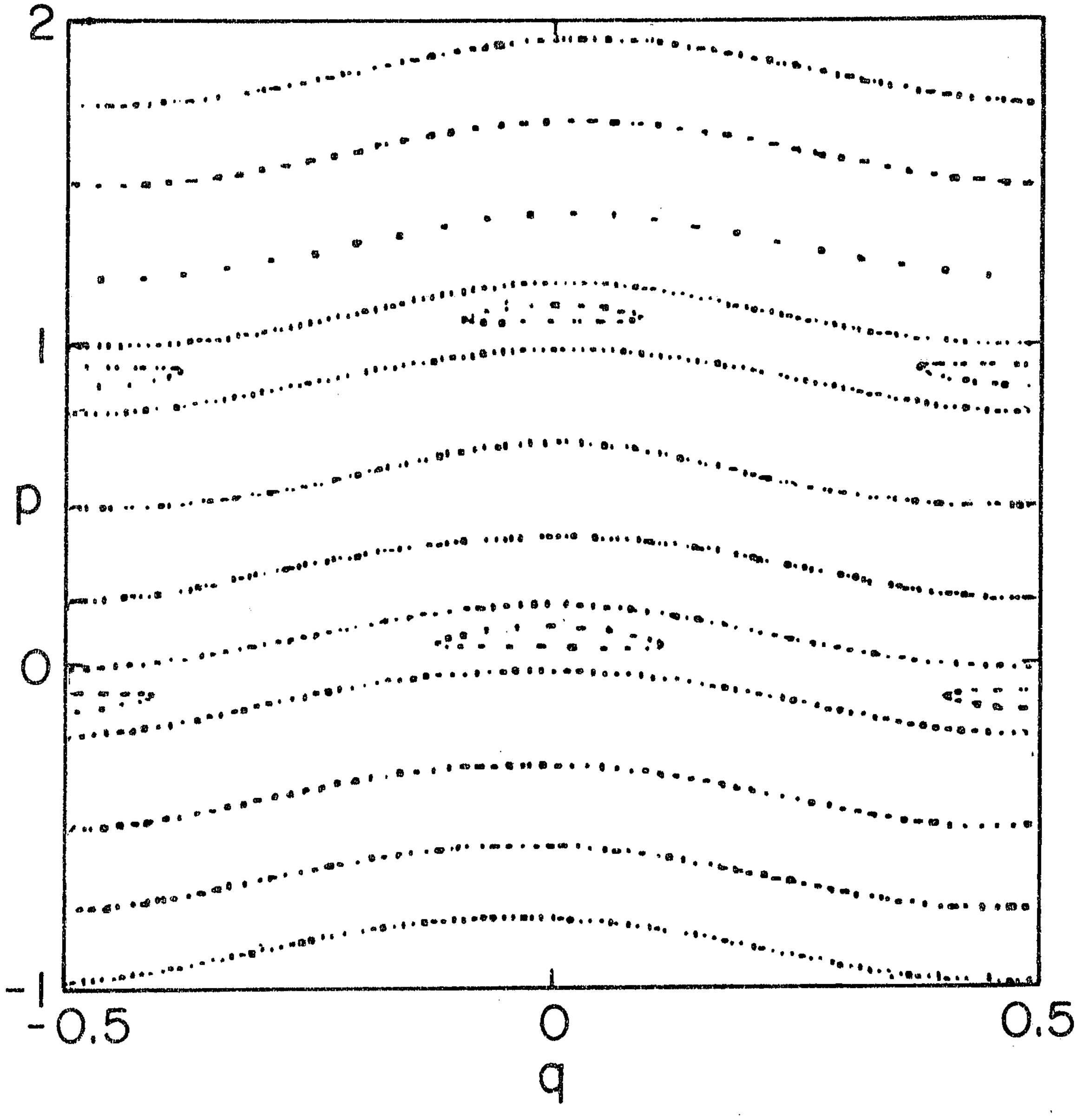

FIG. 4

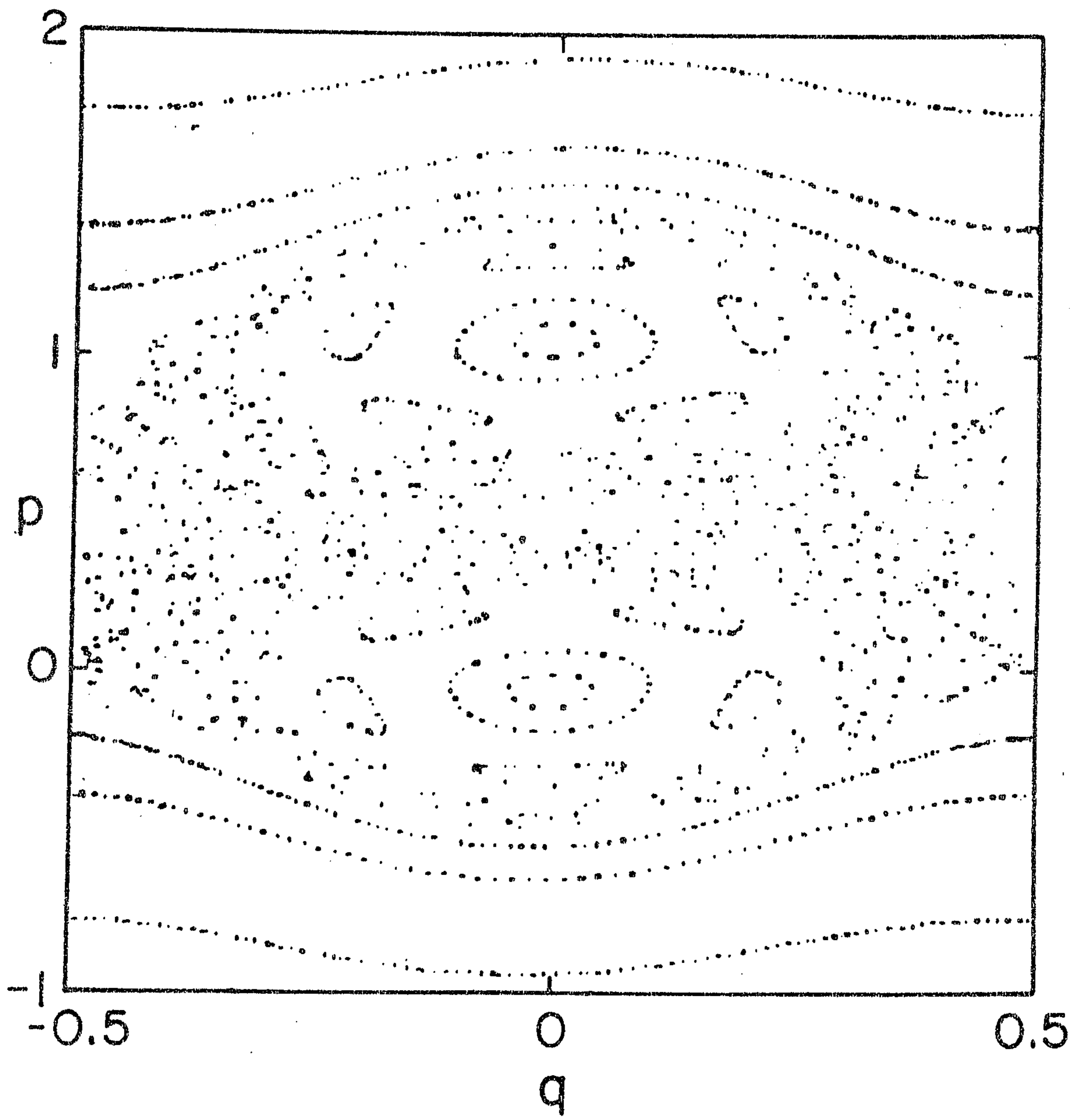

FIG. 2

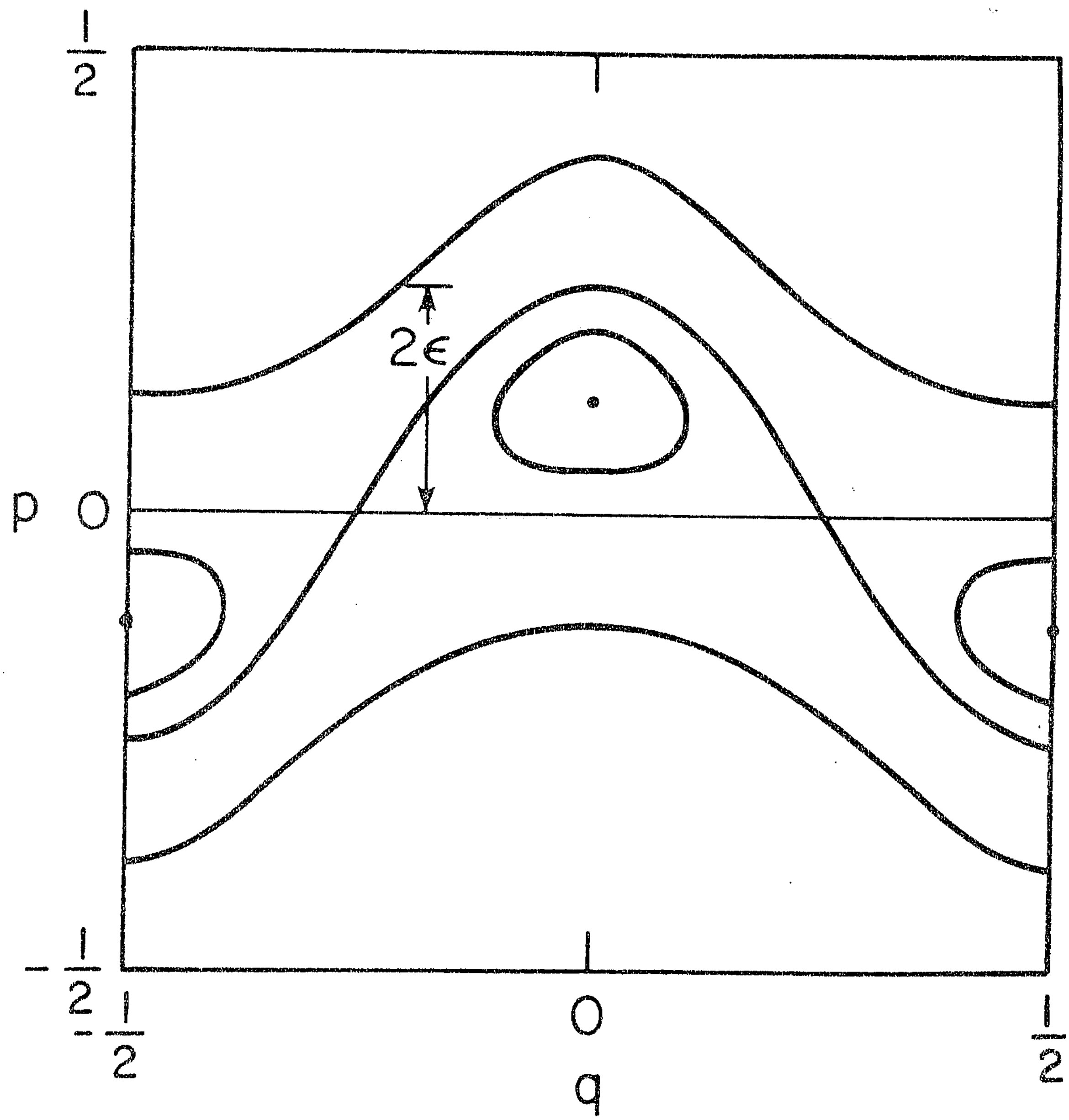

FIG. 3

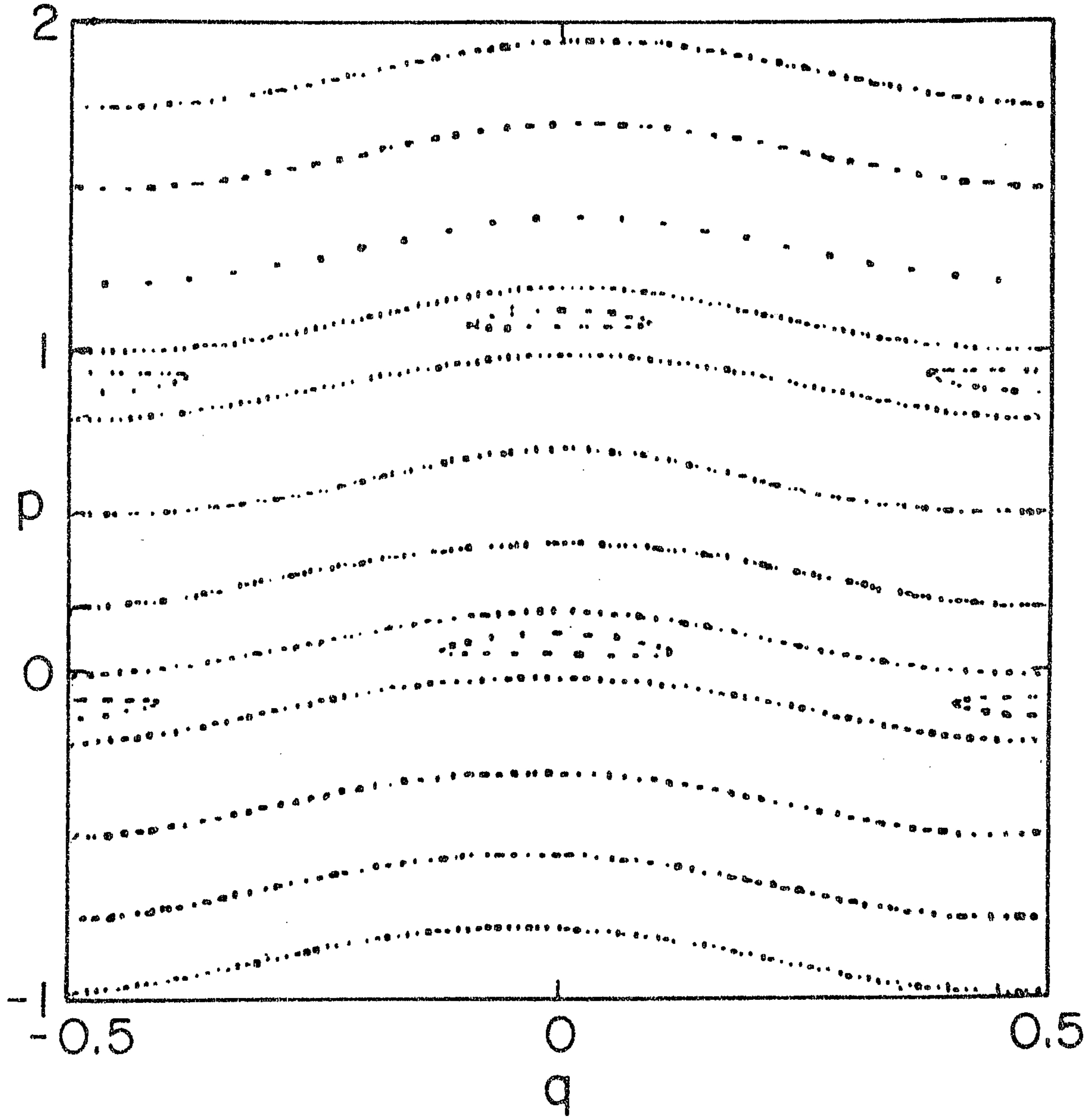

FIG. 4